Research Article
On Weighted Hadamard-Type Singular Integrals and Their Applications
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By means of an expression with a kind of integral operators, some properties of the weighted Hadamard-type singular integrals are revealed. As applications, the solution for certain strongly singular integral equations is discussed and illustrated.

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1. Introduction

The concept of Hadamard-type singular integrals was first introduced by Hadamard [1], and then developed and adopted in applications by many authors (see [2–13]). This type of integrals is expressed as

\[ \text{f.p.} \int_{\Gamma} \frac{f(\tau)}{(\tau-t)^{m+1}} d\tau, \quad t \in \Gamma^0 \]  \hspace{1cm} (1.1)

and its general definition can be found in Lu [11], where \( \Gamma = ab \) is an open smooth curve on a complex plane, \( f \in C^{m+1}(\Gamma) \), and \( m \) is a positive integer. Although from

\[ \text{f.p.} \int_{\Gamma} \frac{f(\tau)}{(\tau-t)^{m+1}} d\tau = \frac{1}{m!} \text{p.v.} \int_{\Gamma} \frac{f^{(m)}(\tau)}{\tau-t} d\tau + \sum_{r=0}^{m-1} \frac{(m-r-1)!}{m!} \left[ \frac{f^{(r)}(a)}{(a-t)^{m-r}} - \frac{f^{(r)}(b)}{(b-t)^{m-r}} \right] \]  \hspace{1cm} (1.2)

we would get some characteristics of this type of integrals, many of them such as mapping
properties still need to have a further investigation, especially for its “weighted type”:

\[
\text{f.p. } \int_{\Gamma} \frac{w(\tau)f(\tau)}{(\tau - t)^{m+1}} d\tau, \quad t \in \Gamma^0,
\]

where \(w(t)\) is an integrable function. In many cases, \(w(t)\) is a fundamental function derived from some mixed boundary problem and therefore may not be smooth enough or even have certain singularities (see [6, 10, 14]).

It is found that Hadamard-type singular integrals can be expressed effectively by a kind of integral operators which we will define and discuss in the next section. So, in Section 3 of the present paper we directly use this expression as the definition of Hadamard-type singular integrals and it appears that the definition is more advantageous than the traditional one. In this paper, some useful results are developed and then in the final section we use them for the solution of certain strongly singular integral equations. Meanwhile, we illustrate some examples as well.

Throughout the paper we always assume that \(m\) is a nonnegative integer; \(\Gamma\) is an open smooth curve on the complex plane oriented from the point \(a\) to the point \(b\), and \(c_0\) is a fixed positive constant such that for all \(t_1, t_2 \in \Gamma\), the arc length \(|t_1 \widehat{t} t_2| \leq c_0 |t_1 - t_2|\); as usual, \(C(\Gamma)\) and \(C^m(\Gamma)\) denote the spaces of continuous and \(m\)-times continuously differentiable complex-valued functions on \(\Gamma\), respectively, \(\|\psi\| = \max_{t \in \Gamma} |\psi(t)|\), \(\|\psi\|_{C^m} = \sum_{k=0}^{m} \|\psi^{(k)}\|\), and the modulus of continuity for \(\psi \in C(\Omega)\) is denoted by \(\omega(\psi, \epsilon)\), where \(\Omega = \Gamma\) or \(\Gamma \times \Gamma\); for convenience, each absolute constant is denoted by \(c\) but takes different values in different places. And, if there is no confusion, we will omit the symbol \(\Gamma\) in some notations of function classes such as \(C(\Gamma)\) and \(C^m(\Gamma)\), and so forth.

2. Some integral inequalities

It is clear that the kind of integral operators introduced in [15] has a close relation to Hadamard-type integrals. Here we restate their definition as follows.

**Definition 2.1.** Let \(w\) and \(\varphi\) be integrable functions on \(\Gamma\) and assume \(\varphi\) is \(m\) times differentiable at \(t_0 \in \Gamma\). If the integral

\[
k! \int_{\Gamma} w(\tau) \varphi(\tau) - P_k(\varphi; \tau, t_0) \frac{1}{(\tau - t_0)^{k+1}} d\tau
\]

exists, then we denote it by \(T_k(w, \varphi)(t_0)\), where

\[
P_k(\varphi; \tau, t_0) = \varphi(t_0) + \varphi'(t_0)(\tau - t_0) + \frac{\varphi''(t_0)}{2!}(\tau - t_0)^2 + \cdots + \frac{\varphi^{(k)}(t_0)}{k!}(\tau - t_0)^k
\]

and \(k = 0, 1, \ldots, m\). If \(k = 0\), \(T_k\) is written as \(T\).

In this section, we will mainly discuss this kind of operators in some smooth function classes, which are given by the following definition.
Definition 2.2. Let \( y \) be an oriented open smooth curve and let \( \Lambda_n(y) \) denote the function class

\[
\left\{ f \in C(y) : \int_0^1 \frac{\omega_y(f,x)}{x} \ln^{n-1} \frac{1}{x} \, dx < \infty \right\},
\]

(2.3)

where \( \omega_y(\varphi,x) = \max\{|\varphi(t') - \varphi(t'')| : |t' - t''| \leq x, t', t'' \in \gamma \} \) and \( n \) is a positive integer. For differentiable function classes, we let

\[
\Lambda^m_n(y) = \left\{ f \in C^m(y) : f^{(m)} \in \Lambda_n(y) \right\}.
\]

(2.4)

For \( t_0 \in \Gamma \), if we say \( f \in C^m(t_0, \Gamma) \) or \( f \in \Lambda^m_n(t_0, \Gamma) \), it means that \( f \) is an integrable function on \( \Gamma \) and there is a neighborhood \( \mathcal{C} \subset \Gamma \) of \( t_0 \) such that \( f \in C^m(\mathcal{C}) \) or \( f \in \Lambda^m_n(\mathcal{C}) \).

Some properties of modulus of continuity will be used repeatedly and we list them in the following lemma. Their proofs are trivial (cf. [16, Chapter 3]).

