Research Article
On Local $\alpha$-Times Integrated $C$-Semigroups

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This paper presents several characterizations of a local $\alpha$-times integrated $C$-semigroup $\{T(t); 0 \leq t < \tau\}$ by means of functional equation, subgenerator, and well-posedness of an associated abstract Cauchy problem. We also discuss properties concerning the nondegeneracy of $T(\cdot)$, the injectivity of $C$, the closability of subgenerators, the commutativity of $T(\cdot)$, and extension of solutions of the associated abstract Cauchy problem.

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1. Introduction

Let $X$ be a complex Banach space and let $B(X)$ be the Banach algebra of all bounded (linear) operators on $X$. Let $j_{-1} := \delta_0$, the Dirac measure at 0, and for $r > -1$, let $j_r : [0, \infty) \to \mathbb{R}$ be defined as $j_r(t) := t/(\Gamma(r + 1))$, $t \geq 0$, where $\Gamma(\cdot)$ is the Gamma function.

Let $C \in B(X)$ and $\tau \in (0, \infty]$. A strongly continuous family $\{T(t); 0 \leq t < \tau\} \subset B(X)$ is called a local $\alpha$-times ($\alpha \geq 0$) integrated $C$-semigroup on $X$ if it satisfies $T(t)C = CT(t)$ for $0 \leq t < \tau$, $T(0) = 0$, and

$$T(s)T(t)x = \left( \int_0^{s+t} - \int_0^s - \int_0^t \right) j_{\alpha-1}(s+t-r)CT(r)x dr$$

$$= \int_0^s [j_{\alpha-1}(r)CT(s+t-r) - j_{\alpha-1}(s+t-r)CT(r)]x dr$$

$$= \int_0^t [j_{\alpha-1}(r)CT(s+t-r) - j_{\alpha-1}(s+t-r)CT(r)]x dr$$

(1.1)

for $x \in X, 0 \leq s, t \leq s + t < \tau$. In case $\tau = \infty$, a local $\alpha$-times integrated $C$-semigroup is named an $\alpha$-times integrated $C$-semigroup (see [1] for general $\alpha \in [0, \infty)$, and [2] for...
the case \( \alpha \in \mathbb{N} \). When \( C = I \), the identity operator, \( T(\cdot) \) is called an \( \alpha \)-times integrated semigroup (cf. \([3, 4]\)).

We say that \( \{ T(t); 0 \leq t < \tau \} \) is a local (0-times integrated) \( C \)-semigroup (cf. \([5–11]\)) if \( T(0) = C \) and

\[
T(t)T(s) = T(s + t)C \quad \forall 0 \leq t, s \leq s + t < \tau.
\] (1.2)

In case \( \tau = \infty \), a local \( C \)-semigroup is called a \( C \)-semigroup (cf. \([12–15]\)).

Local \( \alpha \)-times integrated \( C \)-semigroups were first studied in \([16]\) for the case \( \alpha = n \in \mathbb{N} \) and under the assumption that \( C \) is injective and \( T(\cdot) \) satisfies the condition

\[
T(t)x = 0 \quad \forall 0 < t < \tau \text{ implies } x = 0.
\] (1.3)

Clearly, (1.3) is implied by the following condition:

\[
T(t)x = 0 \quad \forall 0 < t < \frac{\tau}{2} \text{ implies } x = 0.
\] (1.4)

For the case \( \tau = \infty \), both conditions (1.3) and (1.4) become the ordinary definition of nondegeneracy, that is,

\[
T(t)x = 0 \quad \forall t > 0 \text{ implies } x = 0.
\] (1.5)

When \( \tau < \infty \) and \( \alpha = 0 \), (1.4) is strictly stronger than (1.3) and is equivalent to that \( C \) is injective (cf. \([6]\)). It will be seen that in the case \( \alpha > 0 \), (1.4) still implies (1.3) and the injectivity of \( C \) (Lemma 4.1). These facts suggest that a proper definition of nondegeneracy for a local \( \alpha \)-times integrated \( C \)-semigroup seems to be (1.4). In the present paper, we use this definition.

The aim of this paper is to analyze in detail several characterizations for degenerate and nondegenerate local \( \alpha \)-times integrated \( C \)-semigroups, by means of functional equation, subgenerator, and well-posedness of an associated abstract Cauchy problem.

In Section 2, we give the following general characterization of local \( \alpha \)-times integrated \( C \)-semigroups in terms of functional equations:

\[
T(0) = \delta_{0,a} C, \quad T(t)C = CT(t),
\]

\[
S(s)[T(t) - j_a(t)C] = [T(s) - j_a(s)C]S(t) \quad \forall 0 \leq s, t \leq s + t < \tau,
\] (1.6)

where \( \delta_{a,b} \) is the Kronecker delta and \( S(t) := \int_0^t T(s)ds \), \( 0 \leq t < \tau \) (see Theorem 2.3).

