Research Article

On Bloch-Type Functions with Hadamard Gaps

Stevo Stević

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We give some sufficient and necessary conditions for an analytic function \( f \) on the unit ball \( B \) with Hadamard gaps, that is, for \( f(z) = \sum_{k=1}^{\infty} P_k(z) \) (the homogeneous polynomial expansion of \( f \)) satisfying \( \frac{n_{k+1}}{n_k} \geq \lambda > 1 \) for all \( k \in \mathbb{N} \), to belong to the space \( H_p^p(B) = \{ f \mid \sup_{0 < r < 1} (1 - r^2)^{\alpha} \| Rf_r \|_p < \infty, f \in H(B) \} \), \( p = 1, 2, \infty \) as well as to the corresponding little space. A remark on analytic functions with Hadamard gaps on mixed norm space on the unit disk is also given.

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1. Introduction

Let \( B = \{ z \in \mathbb{C}^n : |z| < 1 \} \) be the open unit ball of \( \mathbb{C}^n \), \( \partial B = \{ z \in \mathbb{C}^n : |z| = 1 \} \) its boundary, \( \mathbb{D} \) the unit disk in \( \mathbb{C} \), \( dv \) the normalized Lebesgue measure of \( B \) (i.e., \( \nu(B) = 1 \)), and \( d\sigma \) the normalized rotation invariant Lebesgue measure of \( S \) satisfying \( \sigma(\partial B) = 1 \). We denote the class of all holomorphic functions on the unit ball by \( H(B) \).

For \( f \in H(B) \) with the Taylor expansion \( f(z) = \sum_{|\beta| \geq 0} a_{\beta} z^\beta \), let \( \mathcal{R}f(z) = \sum_{|\beta| \geq 0} |\beta| a_{\beta} z^\beta \) be the radial derivative of \( f \), where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) is a multi-index and \( z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n} \).

It is well known that \( \mathcal{R}f(z) = \sum_{j=1}^{n} z_j (\partial f/\partial z_j)(z) = \sum_{k=0}^{\infty} k P_k(z) \), if \( f(z) = \sum_{k=0}^{\infty} P_k(z) \).

As usual, we write

\[
\| f_r \|_p = \left( \int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} \tag{1.1}
\]

if \( p \in (0, \infty) \), and where \( f_r(\zeta) = f(r\zeta) \). If \( p = \infty \), then \( \| f \|_\infty = \sup_{z \in B} |f(z)| \).
2 Abstract and Applied Analysis

Let $\alpha > 0$. The $\alpha$-Bloch space $B^\alpha = B^\alpha(B)$ is the space of all holomorphic functions $f$ on $B$ such that

$$b_\alpha(f) = \sup_{z \in B} (1 - |z|^2)^\alpha |\mathcal{R} f(z)| < \infty.$$  \hfill (1.2)

It is clear that $B^\alpha$ is a normed space under the norm $\| f \|_{B^\alpha} = |f(0)| + b_\alpha(f)$, and $B^{\alpha_1} \subset B^{\alpha_2}$ for $\alpha_1 < \alpha_2$. Let $B^{\alpha}_0$ denote the subspace of $B^\alpha$ consisting of those $f \in B^\alpha$ for which $(1 - |z|^2)^\alpha |\mathcal{R} f(z)| \rightarrow 0$ as $|z| \rightarrow 1$. This space is called the little $\alpha$-Bloch space. For $\alpha = 1$, the $\alpha$-Bloch space and the little $\alpha$-Bloch space become Bloch space $B$ and the little Bloch space $B_0$. Some characterizations of these spaces can be found, for example, in the following papers [1–6].

We say that an analytic function $f$ on the unit disk $\mathbb{D}$ has Hadamard gaps if $f(z) = \sum_{k=1}^\infty a_k z^k$ where $n_{k+1}/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$.

In [7], Yamashita proved the following result.

