Research Article

$q$-Genocchi Numbers and Polynomials Associated with Fermionic $p$-Adic Invariant Integrals on $\mathbb{Z}_p$

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The main purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials. In particular, by using the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, we construct $p$-adic Genocchi numbers and polynomials of higher order. Finally, we derive the following interesting formula:

$$G_n^{(k)} + q^n x^n = 2^k k! \sum_{l=0}^{k} \sum_{d_1, d_2, \ldots, d_k \leq k-1} \sum_{d_l \in \mathbb{N}} (-1)^l (t + x)^n,$$

where $G_n^{(k)}(x)$ are the $q$-Genocchi polynomials of order $k$.

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1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$.

In $\mathbb{C}$, the ordinary Euler polynomials are defined as

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi).$$  \tag{1.1}$$

In the case $x = 0$, $E_n(0) = E_n$ are called Euler numbers, see [1–13]. Let $\delta_{0,n}$ be the Kronecker symbol. From (1.1) we derive the following relation:

$$E_0 = 1, \quad (E + 1)^n + E_n = 2\delta_{0,n}, \quad n \in \mathbb{N}.  \tag{1.2}$$
Here, we use the technique method notation by replacing $E^n$ by $E_n$ ($n \geq 0$), symbolically. The first few are $1, -1/2, 0, 1/4, \ldots$, and $E_{2k} = 0$ for $k = 1, 2, \ldots$. A sequence consisting of the Genocchi numbers $G_n$ satisfies the following relations:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi), \quad (1.3)$$

see [11, 12]. It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = G_{2k+1} = 0$, $k = 1, 2, 3, \ldots$, and even coefficients are given by

$$G_{2n} = 2(1 - 2^{2n}) B_{2n} = 2nE_{2n-1}(0), \quad (1.4)$$

where $B_n$ is Bernoulli numbers. The first few Genocchi numbers for even integers are $-1, 1, -3, 17, -155, 2073, \ldots$ The first few prime Genocchi numbers are $-3$ and $17$, which occur at $n = 6$ and $8$. There are no others with $n < 10^5$. We now define the Genocchi polynomials as follows:

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \quad (1.5)$$

Thus, we note that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \quad (1.6)$$

In this paper, we use the following notations: $[x]_q = (1 - q^x)/(1 - q)$ and $[x]_{-q} = (1 + (-q)^x)/(1 + q)$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-deformed fermionic integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (1.7)$$

see [1-4]. The fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ can be obtained as $q \to 1$. That is,

$$I_1(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \quad (1.8)$$

From (1.8), we easily derive the following integral equation related to fermionic invariant $p$-adic integral on $\mathbb{Z}_p$:

$$I_1(f_1) + I_1(f) = 2f(0), \quad (1.9)$$

where $f_1(x) = f(x + 1)$, see [5].

The purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials by using the fermionic multivariate $p$-adic invariant integral on $\mathbb{Z}_p$. In addition, we will investigate some interesting identities related to Genocchi numbers and polynomials.
2. Genocchi numbers associated with fermionic p-adic invariant integral on $\mathbb{Z}_p$

From (1.9) we can derive

$$ t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!}, $$

$$ t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, $$

where $G_n(x)$ are Genocchi polynomials. It is easy to check that

$$ t \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2t}{e^t + e^{-t}} = t \sech t = \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} 2^l G_l \right) \frac{t^n}{n!}. $$

By comparing the coefficient on both sides in (2.1), we easily see that

$$ \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(x) = \frac{G_{n+1}(x)}{n+1}. $$

Therefore, we obtain the following proposition.