In Sections 3 and 4, we will define subgenerator and generator of a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup \( T(\cdot) \). Then, we discuss some properties concerning the nodegeneracy of \( T(\cdot) \), the injectivity of \( C \), the closability of subgenerators, and the commutativity of the family \( \{ T(t); 0 \leq t < \tau \} \). For instance, we will see that nondegeneracy is equivalent to the injectivity of \( C \) when \( T(\cdot) \) has a subgenerator \( G \) (Lemma 4.1), and nondegeneracy implies that \( T(\cdot) \) has the generator and \( \{ T(t); 0 \leq t < \tau \} \) is a commutative family (Theorem 3.5 and Proposition 4.6). Notice that (1.1) implies that \( T(t)T(s) = T(s)T(t) \) holds for any pair of \( s, t \geq 0 \) which satisfies \( s + t < \tau \), but, when \( T(\cdot) \) is degenerate, in general, the commutativity does not hold for \( \tau < s + t < 2\tau \) (see \([6]\) for an example).
We also prove a characterization (Theorem 4.15) for nondegenerate local \( \alpha \)-times integrated \( C \)-semigroups, which states that \( \{T(t); 0 \leq t < \tau\} \) is a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup if and only if \( C \) is injective and there is a closed operator \( G \) satisfying

\[
T(t)x - j_a(t)Cx = \begin{cases} S(t)Gx, & x \in D(G); \\ GS(t)x, & x \in X \end{cases}
\]  

(1.7)

for all \( 0 \leq t < \tau \). In this case, \( C^{-1}GC \) is the generator of \( T(\cdot) \).

In Section 5, we discuss the relation between a local \( \alpha \)-times integrated \( C \)-semigroup with generator \( A \) and the associated abstract Cauchy problem:

\[
\begin{align*}
  &u'(t) = Au(t) + Cf(t), & 0 < t < \tau; \\
  &u(0) = 0.
\end{align*}
\]

(ACP\( (A;Cf,0) \))

Let \( C \in B(X) \) be injective and \( \alpha \geq 0 \), and let \( A \) be a closed linear operator such that \( CA \subset AC \). It will be shown (see Theorem 5.1) that the abstract Cauchy problem ACP\( (A; j_aCx,0) \) has a unique solution \( u_x \) for every \( x \in X \) if and only if \( A \) is a subgenerator of a local \( \alpha \)-times integrated \( C \)-semigroup \( T(\cdot) \). Moreover, the solution is given by \( u_x(t) = \int_0^t T(s)x \, ds \).

In Section 6, we apply Theorem 4.15 to show that the generator \( A \) of a local \( \alpha \)-times integrated \( C \)-semigroup on \([0,\tau)\) also generates a local \( (\alpha + n) \)-times integrated \( C^2 \)-semigroup on \([0,2\tau)\) for any integer \( n \) which is not less than \( \alpha \) (see Theorem 6.1). This is a generalization to \( \alpha \)-times integrated \( C \)-semigroups of a result in [17] on \( n \)-times integrated semigroups. This generalization (for the case \( \alpha = n \)) has been proved in [16] by different approach, and the case \( n = 0 \) was treated in [10].

As is well known, there is the Hille-Yosida generation theorem for a \( (C_0) \)-semigroup in terms of the resolvent of the generator (or equivalently, the Laplace transform of the \( (C_0) \)-semigroup). For an exponentially bounded nondegenerate \( \alpha \)-times integrated \( C \)-semigroup, we also have a Hille-Yosida type generation theorem in terms of the \( C \)-resolvent of the generator (or equivalently, the Laplace transform of the \( C \)-semigroup) (cf. [1, 2]). For nonexponentially bounded \( C \)-semigroups and local \( C \)-semigroups, the Laplace transform does not exist. In this case, there is a Hille-Yosida type generation theorem in terms of the asymptotic \( C \)-resolvent of the generator (cf. [9, 7]). See also [18] for a similar Hille-Yosida type generation theorem for nondegenerate local \( C \)-cosine functions. Finally, we remark that it is also possible to establish a similar Hille-Yosida type generation theorem for a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup with \( \alpha > 0 \).

2. Degenerate local \( \alpha \)-times integrated \( C \)-semigroups

Let \( h : [0,b] \to \mathbb{C} \) be integrable and let \( f : [0,b] \to X \) be Bochner integrable, where \( b > 0 \). The convolution of \( h \) and \( f \) is the function \( h * f \) defined by \( (h * f)(t) := \int_0^t h(t - s)f(s)ds \), \( 0 \leq t \leq b \) whenever the integral is well-defined at every point \( t \in [0,b] \). When \( h = j_{-1} \), the Dirac measure, we define \( (j_{-1} * f)(t) := f(t) \) for \( t \in [0,b] \). We will need the following lemma: (a) can be verified by using the Laplace transform and (b) is a modification of Titchmarsh’s theorem (cf. [19, Corollary 2.2.5]).
Lemma 2.1. The following hold for $r,s \geq -1$.

(a) $j_r \ast j_s = j_{r+s+1}$.
(b) Let $f : [0,b] \to X$ be Bochner integrable. If $j_r \ast f \equiv 0$ on $[0,b]$, then $f = 0$ almost everywhere.

We will also need the following lemma whose proof we omit.

Lemma 2.2. Let $\alpha \geq 0$ and let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function satisfying $T(0) = 0$. Let $S(t)x := \int_0^t T(s)x ds$ for all $x \in X$ and $0 \leq t < \tau$. Then, $S(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup if and only if $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup.