Theorem 1.1. Assume that $f$ is an analytic function on $\mathbb{D}$ with Hadamard gaps. Then for $\alpha > 0$, the following two propositions hold:

(a) $f \in B^{\alpha}(\mathbb{D})$ if and only if $\limsup_{k \to \infty} |a_k| n_k^{1-\alpha} < \infty$;

(b) $f \in B^{\alpha}_0(\mathbb{D})$ if and only if $\lim_{k \to \infty} |a_k| n_k^{1-\alpha} = 0$.

An analytic function on $B$ with the homogeneous expansion $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ (here, $P_{n_k}$ is a homogeneous polynomial of degree $n_k$) is said to have Hadamard gaps if $n_{k+1}/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$. In [8], among others, Choa generalizes the main result in [9], proving the following result.

Theorem 1.2. Assume that $p \in (0, \infty)$ and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ is an analytic function on $B$ with Hadamard gaps. Then the following statements are equivalent:

(a) $\| f \|_{X_p} = (\int_B |\mathcal{R} f(z)|^p (1 - |z|^2)^{p-1} d\nu(z))^{1/p} < \infty$;

(b) $\sum_{k=1}^\infty \| P_{n_k} \|_p < \infty$.

This result motivates us to find some characterizations for certain function spaces of analytic functions on the unit ball, in terms of the sequence $(\| P_{n_k} \|_p)_{k \in \mathbb{N}}$.

Now note that the quantity $b_\alpha$ in the definition of the $\alpha$-Bloch spaces can be written in the following form:

$$b_\alpha(f) = \sup_{0 < r < 1} (1 - r^2)^\alpha \sup_{\zeta \in S} |\mathcal{R} f(r\zeta)| = \sup_{0 < r < 1} (1 - r^2)^\alpha M_\infty(\mathcal{R} f, r).$$  \hfill (1.3)

On the other hand, the quantity $b_\alpha$ can be considered as the limit case of the following quantities:

$$\| f \|_{\mathcal{R}^\alpha} = \sup_{0 < r < 1} (1 - r^2)^\alpha \| \mathcal{R} f_r \|_p,$$  \hfill (1.4)

as $p \to \infty$. Note that for every $f \in H(B)$ and $p \in (0, \infty)$,

$$\sup_{0 < r < 1} (1 - r^2)^\alpha \| \mathcal{R} f_r \|_p \leq \sup_{0 < r < 1} (1 - r^2)^\alpha \| \mathcal{R} f_r \|_\infty.$$  \hfill (1.5)
Hence, in this paper we also consider analytic functions with Hadamard gaps on the following spaces:

$$\mathcal{B}_p^\alpha = \left\{ f \mid \sup_{0 < r < 1} (1 - r^2)^\alpha \| \mathcal{R} f_r \|_p < \infty, \ f \in H(B) \right\},$$

$$\mathcal{B}_{p,0}^\alpha = \left\{ f \mid \lim_{r \to 1} (1 - r^2)^\alpha \| \mathcal{R} f_r \|_p = 0, \ f \in H(B) \right\}. \quad (1.6)$$

Motivated by Theorem 1.1 in this paper, we study analytic functions with Hadamard gaps, which belong to $\mathcal{B}_p^\alpha$ or $\mathcal{B}_{p,0}^\alpha$ space when $p = 1, 2, \infty$. Some characterizations for these classes of functions on the unit ball are given in terms of the sequence $(\| P_{n_k} \|_p)_{k \in \mathbb{N}}$.

The following are the main results.

**Theorem 1.3.** Assume that $\alpha > 0$, $p = 1, 2, \infty$, and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ is an analytic function on $B$ with Hadamard gaps. Then the following statements are equivalent:

(a) $f \in \mathcal{B}_p^\alpha$;

(b) $\limsup_{k \to \infty} \| P_{n_k} \|_p n_k^{1-\alpha} < \infty$.

**Theorem 1.4.** Assume that $\alpha > 0$, $p = 1, 2, \infty$, and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ is an analytic function on $B$ with Hadamard gaps. Then the following statements are equivalent:

(a) $f \in \mathcal{B}_{p,0}^\alpha$;

(b) $\lim_{k \to \infty} \| P_{n_k} \|_p n_k^{1-\alpha} = 0$.

Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant $C$ such that $B/C \leq A \leq CB$.