Lemma 2.3. Let \( \omega(x) \) be a modulus of continuity. Then

\[
\omega(x) \leq \frac{2}{\ln 2} \int_0^x \frac{\omega(y)}{y} \, dy, \quad x > 0,
\]

(2.5)

\[
\omega(x) \ln \frac{1}{x} \leq 2 \int_0^x \frac{\omega(y)}{y} \, dy, \quad 0 < x \leq 1,
\]

(2.6)

\[
x \int_0^l \frac{\omega(y)}{y^2} \, dy \leq 2 \omega(x) \ln \frac{1}{x}, \quad 0 < x \leq l,
\]

(2.7)

\[
\int_0^1 \frac{\omega(y)}{y} \, dy \leq (l + 1) \int_0^1 \frac{\omega(y)}{y} \, dy, \quad l > 1.
\]

(2.8)

Now we suppose \( \varphi \in C(\Gamma) \cap C^m(y) \), and set \( \Psi_k(\tau, t) = k!(\varphi(\tau) - P_k(\varphi; \tau, t))/(\tau - t)^{k+1} \), where \( y \) is a subarc of \( \Gamma \) and \( (\tau, t) \in \Gamma \times y \). Then it is easy to verify that

\[
\frac{\partial}{\partial t} \Psi_k(\tau, t) = \Psi_{k+1}(\tau, t), \quad k = 0, 1, \ldots, m - 1
\]

(2.9)

for \( (\tau, t) \in \Gamma \times y \) but \( \tau \neq t \), and

\[
|\Psi_k(\tau, t)| \leq c \max_{\tau \in y} \frac{|\varphi^{(k+1)}(\tau)|}{\omega_y(\varphi^{(m)}, |\tau - t|)},
\]

(2.10)

for \( (\tau, t) \in y \times y \) (cf. [15]). So, if \( \varphi \in \Lambda^m_1(\Gamma) \), \( T_k \varphi = T_k(w, \varphi) \) is differentiable when \( k = 0, 1, \ldots \) or \( m - 1 \) and \( T_m \varphi \) is integrable. Furthermore, we have the following theorem.

Theorem 2.4. Let \( w \) be an integrable function on \( \Gamma \), \( \varphi \in \Lambda^m_1(\Gamma) \), and \( 0 < x \leq 1 \). If \( w \) is bounded, then

\[
\omega(T_m \varphi, x) \leq c \|w\| \left( \omega(\varphi^{(m)}, x) \ln \frac{1}{x} + \int_0^x \frac{\omega(\varphi^{(m)}, y)}{y} \, dy \right),
\]

(2.11)
and if \( w \in \Lambda_1(\Gamma) \) satisfying \( w(a) = w(b) = 0 \), then

\[
\omega(T_m \varphi, x) \leq c \| w \|_{\Lambda_1} \left( \int_0^x \frac{\omega(\varphi^{(m)}, y)}{y} \, dy + x \int_x^1 \frac{\omega(\varphi^{(m)}, y)}{y^2} \, dy \right),
\]

where \( \| w \|_{\Lambda_1} = \| w \| + \int_0^1 \frac{\omega(w, y)}{y} \, dy \) and \( c \) is a positive number related to \( m \) and \( \Gamma \).

\textbf{Proof.} It is equivalent to prove that for \( t_1, t_2 \in \Gamma \)

\[
| T_m(\varphi)(t_1) - T_m(\varphi)(t_2) | \leq c \| w \|_{\Lambda_1} \left( \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} \, dy + \int_\delta^L \frac{\omega(\varphi^{(m)}, y)}{y^2} \, dy \right) \tag{2.9}'
\]

if \( w \) is bounded and

\[
| T_m(\varphi)(t_1) - T_m(\varphi)(t_2) | \leq c \| w \|_{\Lambda_1} \left( \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} \, dy + \int_\delta^L \frac{\omega(\varphi^{(m)}, y)}{y^2} \, dy \right) \tag{2.10}'
\]

if \( w \in \Lambda_1(\Gamma) \) satisfying \( w(a) = w(b) = 0 \), where \( \delta = | t_2 - t_1 | \) and \( L = | \Gamma | \). For convenience, we assume \( 0 < \delta < 1 \) and \( a < t_1 < t_2 < b \) Here \( t_1 < t_2 \) means that \( t_1 \) precedes \( t_2 \).

(i) If \( | t_1 - a | > \delta \) and \( | b - t_2 | \leq \delta \) or \( | t_1 - a | \leq \delta \) and \( | b - t_2 | > \delta \), we let

\[
T_m(\varphi)(t_1) - T_m(\varphi)(t_2) = m! \left( \int_{t_1}^{t_2} w(\tau) [ \Psi_m(\tau, t_2) - \Psi_m(\tau, t_1) ] \, d\tau \right) = I_1 + I_2. \tag{2.13}
\]

Because of the similarity, we assume \( | t_1 - a | > \delta \) and \( | b - t_2 | \leq \delta \). In this case, from (2.10) and \( | \hat{t}_1 b | = | \hat{t}_1 t_2 | + | \hat{t}_2 b | \leq 2c_0 \delta \), we have

\[
| I_2 | = m! \left| \int_{\hat{t}_1}^{\hat{t}_2} w(\tau) [ \Psi_m(\tau, t_2) - \Psi_m(\tau, t_1) ] \, d\tau \right| \leq c \| w \| \int_{\hat{t}_1}^{\hat{t}_2} \frac{\omega(\varphi^{(m)}, | \tau - t_2 |)}{| \tau - t_2 |} + \frac{\omega(\varphi^{(m)}, | \tau - t_1 |)}{| \tau - t_1 |} \, d\tau \tag{2.14}
\]

\[
\leq c \| w \| \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} \, dy.
\]

If we let

\[
h(\tau) = P_m(\varphi; \tau, t_1) - P_m(\varphi; \tau, t_2), \quad \tau \in \Gamma, \tag{2.15}
\]

then

\[
h(\tau) = h(t_2) + \frac{h'(t_2)}{1!} (\tau - t_2) + \cdots + \frac{h^{(m)}(t_2)}{m!} (\tau - t_2)^m \tag{2.16}
\]

\[
\frac{m!}{k!} | h^{(k)}(t_2) | \leq c \binom{m}{k} \omega(\varphi^{(m)}, \delta) \delta^{m-k} \tag{2.17}
\]
for $k = 1, 2, \ldots, m$, therefore

\[
i_1 \equiv m! \left| \int_{\tau_1} w(\tau) \frac{P_m(\varphi; \tau, t_2) - P_m(\varphi; \tau, t_1)}{(\tau - t_2)^{m+1}} d\tau \right|
\]

\[
\leq m! \sum_{k=0}^{m} \frac{|h(k)| \cdot (t_2)}{k!} \int_{\tau_1} \frac{|w(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau|
\]

\[
\leq c\omega(\varphi(\cdot, \cdot), \delta) \sum_{k=0}^{m} \left( \sum_{k} \delta^{m-k} \int_{\tau_1} \frac{|w(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau| \right).
\]