Theorem 2.3. Let $\alpha \geq 0$ and let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function satisfying $T(0) = 0$. Let $S(t)x := \int_0^t T(s)x ds$ for all $x \in X$ and $0 \leq t < \tau$. Then, $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup on $X$ if and only if $T(t)C = CT(t)$ for all $0 \leq t < \tau$ and

$$S(s)[T(t) - j_a(t)C] = [T(s) - j_a(s)C]S(t) \quad \forall 0 \leq s, t \leq s + t < \tau. \quad (2.1)$$

Proof. Suppose $T(\cdot)$ is an $\alpha$-times integrated $C$-semigroup on $X$. Integrating (1.1) with respect to $t$, and using integration by parts, we obtain the following equation:

$$T(s)S(t)x = \int_0^s j_{a-1}(r)C[S(s+t-r) - j_a(s+t-r)CT(r)]x dr$$

$$= \left(\int_t^{s+t} - \int_0^s\right) j_{a-1}(s+t-r)CS(r)x dr - j_a(t)CS(t)x. \quad (2.2)$$

Integrating (1.1) with respect to $s$, we also have

$$S(s)T(t)x = \int_0^t \left[ j_{a-1}(r)CS(s+t-r) - j_a(s+t-r)CT(r) \right]x dr$$

$$= \left(\int_s^{s+t} - \int_0^t\right) j_{a-1}(s+t-r)CS(r)x dr - j_a(s)CS(t)x \quad (2.3)$$

for $x \in X$ and $0 \leq s, t \leq s + t < \tau$. Comparing (2.2) and (2.3), we obtain

$$T(s)S(t)x + j_a(t)CS(t)x = \left(\int_0^{s+t} - \int_0^t - \int_0^s\right) j_{a-1}(s+t-r)CS(r)x dr$$

$$= S(s)T(t)x + j_a(s)CS(t)x. \quad (2.4)$$

Since $T(\cdot)$ commutes with $C$, so does $S(\cdot)$. Therefore, (2.1) holds.

Conversely, suppose that $T(\cdot)$ satisfies (2.1). By Lemma 2.2, it suffices to show that $S(\cdot)$ is an $(\alpha + 1)$-times integrated $C$-semigroup. First, we replace $s$ by $s+t-r$ and $t$ by $r$ in (2.1). Then, we have for $x \in X$

$$S(s+t-r)T(r)x - T(s+t-r)S(r)x = S(s+t-r)j_a(r)Cx - j_a(s+t-r)CS(r)x. \quad (2.5)$$
By integrating the right-hand side with respect to $r$ from 0 to $t$, we obtain from $CT(\cdot) = T(\cdot)C$ that

$$
\int_0^t S(s + t - r) j_\alpha(r)Cx \, dr - \int_0^t j_\alpha(s + t - r)CS(r)x \, dr
$$

$$
= \int_s^{s+t} S(r) j_\alpha(s + t - r)Cx \, dr - \int_0^t j_\alpha(s + t - r)CS(r)x \, dr
$$

$$
= \left( \int_0^{s+t} - \int_0^s - \int_0^t \right) j_\alpha(s + t - r)CS(r)x \, dr.
$$

(2.6)

On the other hand, from the left-hand side, we have

$$
\int_0^t S(s + t - r)T(r)x \, dr - \int_0^t T(s + t - r)S(r)x \, dr
$$

$$
= S(s + t - r)S(r)x|_0^t + \int_0^t T(s + t - r)S(r)x \, dr - \int_0^t T(s + t - r)S(r)x \, dr
$$

$$
= S(s)S(t) - S(s + t)S(0) = S(s)S(t)
$$

(2.7)

for $0 \leq t, s < s + t < \tau$. Therefore, $S(\cdot)$ is an $(\alpha + 1)$-times integrated $C$-semigroup. The result follows from Lemma 2.2.

**Corollary 2.4.** Let $\alpha > 0$, $\beta \geq -1$. If $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup, then $j_\beta * T(\cdot)$ is an $(\alpha + \beta + 1)$-times integrated $C$-semigroup.

**Proof.** Let $U(t) := j_\beta * T(t)$ for all $0 \leq t < \tau$. Using Lemma 2.1(a) and Theorem 2.3, we have for every $0 \leq s, t \leq s + t < \tau$ and $x \in X$,

$$
[U(s) - j_{\alpha+\beta+1}(s)C] \int_0^t U(r)x \, dr
$$

$$
= \int_0^s j_\beta(s-u)[T(u) - j_\alpha(u)C]j_\beta * T(t)x \, du
$$

$$
= \int_0^s j_\beta(s-u)[T(u) - j_\alpha(u)C] \int_0^t j_\beta(t-v)(j_0 * T)(v)x \, dv \, du
$$

$$
= \int_0^s \int_0^t j_\beta(s-u)j_\beta(t-v)[T(u) - j_\alpha(u)C](j_0 * T)(v)x \, dv \, du
$$

$$
= \int_0^s \int_0^t j_\beta(s-u)j_\beta(t-v)(j_0 * T)(u)[T(v) - j_\alpha(v)C]x \, dv \, du
$$

$$
= \int_0^s j_\beta(s-u)(j_0 * T)(u)d \int_0^t j_\beta(t-v)[T(v) - j_\alpha(v)C]x \, dv
$$

$$
= j_\beta * (j_0 * T)(s)[j_\beta * T(t) - j_\beta * j_\alpha(t)C]x
$$

$$
= \int_0^t U(r)dr[U(t) - j_{\alpha+\beta+1}(t)C]x.
$$

(2.8)
Therefore, $U = j_\beta * T$ is an $(\alpha + \beta + 1)$-times integrated $C$-semigroup by Theorem 2.3 again. □

3. $(C,\alpha)$-subgenerators

Let $T(\cdot) : [0, \tau) \to B(X)$ be a strongly continuous function. We consider properties of those linear operators $G$ which satisfy $R(S(t)) \subset D(G)$ and $S(t)G \subset GS(t) = T(t)x - j_\alpha(t)C$, that is, the following two conditions hold:

$$T(t)x - j_\alpha(t)Cx = S(t)Gx \quad \text{for } x \in D(G), \quad 0 \leq t < \tau, \quad (3.1)$$

$$R(S(t)) \subset D(G), \quad T(t)x - j_\alpha(t)Cx = GS(t)x \quad \text{for } x \in X, \quad 0 \leq t < \tau. \quad (3.2)$$

Such an operator $G$ will be called a $(C,\alpha)$-subgenerator of $T(\cdot)$. There may or may not exist $(C,\alpha)$-subgenerators for a given local $\alpha$-times integrated $C$-semigroup and there may be many ones. If there is a $(C,\alpha)$-subgenerator which contains all $(C,\alpha)$-subgenerators of $T(\cdot)$, then we call this maximal $(C,\alpha)$-subgenerator the $(C,\alpha)$-generator of $T(\cdot)$.