2. **Proof of main results**

Before proving the main results of this paper we quote two auxiliary results which are incorporated in the lemmas which follow (see [9, 10]).

**Lemma 2.1.** Assume that $p \in (0, \infty)$. If $(n_k)$ is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \geq \lambda > 1$, for all $k$, then there is a positive constant $K$ depending only on $p$ and $\lambda$ such that

$$1/K \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{in_k \theta} \right|^p d\theta \right)^{1/p} \leq A \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \quad (2.1)$$

for any number $a_k$, $k \in \mathbb{N}$.

**Lemma 2.2.** Assume that $\alpha > 0$, $p > 0$, $n \in \mathbb{N}_0$, $(a_n)_{n \in \mathbb{N}_0}$ is the sequence of nonnegative numbers, $I_n = \{ k \mid 2^n \leq k < 2^{n+1}, \ k \in \mathbb{N} \}$, $t_n = \sum_{k \in I_n} a_k$, and $g(x) = \sum_{n=1}^{\infty} a_n x^n$. Then there is a positive constant $K$ depending only on $p$ and $\alpha$ such that

$$\frac{1}{K} \sum_{n=0}^{\infty} \frac{t_n^p}{2^{na}} \leq \int_0^1 (1 - x)^{a-1} g^p(x) dx \leq K \sum_{n=0}^{\infty} \frac{t_n^p}{2^{na}}. \quad (2.2)$$
Proof of Theorem 1.3. (a) ⇒ (b) (Case $p = 1$). Let $f \in \mathcal{B}_1^\alpha$. Let $f_\zeta(w) = f(\zeta w)$, $\zeta \in S$, where $\zeta$ is fixed and $w \in \mathbb{D}$, be a slice function. By some calculation we see that

$$f'_\zeta(w) = \zeta_1 \frac{\partial f}{\partial z_1}(w\zeta) + \cdots + \zeta_n \frac{\partial f}{\partial z_n}(w\zeta) = \frac{1}{w} \mathcal{R} f(w\zeta). \tag{2.3}$$

From (2.3) and since $f'_\zeta(w) = \sum_{k=1}^\infty n_k P_{nk}(\zeta) w^{n_k-1}$, we have that

$$\int_S n_k |P_{nk}(\zeta)| \, d\sigma(\zeta) = \int_S \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\eta f'_\zeta(\eta)}{\eta^{n_k+1}} \, d\eta \right| \, d\sigma(\zeta) \leq \frac{1}{2\pi} \int_{\partial \mathbb{D}} \int_S \left| \frac{\mathcal{R} f(\zeta \eta)}{\eta^{n_k+1}} \right| \, d\sigma(\zeta) \, |d\eta| \leq \frac{\|f\|_{\mathcal{B}_1^\alpha}}{(1-r)^\alpha r^{n_k}}, \tag{2.4}$$

which implies that

$$n_k r^{n_k} \|P_{nk}\|_1 \leq \frac{\|f\|_{\mathcal{B}_1^\alpha}}{(1-r)^\alpha}, \tag{2.5}$$

for every $k \in \mathbb{N}$ and $r \in (0,1)$. Choosing $r = 1 - (1/n_k)$, we obtain $n_k^{1-\alpha} \|P_{nk}\|_1 \leq C$, as desired.

(b) ⇒ (a) (Case $p = 1$). Assume $\limsup_{k \to \infty} \|P_{nk}\|_1 n_k^{1-\alpha} < \infty$. We have that

$$\|f\|_{\mathcal{B}_1^\alpha} = \sup_{0 < r < 1} (1-r^2)^\alpha \int_S \left| \mathcal{R} f(r\zeta) \right| \, d\sigma(\zeta) \leq \sup_{0 < r < 1} (1-r^2)^\alpha \int_S \sum_{k=1}^\infty n_k P_{nk}(\zeta) r^{n_k} \, d\sigma(\zeta) \leq \sup_{0 < r < 1} (1-r^2)^\alpha \sum_{k=1}^\infty n_k \|P_{nk}\|_1 r^{n_k} \leq \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty \left( \sum_{n_k \leq n} n_k \right) r^{n} \leq C \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty n^{\alpha} r^{n} \leq C,$$