(2.18)

For $\tau \in \tau_1, c_0 |\tau - t_2| \geq |\tau t_2| = |\tau t_1| + |t_1 t_2| \geq |t_2 - t_1| = \delta$, that is, $\delta/|\tau - t_2| \leq c_0$, so that the above inequality becomes

\[
i_1 \leq c\omega(\varphi(\cdot, \cdot), \delta) \int_{\tau_1} \frac{|w(\tau)|}{|\tau - t_2|} |d\tau|.
\]

(2.19)

If $w$ is bounded, then, by some computation, we have

\[
i_1 \leq c\|w\|\omega(\varphi(\cdot, \cdot), \delta) \ln \frac{L}{\delta}.
\]

(2.20)

Since $|b - t_2| \leq |t_2 - t_1| \leq c_0 |\tau - t_2|, |\tau - b| \leq |\tau - t_2| + |t_2 - b| \leq (1 + c_0) |\tau - t_2|$ and it follows that, if $w \in \Lambda_1$ and $w(b) = 0$,

\[
\int_{\tau_1} \frac{|w(\tau)|}{|\tau - t_2|} |d\tau| = \int_{\tau_1} \frac{|w(\tau) - w(b)|}{|\tau - t_2|} |d\tau|
\]

\[
\leq \int_{\tau_1} \frac{\omega(w, (1 + c_0) |\tau - t_2|)}{|\tau - t_2|} |d\tau| \leq c \int_0^1 \frac{\omega(w, y)}{y} dy,
\]

or

\[
i_1 \leq c \int_0^1 \frac{\omega(w, y)}{y} dy \omega(\varphi(\cdot, \cdot), \delta).
\]

(2.22)

On the other hand, by (2.10),

\[
i_2 \equiv m! \int_{\tau_1} w(\tau) \cdot \varphi(\tau) \left[ P_m(\varphi; \tau, t_2) - P_m(\varphi; \tau, t_1) \right] \left[ \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right] d\tau
\]

\[
\leq c\|w\| |t_1 - t_2| \sum_{k=0}^{m} \int_{\tau_1} \frac{\omega(\varphi(\cdot, \cdot), |\tau - t_1|)}{|\tau - t_1|} |\tau - t_1| |d\tau|
\]

\[
\leq c(m + 1)\|w\| |t_1 - t_2| \int_{\tau_1} \frac{\omega(\varphi(\cdot, \cdot), |\tau - t_1|)}{|\tau - t_1|} |\tau - t_1| |d\tau|
\]

\[
\leq c\|w\| \delta \int_0^L \frac{\omega(\varphi(\cdot, \cdot), y)}{y(y + \delta)} dy
\]

\[
\leq c\|w\| \left( \int_0^\delta \frac{\omega(\varphi(\cdot, \cdot), y)}{y} dy + \delta \int_\delta^L \frac{\omega(\varphi(\cdot, \cdot), y)}{y^2} dy \right),
\]

(2.23)
where \( L = |\Gamma| \). Now from

\[
I_1 = \left| m! \int_{a_1}^f w(\tau) \left[ \frac{\varphi(\tau) - P_m(\varphi; \tau, t_2)}{(\tau - t_2)^{m+1}} - \frac{\varphi(\tau) - P_m(\varphi; \tau, t_1)}{(\tau - t_1)^{m+1}} \right] d\tau \right|
\]

\[
= \left| m! \int_{a_1}^f w(\tau) \left\{ \frac{P_m(\varphi; \tau, t_1) - P_m(\varphi; \tau, t_2)}{(\tau - t_2)^{m+1}} \right. \\
+ \left[ \varphi(\tau) - P_m(\varphi; \tau, t_1) \right] \left( \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right) \right\} d\tau \right| 
\leq i_1 + i_2,
\]

we have

\[
I_1 \leq c\|w\| \left( \omega(\varphi^{(m)}), \delta \right) \ln \frac{L}{\delta} + \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} dy
\]

\[
(2.25)
\]

if \( w \) is bounded, where we have used the inequality (2.7), or

\[
I_1 \leq c\|w\|_\Lambda_1 \left( \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} dy + \delta \int_\delta^L \frac{\omega(\varphi^{(m)}, y)}{y^2} dy \right)
\]

\[
(2.26)
\]

if \( w \in \Lambda_1 \) and \( w(b) = 0 \), where we have used the inequality (2.5), and together with (2.14), we obtain (2.9)' and (2.10)'.

(ii) If \( |t_1 - a| > \delta \) and \( |b - t_2| > \delta \), we let

\[
T_m(\varphi)(t_1) - T_m(\varphi)(t_2) = m! \left\{ \int_{a_1}^f + \int_{t_2}^b \right\} w(\tau) \left[ \Psi_m(\tau, t_2) - \Psi_m(\tau, t_1) \right] d\tau
\]

\[
= I_1 + I_2 + I_3.
\]

Similar to the proof of (2.14),

\[
|I_2| \leq c\|w\| \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} dy.
\]

We rewrite \( I_1 + I_3 \) as \( i_1 + i_2 \), where

\[
i_1 = m! \int_{a_1}^f w(\tau) \frac{P_m(\varphi; \tau, t_1) - P_m(\varphi; \tau, t_2)}{(\tau - t_2)^{m+1}} d\tau
\]

\[
+ m! \int_{t_1}^f w(\tau) \frac{P_m(\varphi; \tau, t_1) - P_m(\varphi; \tau, t_2)}{(\tau - t_1)^{m+1}} d\tau,
\]

\[
i_2 = m! \int_{a_1}^f w(\tau) \left[ \varphi(\tau) - P_m(\varphi; \tau, t_1) \right] \left( \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right) d\tau
\]

\[
+ m! \int_{t_1}^f w(\tau) \left[ \varphi(\tau) - P_m(\varphi; \tau, t_2) \right] \left( \frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right) d\tau.
\]

(2.29)
Similar to the proof of (2.23),

$$|i_2| \leq c \|w\| \left( \int_0^\delta \omega(\varphi^{(m)}, y) \frac{dy}{y} + \delta \int_\delta^L \omega(\varphi^{(m)}, y) \frac{dy}{y^2} \right).$$

(2.30)