It will be seen in Theorem 3.5(c) that if $C$ is injective and if there is a closed $(C,\alpha)$-subgenerator $G$ of $T(\cdot)$, then $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup and $A := C^{-1}GC$ is its $(C,\alpha)$-generator. $(C,\alpha)$-subgenerators and $(C,\alpha)$-generator of a local $\alpha$-times integrated $C$-semigroup will be called simply subgenerators and generator, respectively.

**Lemma 3.1.** Let $C \in B(X)$ be injective and let $T(\cdot) : [0, \tau) \to B(X)$ be strongly continuous. If an operator $G$ satisfies condition (3.1), then it satisfies the following condition:

$$u \equiv 0 \text{ is the only solution of the equation } u(t) = G(1 * u)(t), \quad 0 \leq t < \tau. \quad (3.3)$$

In particular, (3.3) holds for any $(C,\alpha)$-subgenerator $G$ of $T(\cdot)$.

**Proof.** Let $u$ be a solution of $u(t) = G \int_0^t u(s)ds$. By (3.1), we have

$$S * u = S * G(1 * u) = [T - j_\alpha C] * (1 * u)$$

$$= [S - j_{\alpha+1} C] * u = S * u - j_{\alpha+1} C * u. \quad (3.4)$$

This proves $j_{\alpha+1} C * u \equiv 0$. It follows from Lemma 2.1(b) and the continuity of $u$ that $Cu \equiv 0$ and hence $u \equiv 0$. □

**Remark 3.2.** Whenever $C$ is injective, Lemma 3.1 implies that an operator $G$ can be a $(C,\alpha)$-subgenerator of at most one strongly continuous local $\alpha$-times integrated $C$-semigroup $T(\cdot)$.

**Lemma 3.3.** Let $T(\cdot) : [0, \tau) \to B(X)$ be strongly continuous. If $CT(t) = T(t)C$ for $0 \leq t < \tau$, and if $T(\cdot)$ has a $(C,\alpha)$-subgenerator $G$, then $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup with $G$ a subgenerator.
Then the following two conditions are equivalent:
continuous function and $G$ that

Therefore, using (3.6) and (3.7), we have for $0 \leq s, t < \tau$ and $x \in X$

\[
[T(t) - j_{a}(t)C]S(s)x = S(t)GS(s)x = S(t)[T(s) - j_{a}(s)C]x. \quad (3.5)
\]

Hence it follows from Theorem 2.3 that $T(\cdot)$ is an $\alpha$-times integrated $C$-semigroup. □

**Proposition 3.4.** Let $C \in B(X)$ be an injection. Let $T(\cdot): [0, \tau) \rightarrow B(X)$ be a strongly continuous function and $G$ be a closed operator satisfying (3.2) and (3.3). Suppose that $B$ is a closed operator such that $BG \subset GB$, that is, $D(BG) \subset D(GB)$ and $BG = GB$ on $D(BG)$, and such that $S(t)D(B) \subset D(B)$ for all $0 \leq t < \tau$, and $BS(\cdot)x \in C([0, \tau), X)$ for all $x \in D(B)$. Then the following two conditions are equivalent:

(a) $CB \subset BC$;
(b) $S(t)B \subset BS(t)$ and $G(1 \ast S)(t)D(B) \subset D(B)$ for all $0 \leq t < \tau$.

**Proof.** (a)⇒(b). Integrating (3.2), we have from the closedness of $G$ that

\[S(t)x - j_{a+1}(t)Cx = (1 \ast GS)(t)x = G(1 \ast S)(t)x \quad \text{for } x \in X. \quad (3.6)\]

Let $x \in D(B)$. By assumption, $S(t)x \in D(B)$. Also, by (a) we have $j_{a+1}(t)Cx \in D(B)$ and $B j_{a+1}(t)Cx = j_{a+1}(t)CBx$ for $0 \leq t < \tau$. Hence it follows from (3.6) that $G(1 \ast S)(t)x \in D(B)$ for all $0 \leq t < \tau$. Then, by the closedness of $B$ and the assumption on $B$ we obtain that

\[BG(1 \ast S)(t)x = GB(1 \ast S)(t)x = G(1 \ast BS)(t)x \quad \forall 0 \leq t < \tau. \quad (3.7)\]

Therefore, using (3.6) and (3.7), we have for $x \in D(B)$ and $0 \leq t < \tau$,

\[S(t)Bx - G(1 \ast S)(t)Bx = j_{a+1}(t)CBx = B j_{a+1}(t)Cx\]
\[= B[S(t)x - G(1 \ast S)(t)x] \quad (3.8)\]
\[= BS(t)x - G(1 \ast BS)(t)x.\]

This implies $S(t)Bx - BS(t)x = G1 \ast [S(\cdot)B - BS(\cdot)](t)x$ for all $0 \leq t < \tau$. Since $u = S(\cdot)Bx - BS(\cdot)x$ is a strongly continuous solution of $u = G1 \ast u$, it follows from (3.3) that $S(\cdot)Bx - BS(\cdot)x \equiv 0$ for all $x \in D(B)$. Therefore, (b) holds.