where we have used the fact that there is a positive constant $C$ independent of $n$ such that $\sum_{n_k \leq n} n_k^{\alpha} \leq C n^{\alpha}$ (here is used the assumption that $n_{k+1}/n_k \geq \lambda > 1$) and the following well-known estimate:

$$\sum_{n=1}^\infty n^{\alpha} r^{n} \leq C(1-r)^{-(\alpha+1)}, \tag{2.7}$$

$\alpha > 0$, $r \in [0,1)$; see, for example, [11].
Case $p = 2$. Since
\[
\|f\|_{B^2} = \sup_{0<r<1} (1-r^2)^a \left( \sum_{k=1}^{\infty} n_k^2 \|P_{nk}\|_{2r^{2nk}}^2 \right)^{1/2}
\] (2.8)
we have that
\[
\sup_{0<r<1} (1-r^2)^a n_k \|P_{nk}\|_{2r^{nk}} \leq \|f\|_{B^2} \leq \sup_{0<r<1} (1-r^2)^a \sum_{k=1}^{\infty} n_k \|P_{nk}\|_{2r^{nk}},
\] (2.9)
from which the result follows similar to the case $p = 1$.

Now we show that $(a) \iff (b)$ for case $p = \infty$. As above, the function $f_\zeta(w) = \sum_{k=1}^{\infty} P_{nk}(\zeta) w^{nk}$, where $w = re^{i\theta}$, is a lacunary series in $D$ and
\[
(1-r^2)^a \mathcal{R} f(r\zeta) = re^{i\theta} (1-r^2)^a f_\zeta e^{-i\theta}(re^{i\theta}),
\] (2.10)
from which by Theorem 1.1 the equivalence follows.

Proof of Theorem 1.4. (a)$\implies$(b) (Case $p = 1$). Let $f \in B_{1,0}^a$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that
\[
(1-r^2)^a \int_S |\mathcal{R} f(r\zeta)| d\sigma(\zeta) < \varepsilon,
\] (2.11)
whenever $\delta < r < 1$. From (2.4), (2.11), and rotational invariance of $d\sigma$, we have that
\[
\int_S n_k |P_{nk}(\zeta)| d\sigma(\zeta) \leq \frac{1}{2\pi} \int_{\partial\mathbb{D}} \int_S |\mathcal{R} f(\zeta \eta)| |\eta^{n_{k+1}}| d\sigma(\zeta) |d\eta|
\leq \frac{1}{2\pi} \int_{\partial\mathbb{D}} \int_S \frac{(1-r^2)^a |\mathcal{R} f(\zeta \eta)|}{(1-r^2)^a r^{nk+1}} d\sigma(\zeta) |d\eta|
\leq \frac{\varepsilon}{(1-r)^a r^{nk}},
\] (2.12)
which implies that
\[
n_k r^{nk} \|P_{nk}\|_1 \leq \frac{\varepsilon}{(1-r)^a}
\] (2.13)
for every $k \in \mathbb{N}$ and $r \in (\delta,1)$. Choosing $r = 1 - (1/n_k)$, we obtain
\[
n_k \|P_{nk}\|_1 \leq C \varepsilon n_k^a,
\] (2.14)
from which (b) follows in this case.

(b)$\implies$(a) (Case $p = 1$). Assume that $\lim_{k \to \infty} \|P_{nk}\|_1 n_k^{-1-a} = 0$, then for every $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that
\[
\|P_{nk}\|_1 \leq \varepsilon n_k^{a-1}, \quad \text{for } k \geq k_0.
\] (2.15)
We may assume that \( k_0 = 1 \). From this and by the proof of Theorem 1.3, \((b) \Rightarrow (a)\) (Case \( p = 1 \)), we have that
\[
(1 - r^2)^{\alpha} \| R f_r \|_1 \leq \sup_{0 < r < 1} (1 - r^2)^{\alpha + 1} \sum_{n=1}^{\infty} \left( \sum_{\lambda \in n} n_k \| P_{n_k} \|_1 \right) r^n \leq C \varepsilon \sup_{0 < r < 1} (1 - r^2)^{\alpha + 1} \sum_{n=1}^{\infty} n^\alpha r^n \leq C \varepsilon,
\]
from which the implication follows.