Using (2.16), we rewrite $i_1$ as

$$m! \sum_{k=0}^{m-1} \left[ \frac{h^{(k)}(t_2)}{k!} \int_{at_1} \frac{w(\tau) d\tau}{(\tau - t_2)^{m-k+1}} + \frac{h^{(k)}(t_1)}{k!} \int_{t_2b} \frac{w(\tau) d\tau}{(\tau - t_1)^{m-k+1}} \right] + \left( \int_{at_1} \frac{w(\tau) d\tau}{\tau - t_2} + \int_{t_2b} \frac{w(\tau) d\tau}{\tau - t_1} \right) [\varphi^{(m)}(t_1) - \varphi^{(m)}(t_2)] = i_{11} + i_{12}.$$

(2.31)

Notice that

$$\left| \int_{at_1} \frac{d\tau}{\tau - t_2} \right| \leq c \delta^{-m+k}, \quad \left| \int_{t_2b} \frac{d\tau}{\tau - t_1} \right| \leq c \delta^{-m+k}$$

for $k = 0, 1, \ldots, m - 1$. Hence, by using the inequality (2.17), we have

$$|i_{11}| \leq c \|w\| \omega(\varphi^{(m)}, \delta).$$

(2.33)

Similar to the proof of (2.20),

$$i_{12} \leq c \|w\| \omega(\varphi^{(m)}, \delta) \ln \frac{L}{\delta}.$$

(2.34)

But if $w \in \Lambda_1$ and $w(a) = w(b) = 0$, then

$$\left| \int_{at_1} \frac{w(\tau) d\tau}{\tau - t_2} + \int_{t_2b} \frac{w(\tau) d\tau}{\tau - t_1} \right| \leq c \left( \|w\| + \int_0^1 \frac{\omega(w, y)}{y} dy \right)$$

(2.35)

(see [15, Section 6]), and thus

$$|i_{12}| \leq c \left( \|w\| + \int_0^1 \frac{\omega(w, y)}{y} dy \right) \omega(\varphi^{(m)}, \delta).$$

(2.36)

Now from $|i_1| \leq |i_{11}| + |i_{12}|$, we obtain

$$|i_1| \leq c \|w\| \omega(\varphi^{(m)}, \delta) \left( \ln \frac{L}{\delta} + 1 \right)$$

(2.37)

if $w$ is bounded and obtain

$$|i_1| \leq c \left( \|w\| + \int_0^1 \frac{\omega(w, y)}{y} dy \right) \omega(\varphi^{(m)}, \delta)$$

(2.38)

if $w \in \Lambda_1$ and $w(a) = w(b) = 0$, and these, together with (2.28) and (2.30), lead to (2.9)' and (2.10)'.

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(iii) If $|t_1 - a| \leq \delta$ and $|b - t_2| \leq \delta$, then $|\Gamma| = |a\hat{t}_1| + |t_1\hat{t}_2| + |t_2 b| \leq 3c_0 \delta$. From (2.10),

$$
T_m(\varphi)(t_1) - T_m(\varphi)(t_2) \leq c \|w\| \int_{\Gamma} \left[ \frac{\omega(\varphi^{(m)}, |\tau - t_2|)}{|\tau - t_2|} + \frac{\omega(\varphi^{(m)}, |\tau - t_1|)}{|\tau - t_1|} \right] |d\tau|
$$

and thus (2.9)' and (2.10)' are also valid.

Now we have proved that (2.9)' and (2.10)' are true in all cases and the constant $c > 0$ depending on $m$ and $\Gamma$ can be derived from the process of the proof.

The proof is completed.

Generally, for $T_k$, $k = 0, 1, \ldots, m$, we have the following results.

**Theorem 2.5.** Assume $w$, $\varphi$, and $w\varphi$ are all integrable and $t_0 \in \Gamma$. If $w$ is bounded on some neighborhood of $t_0$ on $\Gamma$ and $\varphi \in \Lambda_1^m(t_0, \Gamma)$, then $T(w, \varphi)$ is $m$-time continuously differentiable at $t_0$ and

$$
T_k(w, \varphi)(t_0) = \frac{d^k}{dt^k} T(w, \varphi)(t_0)
$$

for $k = 1, 2, \ldots, m$.

**Proof.** It is obvious that (2.40) is true for $k = 1, 2, \ldots, m - 1$. So, we need only to prove

$$
\frac{d}{dt} T_{m-1}(w, \varphi)(t_0) = T_m(w, \varphi)(t_0)
$$

and $T_m(w, \varphi)$ is continuous at $t_0$.

According to the given conditions, there is a subarc $\gamma \subset \Gamma$ with $t_0 \in \gamma$ but $t_0 \in \Gamma \setminus \gamma^0$ such that $w$ is bounded on $\gamma$ and $\varphi \in \Lambda_1^m(\gamma)$, where $\gamma^0$ denotes the inner points of $\gamma$. Write $T_{m-1}(\varphi)$ as

$$
\left( \int_{\Gamma \setminus \gamma} + \int_{\gamma} \right) w(\tau) \Psi_{m-1}(\tau, t)d\tau = I_1(t) + I_2(t).
$$

Then $I_1(t)$ is continuously differentiable at $t_0$ and

$$
I_1(t_0) = \int_{\Gamma \setminus \gamma} w(\tau) \Psi_m(\tau, t_0) d\tau.
$$

For $I_2$, we consider

$$
\left( \int_{\gamma \setminus \hat{t}t} + \int_{\hat{t}t} \right) w(\tau) \left[ \frac{\Psi_{m-1}(\tau, t) - \Psi_{m-1}(\tau, t_0)}{t - t_0} - \Psi_m(\tau, t_0) \right] d\tau = i_1 + i_2,
$$

where $\hat{t}$ and $\hat{t}_0$ are points in $\gamma$. 


where $t \in \gamma$ and, without loss of generality, we assume $t_0 < t$. Notice that

$$i_1 = \int_{y \in \delta t} w(\tau) \left[ \frac{1}{t - t_0} \int_{c} \left[ \Psi_m(\tau, \zeta) - \Psi_m(\tau, t_0) \right] d\zeta \right] d\tau$$

$$= \frac{1}{t - t_0} \int_{c} \left[ \int_{y \in \delta t} w(\tau) [\Psi_m(\tau, \zeta) - \Psi_m(\tau, t_0)] d\tau \right] d\zeta. \quad (2.45)$$