**Proposition 2.1.** For $k \in \mathbb{Z}_+$,

(i) $\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(x) = G_{n+1}(x)/(n+1)$ (Witt’s formula for Genocchi polynomials);

(ii) $\int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x) = \left( \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} 2^l G_l \right) \frac{t^n}{n!} \right) (1/(2 \zeta(n+1))) / \zeta(n+1)!. $

Let $\mathfrak{o}_C = \{ x \in \mathbb{C}_p \mid |x|_p \leq 1 \}$ be the integer ring of $\mathbb{C}_p$. We note that $i = (-1)^{1/2} \in \mathfrak{o}_C$. By using Taylor expansion, we see that

$$ e^{ix} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. $$

In the p-adic number field, sin $x$ and cos $x$ are defined as

$$ \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, $$

$$ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. $$

From (2.4) and (2.5), we derive

$$ e^{ix} = \cos x + i \sin x. $$

This is equivalent to

$$ \cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. $$
By (2.7), we easily see that

$$\sec t = \frac{2}{e^{it} + e^{-it}} = \int_{Z_p} e^{(2x+1)i} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \int_{Z_p} (2x + 1)^n d\mu_{-1}(x) \frac{i^n t^n}{n!}. \quad (2.8)$$

It is not difficult to show that \( \int_{Z_p} (2x + 1)^{2n+1} d\mu_{-1}(x) = 0 \) for \( n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). From (2.8), we note that

$$\sec t = \sum_{n=0}^{\infty} i^n \int_{Z_p} (2x + 1)^n d\mu_{-1}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \int_{Z_p} (2x + 1)^{2n} d\mu_{-1}(x) \frac{t^{2n}}{(2n)!}. \quad (2.9)$$

Thus, we have

$$t \sec t = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left( \sum_{l=0}^{2n+1} \binom{2n+1}{l} 2^l G_l \right) \frac{t^{2n+1}}{(2n+1)!}. \quad (2.10)$$

Now we consider the fermionic multivariate \( p \)-adic invariant integral on \( Z_p \) as follows:

$$t^k \int_{Z_p} \cdots \int_{Z_p} e^{(x_1 + x_2 + \cdots + x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{t^k}{(e^t + 1) \cdots (e^t + 1)} = \sum_{n=0}^{\infty} G^{(k)}_n \frac{t^n}{n!}. \quad (2.11)$$

where \( G^{(k)}_n \) are the \( n \)th Genocchi number of order \( k \). By comparing the coefficient on both sides in (2.11), we see that \( G^{(k)}_0 = G^{(k)}_1 = \cdots = G^{(k)}_k = 0, \) and

$$\int_{Z_p} \cdots \int_{Z_p} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) (n + k)_k = G^{(k)}_{n+k}, \quad (2.12)$$

where \( (n + k)_k \) is the Jordan factor which is defined by \( (n + k)_k = (n + k) \cdots (n + 1) \). Thus, we note that

$$\int_{Z_p} \cdots \int_{Z_p} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G^{(k)}_{n+k}}{\binom{n+k}{n} n!}, \quad (2.13)$$

for \( k \in \mathbb{N}, \ n \in \mathbb{Z}_+ \).

**Theorem 2.2.** For \( n \in \mathbb{Z}_+, \ k \in \mathbb{N}, \)

$$\int_{Z_p} \cdots \int_{Z_p} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G^{(k)}_{n+k}}{\binom{n+k}{n} n!}. \quad (2.14)$$
The multinomial coefficient is well known as

\[(x_1 + x_2 + \cdots + x_k)^n = \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}. \tag{2.15}\]

Therefore, we obtain the following corollary.

Corollary 2.3. For \(n \in \mathbb{Z}_+, k \in \mathbb{N}\),

\[
\sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} \left( \frac{G_{l_1+1}}{l_1 + 1} \right) \left( \frac{G_{l_2+1}}{l_2 + 1} \right) \cdots \left( \frac{G_{l_k+1}}{l_k + 1} \right) = \frac{\binom{n+k}{n+1}}{n!}. \tag{2.16}\]

For \(q \in \mathbb{C}_p\) with \(|1-q| < 1\), it is not difficult to show that

\[
t \int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2t}{qe^t + 1}. \tag{2.17}\]

Now, we define the \(q\)-extension of the Genocchi numbers as follows:

\[
\frac{2t}{qe^t + 1} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{2.18}\]