(b)⇒(a). Let $x \in D(B)$. By (b) and (3.6), we have

\[j_{a+1}(t)Cx = S(t)x - G(1 \ast S)(t)x \in D(B) \quad \forall 0 \leq t < \tau. \quad (3.9)\]

So, $Cx \in D(B)$. By the closedness of $B$ and the assumption on $B$, this implies that $BG(1 \ast S)(t)x = BS(t)x - B j_{a+1}(t)Cx = S(t)Bx - j_{a+1}(t)BCx$ is strongly continuous on $0 \leq t < \tau$. It follows from the assumption on $B$, the closedness of $B$, and condition (b) that

\[BG(1 \ast S)(t)x = GB(1 \ast S)(t)x = G(1 \ast BS)(t)x = G(1 \ast S)(t)Bx \quad (3.10)\]
for all $0 \leq t < \tau$. Therefore, by (3.6) and (b) again, we obtain that

$$B j_{a+1}(t)Cx = BS(t)x - BG(1 \ast S)(t)x$$
$$= S(t)Cx - G(1 \ast S)(t)Cx = j_{a+1}(t)CBx \quad \forall 0 \leq t < \tau.$$  \hfill (3.11)

This proves (a).

Note that if $B \in B(X)$, the assumption that $S(t)D(B) \subset D(B)$ for all $0 \leq t < \tau$ and $BS(\cdot)x \in C([0, \tau], X)$ for $x \in D(B)$ always holds.

**Theorem 3.5.** Let $C \in B(X)$ be injective, and let $T(\cdot) : [0, \tau) \rightarrow B(X)$ be a strongly continuous function with a closed $(C, \alpha)$-subgenerator $G$. Then, the following hold:

(a) $CT(t) = T(t)C$ for all $0 \leq t < \tau$ (or equivalently, $CS(t) = S(t)C$ for all $0 \leq t < \tau$), so that $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup.

(b) $T(t)T(s) = T(s)T(t)$ for all $0 \leq s, t < \tau$.

(c) $CG \subset GC$, and $C^{-1}GC$ is the generator of $T(\cdot)$.

**Proof.** By the definition of $(C, \alpha)$-subgenerator, we have $R(S(s)) \subset D(G)$ and $S(s)G \subset GS(s)$ for all $s \in [0, \tau)$. Also, by Lemma 3.1, (3.3) holds. Hence the hypothesis and Proposition 3.4 (b) hold with $B$ replaced by $G$, so that Proposition 3.4 (a) also holds with $B$ replaced by $G$, that is, the first part of the above condition (c) is true. Then, the hypothesis and Proposition 3.4 (a) hold with $B$ replaced by $C$, and consequently Proposition 3.4 (b) also holds with $B$ replaced by $C$, that is, $S(t)C = CS(t)$ for all $0 \leq t < \tau$. Then Lemma 3.3 implies that $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup. Finally, applying (a) and Proposition 3.4 with $B$ replaced by $S(s)$ for any $(0 \leq s < \tau)$ yields that Proposition 3.4 (b) also holds with $B$ replaced by $S(s)$, that is, $S(t)S(s) = S(s)S(t)$ for all $0 \leq t < \tau$. Then, by differentiation with respect to $s$ and $t$, we obtain the above condition (b).

To show the second part of (c), we first show that $C^{-1}GC$ is a subgenerator of $T(\cdot)$. Since $G$ is a closed $(C, \alpha)$-subgenerator of $T(\cdot)$ and $G \subset C^{-1}GC$, we have $T(t) - j_a(t)C = GS(t) = C^{-1}GS(t)$ for all $0 \leq t < \tau$. Moreover, if $x \in D(C^{-1}GC)$, then $Cx \in D(G)$ and $GCx \in R(C)$, so that, by (a),

$$C[T(t)x - j_a(t)Cx] = [T(t) - j_a(t)C]Cx = S(t)GCx$$
$$= S(t)CC^{-1}GCx = CS(t)C^{-1}GCx.$$  \hfill (3.12)

It follows from the injectivity of $C$ that $T(t)x - j_a(t)Cx = S(t)C^{-1}GCx$ for all $0 \leq t < \tau$. Therefore, $C^{-1}GC$ is a subgenerator of $T(\cdot)$.

Let $B$ be any subgenerator of $T(\cdot)$. It follows from (3.1) and (3.2) that for every $x \in D(B)$, $j_{a+1}(t)Cx = S(t)x - (1 \ast S)(t)Bx \in D(G)$. This together with (3.2) and the closedness of $G$ implies

$$GS(t)x - Gj_{a+1}(t)Cx = G(1 \ast S)(t)Bx = (1 \ast [T - j_aC])(t)Bx$$
$$= S(t)Bx - j_{a+1}(t)CBx = BS(t)x - j_{a+1}(t)CBx.$$  \hfill (3.13)
Since \( GS(t) = T(t) - j_a(t)C = BS(t) \) by (3.2), we have \( Gj_{a+1}(t)Cx = j_{a+1}(t)CBx \) for all \( 0 \leq t < \tau \). Since \( C \) is injective, this implies \( Bx = C^{-1}GCx \), that is, \( B \subset C^{-1}GC \). Hence \( C^{-1}GC \) is the generator of \( T(\cdot) \).

The next corollary is about the converse of (c) of Theorem 3.5.