By using Theorem 2.4 to the internal integral, we have

$$|i_1| \leq c' \|w\|_y \left[ \omega(\varphi(m), \delta) \left( 1 + \ln \frac{1}{\delta} \right) + \int_0^\delta \frac{\omega(\varphi(m), y)}{y} dy \right], \quad (2.46)$$

where $\|w\|_y = \max_{t \in \gamma} |w(t)|$ and $\delta = |t - t_0|$. If $\tau \in \delta t$, then $|\tau - t| \leq c_0 |t - t_0|$ and $|\tau - t_0| \leq c_0 |t - t_0|$, and from (2.10),

$$\left| \frac{\Psi_{m-1}(\tau, t) - \Psi_{m-1}(\tau, t_0)}{t - t_0} - \Psi_m(\tau, t_0) \right|$$

$$= \left| \frac{\Psi_m(\tau, t)(\tau - t) + \varphi(m)(t) - \Psi_m(\tau, t_0)(\tau - t_0) + \varphi(m)(t_0)}{m(t - t_0)} - \Psi_m(\tau, t_0) \right|$$

$$\leq c \left( \frac{\omega(\varphi(m), |t - t_0|)}{|t - t_0|} + \frac{\omega(\varphi(m), |\tau - t_0|)}{|\tau - t_0|} \right), \quad (2.47)$$

so that

$$|i_2| \leq c' \|w\|_y \left[ \omega(\varphi(m), \delta) + \int_0^\delta \frac{\omega(\varphi(m), y)}{y} dy \right]. \quad (2.48)$$

Since $\varphi(m) \in \Lambda_1(y)$, (2.46), and (2.48) result in $i_1, i_2 \to 0$ when $t \to t_0$, it follows that

$$I'_2(t_0) = \int_{y \in \delta t} w(\tau) \Psi_m(\tau, t_0) d\tau. \quad (2.49)$$

On the other hand, according to Theorem 2.4, $\int_{\gamma} w(\tau) \Psi_m(\tau, t) d\tau$ is continuous on $\gamma$ because $w$ is bounded on the subarc and $\varphi \in \Lambda_m^m(\gamma)$. Now we have proved that $T_{m-1}(w, \varphi)$ is differentiable at $t_0$ and there holds (2.41).

The following corollaries can be verified easily.

**Corollary 2.6.** If $w$ is bounded and $\varphi \in \Lambda_1^m$, then $T(w, \varphi) \in C^m$ and

$$\|T(w, \varphi)\|_{C^m} \leq c \|w\| \left[ \sum_{k=1}^m \|\varphi^{(k)}\| + \int_0^1 \frac{\omega(\varphi^{(m)}, y)}{y} \ln \frac{1}{y} dy \right]. \quad (2.50)$$

**Corollary 2.7.** If $w$ is bounded and $\varphi \in \Lambda_m^{m+1}$, then $T(w, \varphi) \in \Lambda_m^m$ and

$$\|T(w, \varphi)\|_{\Lambda_m^m} \leq c \|w\| \left[ \sum_{k=1}^m \|\varphi^{(k)}\| + \int_0^1 \frac{\omega(\varphi^{(m)}, y)}{y} \ln^n \frac{1}{y} dy \right], \quad (2.51)$$
where $n$ is a positive integer and $\| \cdot \|_{\Lambda^n_m}$ is defined by

$$
\| \psi \|_{\Lambda^n_m} = \| \psi \|_{C^n_m} + \int_0^1 \frac{\omega(\psi^{(m)}(y), y)}{y} \ln^{n-1} \frac{1}{y} dy, \quad \psi \in \Lambda^n_m.
$$

(2.52)

**Remark 2.8.** The space $\Lambda^n_m$ normed by $\| \cdot \|_{\Lambda^n_m}$ is a Banach space and thus the above corollaries imply that, if the “weight” $w$ is bounded, then $T(w, \cdot) \in \mathcal{L}(\Lambda^n_1, C^n_m)$ and $T(w, \cdot) \in \mathcal{L}(\Lambda^n_{m+1}, \Lambda^n_m)$ for $n \geq 1$, where $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from Banach space $X$ to Banach space $Y$.

**Remark 2.9.** Generally, the inequality (2.12) is called Zygmund-type inequality. In this case, if we consider the operator in $C^{m, \lambda}$, then $T(w, \cdot) \in \mathcal{L}(C^{m, \lambda}, C^{m, \lambda})$ or in detail,

$$
\| T(w, \varphi) \|_{C^{m, \lambda}} \leq c \| w \|_{\Lambda^1} \| \varphi \|_{C^{m, \lambda}},
$$

(2.53)

where $C^{m, \lambda}$ is the space of functions in $C^m$ whose $m$th derivative satisfies a Hölder condition with exponent $\lambda \in (0, 1)$, and its norm is defined by

$$
\| \psi \|_{C^{m, \lambda}} = \| \psi \|_{C^m} + \sup_{0 < x \leq 1} \frac{\omega(\psi^{(m)}(x), x)}{x^\lambda}
$$

(2.54)

(cf. [15] and [17, Chapitre II, Section 6]). The positive constant $c$ in inequality (2.53) depends only on $m, \lambda,$ and $\Gamma$.

### 3. Weighted Hadamard-type singular integrals

In this section, we start from the definition of a basic Hadamard-type or finite-part integral, and then give an expression for general ones by means of singularity deletion method (cf. [7, 11]). For convenience, we denote the Hadamard-type integrals of the form (1.3) by $H_m(w, f)(t)$.

**Definition 3.1.** For $t_0 \in \Gamma^0$, the Hadamard-type singular integral or finite part f.p. $\int_\Gamma (d\tau / (\tau - t_0)^{m+1})$ is defined by

$$
f.p. \int_\Gamma \frac{d\tau}{(\tau - t_0)^{m+1}} = \frac{1}{m} \left[ \frac{1}{(a-t_0)^m} - \frac{1}{(b-t_0)^m} \right].
$$

(3.1)

If letting $h_0(t) = \text{p.v.} \int_\Gamma (1/(\tau - t)) d\tau, t \in \Gamma^0$, where p.v. means Cauchy principal value, then we have $h_0(t) = \ln((b-t)/(t-a))$ and

$$
f.p. \int_\Gamma \frac{1}{(\tau - t)^{m+1}} d\tau = \frac{1}{m!} \frac{d^m}{dt^m} h_0(t), \quad t \in \Gamma^0.
$$

(3.2)

So, the integral is well defined.
Definition 3.2. Let \( t_0 \in \Gamma^0 \) and \( f \in C^m(t_0, \Gamma) \). If \( T_m(1, f)(t_0) \) exists, then the \( m \)-order Hadamard-type singular integral or finite part integral of \( f \) is defined by

\[
\text{f.p.} \int_{\Gamma} \frac{f(\tau)}{(\tau - t_0)^{m+1}} d\tau = \frac{1}{m!} T_m(1, f)(t_0) + \sum_{k=0}^{m} \frac{f^{(k)}(t_0)}{k!} h_{m-k}(t_0),
\]

where \( h_r(t) = \text{f.p.} \int \frac{1}{(\tau - t)^{r+1}} d\tau \) with \( r = 1, 2, \ldots \).