By (2.17) and (2.18), we easily see that

\[
\frac{G_{n+1,q}}{n+1} = \int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x). \tag{2.19}\]

With the same motivation to construct the Genocchi polynomials of higher order, we can consider the \(q\)-extension of higher-order Genocchi numbers as follows:

\[
t^k \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1 + x_2 + \cdots + (k-1)x_k} e^{(x_1 + x_2 + \cdots + x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \]

\[
= \frac{t^k 2^k}{(q^t + 1)(q^{e^t} + 1) \cdots (q^{k-1} e^t + 1)} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(k)} \frac{t^n}{n!}, \tag{2.20}\]

where \(G_{n,q}^{(k)}\) are the \(q\)-Genocchi polynomials of order \(k\). The basic \(q\)-natural numbers are defined as

\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}. \tag{2.21}\]

The \(q\)-factorial of \(n\) is defined as

\[
[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q = (1 + q + \cdots + q^{n-1}) \cdots (1 + q) \cdot 1. \tag{2.22}\]
The $q$-binomial coefficient is also defined as
\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}.
\tag{2.23}
\]

Note that $\lim_{q \to 1} \binom{n}{k}_q = (\binom{n}{k}) = (n(n-1)\cdots(n-k+1))/k!$. The $q$-binomial coefficient satisfies the following recursion formula:
\[
\binom{n+1}{k}_q = \binom{n}{k}_q + q^k \binom{n}{k-1}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q.
\tag{2.24}
\]

From this recursion formula, we can derive
\[
\binom{n}{k}_q = \sum_{d_0+d_1+\cdots+d_k=k-1} q^{d_0+1+d_1+\cdots+d_k}.
\tag{2.25}
\]

The $q$-binomial expansion is given by
\[
\prod_{i=1}^{n} (a + bq^{i-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^{n-k} b^k,
\]
\[
\prod_{i=1}^{n} (1 - bq^{i-1})^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q b^k.
\tag{2.26}
\]

By (2.20) and (2.26), we see that
\[
\int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} q^{x_1+2x_2+\cdots+(k-1)x_k} e^{(x_1+x_2+\cdots+x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= t^k 2^k \prod_{i=1}^{k} (1 - (-q^{i-1})e^t)^{-1} e^{xt}
\]
\[
= t^k 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l e^{(l+x)t}
\]
\[
= t^k 2^k \sum_{n=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l (l+x)^n \frac{t^n}{n!}.
\tag{2.27}
\]

Therefore, we obtain the following theorem.

**Theorem 2.4.** For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$, we have
\[
\int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} q^{x_1+2x_2+\cdots+(k-1)x_k} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l (l+x)^n.
\tag{2.28}
\]
By (2.20), it is not difficult to show that

\[
G_{n+k,q}^{(k)}(x) = k! \binom{n+k}{k} \int_{\mathbb{Z}_q} \ldots \int_{\mathbb{Z}_q} q^{x_2+2x_3+\cdots+(k-1)x_k} \times (x + x_1 + x_2 + \ldots + x_k)^k \times d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)(n+k)_k,
\]

where \( n = 0, 1, 2, \ldots \). Therefore, we obtain the following corollary.

**Corollary 2.5.** For \( n \in \mathbb{Z}_+, \ k \in \mathbb{N}, \)

\[
\frac{G_{n+k,q}^{(k)}(x)}{k!\binom{n+k}{k}} = 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l}_q (-1)^l (l+x)^n. \tag{2.30}
\]

**Corollary 2.6.** For \( n \in \mathbb{Z}_+, \ k \in \mathbb{N}, \)

\[
G_{n+k,q}^{(k)}(x) = 2^k k! \binom{n+k}{k} \sum_{l=0}^{\infty} \sum_{d_0+d_1+\cdots+d_k=k-1} q^{d_0+1d_1+\cdots+k-1d_k} (-1)^l (l+x)^n. \tag{2.31}
\]

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**References**