**Corollary 3.6.** Let \( C \in B(X) \) be injective, let \( G \) be a closed operator satisfying \( G \subset C^{-1}GC \), and let \( T(\cdot) : [0, \tau) \rightarrow B(X) \) be a strongly continuous function. If \( C^{-1}GC \) is a \((C, \alpha)\)-subgenerator of \( T(\cdot) \), and if for every \( 0 \leq t < \tau \), there is a dense subspace \( D_t \) of \( X \) such that \( S(t)D_t \subset D(G) \), then \( G \) is also a \((C, \alpha)\)-subgenerator of \( T(\cdot) \). In particular, the conclusion holds when \( C \) has dense range.

**Proof.** \( C^{-1}GC \) and \( T(\cdot) \) satisfy

\[
T(t)x - j_a(t)Cx = S(t)C^{-1}GCx \quad \text{for} \quad x \in D(C^{-1}GC); \tag{3.14}
\]

\[
T(t)x - j_a(t)Cx = C^{-1}GCS(t)x \quad \text{for} \quad x \in X \tag{3.15}
\]

for \( 0 \leq t < \tau \). Since \( G \subset C^{-1}GC \), (3.14) implies that \( G \) satisfies (3.1). Equation (3.15) and the assumption \( CG \subset GC \) imply that for every \( x \in D_t \),

\[
C[T(t) - j_a(t)C]x = GCS(t)x = CGS(t)x. \tag{3.16}
\]

Since \( C \) is injective, this implies \( T(t)x - j_a(t)Cx = GS(t)x \) for \( x \in D_t \). It follows from \( D_t = X \) and the closedness of \( G \) that, for every \( x \in X \), \( S(t)x \in D(G) \), and \( T(t)x - j_a(t)Cx = GS(t)x \) for all \( x \in X \), that is, \( G \) satisfies (3.2). Therefore \( G \) is a closed \((C, \alpha)\)-subgenerator of \( T(\cdot) \).

Since (3.15) shows that \( S(t)Cx = CS(t)x \in D(G) \) for all \( x \in X \) and \( 0 \leq t < \tau \), we can take \( D_t = R(C) \) if \( C \) has dense range.

**Corollary 3.7.** Let \( C \in B(X) \) be injective and let \( T, H : [0, \tau) \rightarrow B(X) \) be strongly continuous functions with closed \((C, \alpha)\)-subgenerators \( G \) and \( K \), respectively. Suppose \( KG \subset GK \) and \( (1 \ast T)(t)D(K) \subset D(K) \) for all \( 0 \leq t < \tau \) and \( K(1 \ast T)(\cdot)x \in C([0, \tau), X) \) for all \( x \in D(K) \). Then \( T(t)H(s) = H(s)T(t) \) for all \( 0 \leq s, t < \tau \).

**Proof.** By Theorem 3.5, we have \( CK \subset KC, CG \subset GC, CS(t) = S(t)C, \) and \( CH(t) = H(t)C \). Using these facts together with \( KG \subset GK \), we obtain from Proposition 3.4 (by taking \( B = K \)) that \( S(t)K \subset KS(t) \) for all \( 0 \leq t < \tau \). Fix a \( t \geq 0 \). Since \( S(t)K \subset KS(t) \) and \( CS(t) = S(t)C \), taking \( B = S(t) \) in Proposition 3.4 we deduce that \( H(s)S(t) = S(t)H(s) \) for all \( 0 \leq s < \tau \). This completes the proof.

4. Generators of nondegenerate local \( \alpha \)-times integrated C-semigroups

The results discussed so far are formulated under the assumption of existence of a \((C, \alpha)\)-subgenerator of a strongly continuous local \( \alpha \)-times integrated C-semigroup \( T(\cdot) \). In this section, we will see that subgenerators and generator do exist if \( T(\cdot) \) is a nondegenerate local \( \alpha \)-times integrated C-semigroup.

**Lemma 4.1.** Let \( T(\cdot) \) be a local \( \alpha \)-times integrated C-semigroup on \([0, \tau)\). The following conditions have the implication relations (c) \( \Rightarrow \) (a) \( \Rightarrow \) (b):

\[
\alpha \text{-times integrated C-semigroup} \end{equation}
(a) \( T(\cdot) \) is nondegenerate;
(b) \( C \) is injective;
(c) \( u \in C([0, \tau/2), X) \) and \( T \ast u \equiv 0 \) imply \( u \equiv 0 \).

Moreover, when \( T(\cdot) \) has a subgenerator, these three conditions are equivalent.

**Proof.** (a)⇒(b). If \( Cx = 0 \), then from (1.1) we see that \( T(s)T(t)x = 0 \) for all \( 0 < s, t < \tau/2 \), which implies \( x = 0 \) by our definition of nondegeneracy. Hence \( C \) is injective.

(c)⇒(a). If \( x \in X \) is such that \( T(t)x = 0 \) for all \( 0 < t < \tau/2 \), then for \( u \equiv x \) we have \( (T \ast u)(t) = (1 \ast T)(t)x = 0 \) for all \( 0 < t < \tau/2 \). Thus, (a) follows from (c).