According to Theorem 2.5, if \( f \in \Lambda^m(t_0, \Gamma) \) at \( t_0 \in \Gamma^0 \), then \( T_m(1, f) \) exists on some neighborhood of \( t_0 \) and \( T_m(1, f) \in C(t_0, \Gamma) \) as well. Thus, from (3.3), we obtain the following.

Theorem 3.3. If \( f \in \Lambda^m(t_0, \Gamma) \), then \( H_m(1, f) \in C(t_0, \Gamma) \), where \( t_0 \in \Gamma^0 \).

As a special case, we have the following.

Corollary 3.4. If the function \( w \) is integrable on \( \Gamma \) and arbitrary times differentiable on \( \Gamma^0 \), then \( H_r(1, w) \in C^\infty(\Gamma^0) \) and

\[
\text{f.p.} \int_{\Gamma} \frac{w(\tau)}{(\tau - t)^{r+1}} d\tau = \frac{1}{r!} \frac{d^r}{dt^r} \left[ \text{p.v.} \int_{\Gamma} \frac{w(\tau)}{\tau - t} d\tau \right], \quad t \in \Gamma^0, \tag{3.4}
\]

where \( r = 0, 1, \ldots \).

Now we consider “weighted” Hadamard-type integrals of the form \( H_m(w, f) \).

Definition 3.5. Let \( w \) be an integrable function on \( \Gamma \), \( f \in C^m(t_0, \Gamma) \), and \( t_0 \in \Gamma^0 \). If \( T_m(w, f) \), \( h_{r,w} = H_r(1, w) \) with \( r = 0, 1, 2, \ldots, m \) are all existent at \( t_0 \), then the \( m \)-order “weighted” Hadamard-type singular integral or finite part integral of \( f \) at \( t_0 \) is defined by

\[
\text{f.p.} \int_{\Gamma} \frac{w(\tau) f(\tau)}{(\tau - t_0)^{m+1}} d\tau = \frac{1}{m!} T_m(w, f)(t_0) + \sum_{k=0}^{m} \frac{f^{(k)}(t_0)}{k!} h_{m-k,w}(t_0). \tag{3.5}
\]

Theorem 3.6. Assume the function \( w \) is integrable on \( \Gamma \) and arbitrary times differentiable on \( \Gamma^0 \). If \( f \in \Lambda^m_1(\Gamma) \), then \( H_m(w, f) \) exists on \( \Gamma^0 \). Furthermore, \( H_m(w, f) \in C(\Gamma^0) \) and

\[
H_m(w, f)(t) = \frac{1}{m!} \frac{d^m}{dt^m} \left( \text{p.v.} \int_{\Gamma} \frac{w(\tau) f(\tau)}{\tau - t} d\tau \right), \quad t \in \Gamma^0. \tag{3.6}
\]

Proof. It is clear that \( H_m(w, f)(t) \) exists at all \( t \in \Gamma^0 \) and \( H_m(w, f) \in C(\Gamma^0) \) as well. In the following we give the proof of (3.6).

From the given conditions, by Theorem 2.5 and Corollary 3.4, we see that \( T(w, \varphi) \in C^m(\Gamma^0) \) and \( h_{0,w} \in C^{\infty}(\Gamma^0) \). Hence, from

\[
\text{p.v.} \int_{\Gamma} \frac{w(\tau) f(\tau)}{\tau - t} d\tau = T(w, f)(t) + f(t) h_{0,w}(t), \tag{3.7}
\]

we have

\[
\frac{d^m}{dt^m} \left( \text{p.v.} \int_{\Gamma} \frac{w(\tau) f(\tau)}{\tau - t} d\tau \right) = T_m(w, f)(t) + \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(t) h_{0,w}(t), \quad t \in \Gamma^0. \tag{3.8}
\]
Example 3.7. Let \( w_1(t) = (t-a)^{-\alpha}(b-t)^{-\beta} \) with \( 0 < \alpha, \beta < 1 \) and \( \alpha + \beta = 1 \). We have
\[
\text{p.v.} \frac{1}{\pi} \int_{\Gamma} \frac{w_1(\tau)}{\tau - t} d\tau = \frac{\cos \pi \alpha}{\sin \pi \alpha} w_1(t),
\]
(cf. [11]), and hence
\[
\text{f.p.} \int_{\Gamma} \frac{w_1(\tau)f(\tau)}{(\tau - t)^{m+1}} d\tau = \frac{1}{m!} T_m(w_1,f)(t) + \frac{\pi \cos \pi \alpha}{m! \sin \pi \alpha} \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) w_1^{(k)}(t) f^{(m-k)}(t)
\]
for \( f \in \Lambda_1^m \).

Now we let \( \sigma(t) = (t-a)(b-t) \). From
\[
\frac{\sigma(t)}{\tau - t} = \frac{\sigma(\tau)}{\tau - t} + (\tau + t) - (a + b)
\]
we have
\[
\sigma(t) T_m(w_1,f)(t) = T_m(\sigma w_1,f)(t) + m! \int_{\Gamma} w_1(\tau) p(\tau,t) \frac{f(\tau) - P_m(f;\tau,t)}{(\tau - t)^m} d\tau
\]
and thus
\[
| T_m(w_1,f)(t) | \leq \frac{c}{\sigma(t)} \left( \| f^{(m)} \| + \int_{0}^{1} \frac{\omega(f^{(m)},y)}{y} dy \right), \quad t \in \Gamma^0,
\]
where \( p(\tau,t) = (\tau + t) - (a + b) \) and the constant \( c \) depends on \( w_1, \Gamma, \) and \( m \). Therefore,
\[
| H_m(w_1,f)(t) | \leq c \left( \| f^{(m)} \|_{\Lambda_1^m} + \frac{\cos \pi \alpha}{\sin \pi \alpha} \left| \frac{w_1(t)}{\sigma(t)} \right| \| f \|_{C^m} \right), \quad t \in \Gamma^0.
\]
If \( \alpha = 1/2 \), then \( H_m(w_1,f)(t) = (1/m!) T_m(w_1,f)(t) \) and
\[
\left| \text{f.p.} \int_{\Gamma} \frac{w_1(\tau)f(\tau)}{(\tau - t)^{m+1}} d\tau \right| \leq \frac{c}{\sigma(t)} \left( \| f^{(m)} \| + \int_{0}^{1} \frac{\omega(f^{(m)},y)}{y} dy \right), \quad t \in \Gamma^0.
\]