Case \( p = 2 \). By using (2.9) the result follows similar to the Case \( p = 1 \). The proof is omitted.

Finally, in view of (2.10) and employing Theorem 1.1(b) it is easy to see that \((a) \Leftrightarrow (b)\) for case \( p = \infty \). □

3. The case of mixed norm space

In this section, we give a note concerning analytic functions with Hadamard gaps on the mixed norm space. The mixed norm space \( H_{p,q,\alpha}(B) \), \( p, q > 0 \), and \( \alpha \in (-1, \infty) \), consists of all \( f \in H(B) \) such that
\[
\| f \|_{p,q,\alpha} = \left( \int_0^1 |f(r\zeta)|_p^q (1 - r)^{\alpha} \, dr \right)^{1/q} < \infty.
\]
From [12, Theorem 4] the following result holds.

**Theorem 3.1.** Assume that \( p \in (0, \infty) \), \( \alpha > -1 \) and \( f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \) is an analytic function on \( \mathbb{D} \) with Hadamard gaps. Then \( f^{(m)} \in H_{p,q,\alpha}(\mathbb{D}) \) if and only if \( \sum_{k=0}^{\infty} n_k^{m - \alpha - 1} |a_k|^q < \infty \).

**Proof.** First we consider the case \( m = 0 \). Similar to the proof of [12, Theorem 4] and by Lemmas 2.1 and 2.2, we have that
\[
\| f \|_{p,q,\alpha}^q = \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k r^{n_k} e^{in_k \theta} \right|^p d\theta \right)^{q/p} (1 - r)^{\alpha} \, dr
\]
\[
\approx \int_0^1 \left( \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k} \right)^{q/2} (1 - r)^{\alpha} \, dr
\]
\[
\approx \int_0^1 \left( \sum_{k=1}^{\infty} |a_k|^2 \rho^{n_k} \right)^{q/2} (1 - \rho)^{\alpha} \, d\rho
\]
\[
\approx \sum_{k=0}^{\infty} \frac{1}{2(\alpha+1)k} \left( \sum_{m \in I_k} |a_m|^2 \right)^{q/2} \approx \sum_{k=0}^{\infty} \frac{|a_k|^q}{n_k^{\alpha+1}},
\]
from which the result follows in this case.
Since \( f \) has Hadamard gaps and \( f^{(m)}(z) = \sum_{k=1}^{\infty} a_k n_k (n_k - 1) \cdots (n_k - m + 1) z^{n_k - m} \), it follows that \( f^{(m)} \) has Hadamard gaps too. Applying the just proved result to the function \( f^{(m)} \), we obtain that \( f^{(m)} \in H_{p,q,\alpha}(\mathbb{D}) \) if and only if
\[
\sum_{k=0}^{\infty} \frac{|n_k (n_k - 1) \cdots (n_k - m + 1) a_k|^q}{n_k^{a+1-mq}} < \infty,
\]
finishing the proof. \( \square \)

**Remark 3.2.** Motivated by [12, Theorems 3 and 4], we can conjecture that if \( p \in (0, \infty) \), \( \alpha > -1 \), and \( f(z) = \sum_{k=1}^{\infty} P_{nk}(z) \) is an analytic function on \( B \) with Hadamard gaps, then \( \mathcal{R}^{(m)} f \in H_{p,q,\alpha}(B) \) if and only if \( \sum_{k=0}^{\infty} n_k^{q \alpha} \|P_{nk}\|^q_p < \infty \). Note that the result is true for the case of the weighted Bergman space, that is, when \( p = q \), see [12, Corollary 1]. It is also expected that Theorems 1.3 and 1.4 hold for every \( p \in [1; \infty) \) (for the case \( n = 1 \), see [13]).

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**References**


Stevo Stević: Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 35/I, 11000 Beograd, Serbia

*Email addresses: ssstevic@ptt.yu; sstevo@matf.bg.ac.yu*