Next, suppose there is a subgenerator. We show “(b)⇒(c).” If \( u \in C([0, \tau/2), X) \) satisfies \( T \ast u \equiv 0 \), then \( S \ast u \equiv 1 \ast (T \ast u) \equiv 0 \). It follows from (3.2) that

\[
0 \equiv GS \ast u = T \ast u - j_\alpha C \ast u = -j_\alpha \ast Cu. \tag{4.1}
\]

By Lemma 2.1(b), we have \( Cu \equiv 0 \). Since \( C \) is injective, this proves \( u \equiv 0 \). Therefore, (b) implies (c) when \( T(\cdot) \) has a subgenerator. \( \square \)

**Lemma 4.2.** Let \( C \in B(X) \) be injective and \( \{T(t); \, 0 \leq t < \tau\} \) be a local \( \alpha \)-times integrated \( C \)-semigroup. If \( x \in X \) is such that \( T(r)x = 0 \) for all \( 0 < r \leq s \) for some number \( s \in (0, \tau) \), then \( T(r)x = 0 \) for all \( 0 < r < \tau \). In particular, if \( T(\cdot) \) is nondegenerate, then \( T(r)x = 0 \) for all \( 0 < r \leq s \) with some number \( 0 < s < \tau \) implies \( x = 0 \).

**Proof.** For an arbitrary \( 0 \leq t < \tau \), choose an \( s_0 \in (0, \min\{s, \tau - t\}) \). The assumption implies \( T(s_0)x = 0 \) and \( (1 \ast T)(s_0)x = 0 \). Then, it follows from Theorem 2.3 that

\[
-j_\alpha(s_0)C(1 \ast T)(t)x = (1 \ast T)(t)[T(s_0) - j_\alpha(s_0)C]x \\
= [T(t) - j_\alpha(t)C](1 \ast T)(s_0)x = 0. \tag{4.2}
\]

Since \( C \) is injective, this implies that \( (1 \ast T)(t)x = 0 \) for all \( 0 \leq t < \tau \), and hence \( T(t)x = 0 \) for all \( 0 \leq t < \tau \). \( \square \)

We are ready to show the existence of subgenerators and generator for a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup.

**Definition 4.3.** Let \( C \in B(X) \) and let \( T(\cdot) \) be a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup. We define for every \( 0 < t < \tau \) a linear operator \( G_t : D(G_t) \to X \) by

\[
D(G_t) := \left\{ \sum_{k=1}^{n} S(t_k)x_k; \, 0 \leq t_k < t, \, x_k \in X, \, k = 1, 2, \ldots, \, n = 1, 2, \ldots \right\}, \tag{4.3}
\]

\[
G_ty := \sum_{k=1}^{n} [T(t_k) - j_\alpha(t_k)C]x_k \quad \text{for} \quad y = \sum_{k=1}^{n} S(t_k)x_k \in D(G_t). \]

Fix a \( 0 < t < \tau \). We see that \( G_t \) is well-defined. Indeed, if \( \sum_{k=1}^{n} S(t_k)x_k = 0 \), then, by Theorem 2.3, for every \( 0 \leq r < \tau - t \)

\[
S(r) \sum_{k=1}^{n} [T(t_k) - j_\alpha(t_k)C]x_k = \sum_{k=1}^{n} [T(r) - j_\alpha(r)C]S(t_k)x_k = 0. \tag{4.4}
\]
Since \( T(\cdot) \) is nondegenerate, it follows from Lemma 4.2 that \( \sum_{k=1}^{n} [T(t_k) - j_{\alpha}(t_k)C] x_k = 0 \). This proves that \( G_t \) is well-defined. These operators \( G_t \) form an increasing net. Let us define \( G_t : D(G_t) \to X \) by

\[
D(G_t) := \bigcup_{0 < t < \tau} D(G_t),
\]

\[
G_t x := G_t x \quad \text{if} \ x \in D(G_t) \text{ for some } 0 < t < \tau.
\]

**Proposition 4.4.** Let \( T(\cdot) \) be a nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup on \( X \), and let operators \( G_t, G_\tau \) be defined as above.

(i) For \( 0 \leq s < t < \tau \), we have

\[
S(s)X \subset D(G_t), \quad S(s)G_t \subset G_t S(s) = T(s) - j_{\alpha}(s)C. \tag{4.6}
\]

(ii) \( G_t \) is a subgenerator of \( T(\cdot) \), that is,

\[
S(s)X \subset D(G_t), \quad S(s)G_t \subset G_t S(s) = T(s) - j_{\alpha}(s)C \quad \forall 0 \leq s < \tau. \tag{4.7}
\]

**Proof.** (i) Since \( s < t \), by the definition of \( G_t \), we have \( S(s)x \in D(G_t) \) and \( G_t S(s)x = [T(s) - j_{\alpha}(s)C] x \) for all \( x \in X \). To show \( S(s)G_t \subset G_t S(s) = T(s) - j_{\alpha}(s)C \), let \( 0 \leq r < \tau - t \). Then, (1.1) implies that \( S(r) \) commutes with \( T(u) \) and \( S(u) \) for \( 0 \leq u \leq t \). If \( y \in D(G_t) \), then \( y = \sum_{k=1}^{n} S(t_k) x_k \) for some \( t_k \in [0,t) \), \( x_k \in X \), \( k = 1,\ldots,n \). By Theorem 2.3, we have

\[
S(r)S(s)G_t y = S(s)S(r) \sum_{k=1}^{n} [T(t_k) - j_{\alpha}(t_k)C] x_k
\]

\[
= S(s)[T(r) - j_{\alpha}(r)C] \sum_{k=1}^{n} S(t_k) x_k = S(s)[T(r) - j_{\alpha}(r)C] y
\]

\[
= [T(s) - j_{\alpha}(s)C] S(r)y = S(r)[T(s) - j_{\alpha}(s)C] y.
\]

This being true for all \( r \in [0,\tau - t) \), it follows from Lemma 4.2 that \( S(s)G_t y = [T(s) - j_{\alpha}(s)C] y \).