Example 3.8. Let \( w_2(t) = (t-a)^{\alpha}(b-t)^{\beta} \) with \( 0 < \alpha, \beta < 1 \) and \( \alpha + \beta = 1 \). Similarly, we have
\[
\text{p.v.} \frac{1}{\pi} \int_{\Gamma} \frac{w_2(\tau)}{\tau - t} d\tau = -\frac{\cos \pi \alpha}{\sin \pi \alpha} w_2(t) - \frac{t - aa - \beta b}{\sin \pi \alpha},
\]
\[
\text{f.p.} \int_{\Gamma} \frac{w_2(\tau)f(\tau)}{(\tau - t)^{m+1}} d\tau = \frac{1}{m!} T_m(w_2,f)(t) - \frac{\pi \cos \pi \alpha}{m! \sin \pi \alpha} \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) w_2^{(k)}(t) f^{(m-k)}(t)
\]
\[
- \frac{\pi}{m! \sin \pi \alpha} (mf^{(m-1)}(t) + (t - aa - \beta b) f^{(m)}(t))
\]
for $f \in \Lambda^{m}_r$. In this case

$$\left| \text{f.p.} \int_{\Gamma} \frac{w_2(t) f(\tau)}{(\tau - t)^{m+1}} d\tau \right| \leq c \left[ \left\| w_2 \right\| \int_{0}^{1} \frac{\omega(f^{(m)}, y)}{y} dy + \frac{\cos \pi \alpha}{\sin \pi \alpha} \left\| w_2(t) \right\| \| f \|_{C^m}^{m+1} d\tau \right] \left( \| f^{(m-1)} \| + \| f^{(m)} \| \right), \quad t \in \Gamma^0. \tag{3.17}$$

By the way, the operator $T_m(w_2, \cdot) \in \mathcal{L}(C^{m, \lambda})$.

**Remark 3.9.** In fact, we do not redefine Hadamard-type integrals but give an expression of this type of integrals, though Definition 3.5 is more common than the traditional one. In this expression, the main part is the operator $T_m$. Therefore, it would be easier to identify the characteristics of this type of integrals from $T_m$.

### 4. Applications

Consider the strongly singular integral equations of the form

$$\text{f.p.} \frac{1}{\pi} \int_{J} \frac{\varphi(t)}{(t - x)^2} dt + \int_{J} k(x, t) \varphi(t) dt = f(x), \quad x \in (-1, 1) \tag{4.1}$$

which was discussed in [10], where $k, f \in C^{0, \lambda}$ are given, $J = [-1, 1]$, and the unknown function $\varphi$ is required to be integrable but smooth in the inner of $J$.

We let $\varphi = wy$ and define the operators $H$ and $K$ by

$$H y(x) = \text{f.p.} \frac{1}{\pi} \int_{J} \frac{w(t) y(t)}{(t - x)^2} dt,$$

$$K y(x) = \int_{J} w(t) k(x, t) y(t) dt,$$

respectively, where $w(x) = 1/\sqrt{1 - x^2}$. Then the equation becomes

$$(H + K) y = f, \tag{4.1}'$$

and we will see that $y \in C^{1, \lambda}$ if the above equation is solvable.

Corresponding to $H$, we introduce an operator $\hat{H}$ defined by

$$\hat{H} y(x) = -\text{f.p.} \frac{1}{\pi} \int_{J} \frac{\hat{w}(t)}{t - x} dt \int_{0}^{t} y(s) ds, \tag{4.3}$$

where $\hat{w}(x) = \sqrt{1 - x^2}$. Let $\sigma = 1 - x^2$ be a multiplication operator, that is, $\sigma f(x) = (1 - x^2) f(x)$. We have the following.

**Theorem 4.1.** $\sigma H \in \mathcal{L}(C^{1, \lambda}, C^{0, \lambda}), \hat{H} \in \mathcal{L}(C^{0, \lambda}, C^{1, \lambda})$, and

$$H \hat{H} = I_0, \quad \hat{H} H = I_1 - P_1, \tag{4.4}$$
where $I_0$ and $I_1$ are identical operators on $C^{0,\lambda}$ and $C^{1,\lambda}$, respectively,

$$P_1 y(x) = \text{p.v.} \frac{1}{\pi} \int J w(t) y(x + t) dt, \quad y \in C^{1,\lambda}$$

(4.5)

and $0 < \lambda < 1$.

Proof. If we let

$$V \psi(x) = \text{p.v.} \frac{1}{\pi} \int J w(t) \psi(t) dt, \quad \hat{V} \psi(x) = -\text{p.v.} \frac{1}{\pi} \int J \hat{w}(t) \psi(t) dt,$$

(4.6)

then $H = DV$ and $\hat{H} = \hat{V} S$, where $D$ is a differential operator and $S$ is an integral operator defined by $S f(x) = \int_0^x f(t) dt$, $x \in J$. Notice the relations

$$V \hat{V} f(x) = f(x), \quad \hat{V} V f(x) = f(x) - \frac{1}{\pi} \int J w(t) f(t) dt$$

(4.7)

for $f \in C^{0,\lambda}$ (see [11, 14]). Thus we have

$$H \hat{H} f(x) = DV \hat{V} S f(x) = DS f(x) = f(x)$$

(4.8)

for $f \in C^{0,\lambda}$ and

$$\hat{H} H f(x) = \hat{V} S D V f(x) = \hat{V} (V f(x) - V f(0))$$

$$= f(x) - \frac{1}{\pi} \int J w(t) f(t) dt - x V f(0) = f(x) - P_1 f(x),$$

(4.9)

for $f \in C^{1,\lambda}$. Hence, (4.4) is true.

By noting $H f = T_1(w, f)$ and (3.13), we have

$$\sigma H f(x) = T_1(\hat{w}, f)(x) + \int J w(t)(\tau + x) \frac{f(\tau) - P_1 f(\tau; x)}{\tau - x} d\tau$$

(4.10)

and it leads to $\sigma H \in \mathcal{L}(C^{1,\lambda}, C^{0,\lambda})$. On the other hand, $H = \hat{V} S$, but $S \in \mathcal{L}(C^{0,\lambda}, C^{1,\lambda})$ and $\hat{V} \in \mathcal{L}(C^{1,\lambda})$ (see [15]), and thus $\hat{H} \in \mathcal{L}(C^{0,\lambda}, C^{1,\lambda})$. The proof is completed. □

Let $Z^{1,\lambda}_0 = \{ \psi \in C^{1,\lambda} : P_1 \psi = 0 \}$. Then $Z^{1,\lambda}_0$ is a close subspace of $C^{1,\lambda}$ and if restricted in this subspace, $H$ is invertible and its inverse is $\hat{H}$. Therefore, the equation

$$(H + K) y = f, \quad y \in Z^{1,\lambda}_0$$

(4.11)

is equivalent to the following Fredholm integral equation:

$$(I + \hat{K}) y = f^*,$$

(4.12)

where $\hat{K} = \hat{H} K$ and $f^* = \hat{H} f$. Hence, if the Fredholm integral equation is regular, then (4.1)’ has unique solution in $Z^{1,\lambda}_0$. 


Let $p$ be a given polynomial with degree of 1 and $Z_p^{1,\lambda} = p + Z_0^{1,\lambda}$. Notice that $P_1\psi = p$ for $\psi \in Z_p^{1,\lambda}$. Thus, the equation

$$(H + K)y = f, \quad y \in Z_p^{1,\lambda}$$

(4.13)

is equivalent to

$$(I + \hat{K})y = f^* + p.$$  

(4.14)

Since $p$ is arbitrarily given, the solution of (4.1)' has 2 degrees of freedom if the above Fredholm integral equation is regular.

**Theorem 4.2.** Under the given assumptions, (4.1)' is equivalent to (4.14), and if the equation is solvable then the solution has at least 2 degrees of freedom and belongs to $C^{1,\lambda}$.

Usually, we want the solution of (4.1) to be bounded. If we find a solution $y$ from (4.14) satisfying $y(\pm 1) = 0$, then its corresponding solution $\varphi = wy$ of (4.1) is bounded (in fact $\varphi (\pm 1) = 0$, cf. [18]). However, it is possible to choose such a solution from (4.14), because its solution has 2 degrees of freedom. Alternatively, we can define another operator $\hat{H}_0$ by

$$\hat{H}_0 y(x) = -\frac{1 - x^2}{\pi} \text{p.v.} \int_{J} \frac{w(t)}{t - x} dt \int_0^t y(s) ds,$$

(4.15)

and get the solution from the equation

$$(I + \hat{K}_0)y = f_0^*,$$  

(4.16)

where $\hat{K}_0 = \hat{H}_0 K$ and $f_0^* = \hat{H}_0 f$. The reason is stated as follows.

At first, it is easy to verify the following.

**Theorem 4.3.** $\hat{H}_0 \in \mathcal{L}(C^{0,\lambda}, C_0^{1,\lambda})$ and

$$H\hat{H}_0 = I_0, \quad \hat{H}_0 H = I_0^0,$$  

(4.17)

where the space $C_0^{1,\lambda} = \{\psi \in C^{1,\lambda} : \psi(\pm 1) = 0\}$ and $I_0^0$ is an identical operator on it.

Then, from this theorem, we see that (4.16) is equivalent to

$$(H + K)y = f, \quad y \in C_0^{1,\lambda},$$

(4.18)

and hence, the solution of (4.16) is also the solution of (4.1)' but satisfies $y(\pm 1) = 0$.

**Example 4.4.** In order to solve the equation

$$\text{f.p.} \frac{1}{\pi} \int_J \frac{\varphi(t)}{(t - x)^2} dt + \frac{4}{\pi} \int_J (1 + xt)\varphi(t) dt = 2, \quad x \in (-1, 1)$$

(4.19)
we let \( \varphi(x) = \frac{y(x)}{\sqrt{1-x^2}} \), that is,

\[
\text{f.p. } \frac{1}{\pi} \int_{\mathbb{R}} \frac{y(t)}{\sqrt{1-t^2}} \frac{1}{(t-x)^2} dt + \frac{4}{\pi} \int_{\mathbb{R}} \frac{1+xt}{\sqrt{1-t^2}} y(t) dt = 2, \quad x \in (-1, 1)
\]

(4.15') and use \( \hat{H} \) to act on both sides of it. Then, the equation is converted into

\[
y(x) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sqrt{1-t^2}}{1} y(t) dt = -1 + 2x^2 + \theta_0 + \theta_1 x,
\]

where \( \theta_0 \) and \( \theta_1 \) are arbitrary constants. By solving the equation, we have

\[
y(x) = 3\theta_0 - 1 + \frac{7}{2} \theta_1 x + (2 - 4\theta_0)x^2 - 2\theta_1 x^3.
\]

(4.21)

If taking \( \theta_0 = 1 \) and \( \theta_1 = 0 \), we obtain a special solution \( y(x) = 2(1-x^2) \) which satisfies \( y(\pm 1) = 0 \), and thus \( \varphi(x) = 2\sqrt{1-x^2} \) is a bounded as well as unique solution of (4.19). This solution can also be obtained directly by solving the equation

\[
y(x) - \frac{1-x^2}{\pi} \int_{\mathbb{R}} \frac{4+2xt}{\sqrt{1-t^2}} y(t) dt = -2(1-x^2),
\]

(4.22)

which is from (4.15') with \( \hat{H}_0 \) acting on it.

**Remark 4.5.** Obviously, the results on (4.1) obtained in this section are significant for the solution of this kind of equations, and especially, they are helpful for the discussion of error estimation of the approximate solution.

**Remark 4.6.** Theorem 4.1 means \( \sigma H \) is a bounded operator from \( C^{1,\lambda} \) to \( C^{0,\lambda} \) but \( H \) is not. If we consider the operator in the Sobolev spaces \( W^{m,2}_w \) and \( W^{m,2}_{\hat{w}} \), which are the completions of \( C^m \) normed by

\[
\|f\|_{m,w} = \left[ \sum_{r=0}^{m} \int_{\mathbb{R}} w(t) |f^{(r)}(t)|^2 dt \right]^{1/2},
\]

\[
\|f\|_{m,\hat{w}} = \left[ \sum_{r=0}^{m} \int_{\mathbb{R}} \hat{w}(t) |f^{(r)}(t)|^2 dt \right]^{1/2},
\]

(4.23)

respectively, then \( H \in \mathcal{L}(W^{m,2}_w, W^{m-1,2}_w) \) and \( \hat{H} \in \mathcal{L}(W^{m-1,2}_w, W^{m,2}_w) \) as well (cf. [17, 19]). In this case, \( H \) and \( \hat{H} \) are symmetrical.

**References**


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