(ii) follows easily from (i) and the definition of \( G_t \). \( \square \)

**Lemma 4.5.** Suppose \( G \) and \( B \) are subgenerators of \( T(\cdot) \). Define a linear operator \( K : D(G) + D(B) \to X \) by \( Ky := Gx_1 + Bx_2 \) whenever \( y = x_1 + x_2 \) for some \( x_1 \in D(G) \) and \( x_2 \in D(B) \). Then, \( K \) is well-defined and it is also a subgenerator of \( T(\cdot) \).

**Proof.** Suppose \( G \) and \( B \) are two subgenerators of \( T(\cdot) \). If \( y = x_1 + x_2 = z_1 + z_2 \) for some \( x_1, z_1 \in D(G) \) and \( x_2, z_2 \in D(B) \), then (3.1) implies

\[
S(t)(Gx_1 + Bx_2) = [T(t) - j_{\alpha}(t)C] (x_1 + x_2)
\]

\[
= [T(t) - j_{\alpha}(t)C] (z_1 + z_2) = S(t)(Gz_1 + Bz_2) \tag{4.9}
\]
and hence \( T(t)(Gx_1 + Bx_2) = T(t)(Gz_1 + Bz_2) \) for every \( 0 \leq t < \tau \). Since \( T(\cdot) \) is nondegenerate, \( Gx_1 + Bx_2 = Gz_1 + Bz_2 \). Therefore, \( K \) is a well-defined linear operator which satisfies (3.1). Clearly, \( K \) contains both \( G \) and \( B \). Hence

\[
T(t) - j_\alpha(t)C = GS(t) = KS(t) \quad \text{for} \quad 0 \leq t < \tau,
\]

that is, (3.2) holds for \( K \).

**Proposition 4.6.** Let \( T(\cdot) \) be a local \( \alpha \)-times integrated \( C \)-semigroup.

(i) If \( T(\cdot) \) has a subgenerator, then \( T(\cdot) \) has a maximal subgenerator which contains all subgenerators of \( T(\cdot) \); it is called the generator of \( T(\cdot) \).

(ii) If \( T(\cdot) \) is nondegenerate, then \( T(\cdot) \) has a generator.

(iii) Suppose \( T(\cdot) \) is nondegenerate. Any subgenerator \( G \) is closable and its closure \( \overline{G} \) is also a subgenerator of \( T(\cdot) \), and \( A := C^{-1}\overline{G}C \) is the generator of \( T(\cdot) \). In particular, the operator \( G_\tau \) is closable and \( A := C^{-1}\overline{G}C \) is the generator of \( T(\cdot) \).

**Proof.** (i) Suppose \( B \) is a subgenerator of \( T(\cdot) \). Let \( \mathcal{S} \) be the set of all subgenerators of \( T(\cdot) \). Then, \( B \in \mathcal{S} \). If \( G \in \mathcal{S} \), the definition of subgenerator implies \( S(t)X \subseteq D(G) \).

Let \( \{G_i\}_{i \in I} \) be an arbitrary chain in \((\mathcal{S}, \subseteq)\). Define \( G := \bigcup_{i \in I} D(G_i) \rightarrow X \) by \( Gx := G_ix \) for \( x \in G_i \) for some \( i \in I \). It is clear that \( G \) is well-defined and \( D(G) = \bigcup_{i \in I} G_i \). If \( x \in D(G) \), say \( x \in D(G_i) \) for an \( i \in I \), then

\[
S(t)Gx = S(t)G_ix = T(t)x - j_\alpha(t)Cx = G_ixS(t)x = GS(t)x \quad \forall t \geq 0.
\]

Therefore, \( G \) is a subgenerator of \( T(\cdot) \) and so is an upper bound of the chain \( \{G_i\}_{i \in I} \). By the Zorn’s lemma, \( \mathcal{S} \) has a maximal subgenerator, say \( G \).

We claim that \( G \) contains all subgenerators. Suppose there were \( B \in \mathcal{S} \) such that \( D(B) \not\subseteq D(G) \). Then, the operator \( K \) as defined in Lemma 4.5 is a subgenerator which is a proper extension of \( G \). This contradicts the maximality of \( G \) and so we must have \( D(B) \subseteq D(G) \) for any subgenerator \( B \) of \( T(\cdot) \).

(ii) follows from (i) and Proposition 4.4(ii).

(iii) Let \( \{x_n\} \) be a sequence in \( D(G) \) such that \( x_n \rightarrow 0 \) and \( Gx_n \rightarrow y \) as \( n \rightarrow \infty \) for some \( y \in X \). It follows from (3.1) that for every \( 0 \leq t < \tau \)

\[
S(t)y = \lim_{n \rightarrow \infty} S(t)Gx_n = \lim_{n \rightarrow \infty} [T(t) - j_\alpha(t)C]x_n = 0.
\]

Since \( T(\cdot) \) is nondegenerate, this implies \( y = 0 \). Therefore, \( G \) is closable. Finally, let \( y \in D(\overline{G}) \) and \( 0 \leq t < \tau \). Then, there is a sequence \( \{y_n\} \) in \( D(G) \) such that \( (y_n, Gy_n) \rightarrow (y, \overline{G}y) \) as \( n \rightarrow \infty \). By (3.1), we have

\[
S(t)\overline{G}y = \lim_{n \rightarrow \infty} S(t)Gy_n = \lim_{n \rightarrow \infty} [T(t) - j_\alpha(t)C]y_n = [T(t) - j_\alpha(t)C]y.
\]

Since \( \overline{G} \) is an extension of \( G \), we also have that \( \overline{G}S(t) = GS(t) = T(t) - j_\alpha(t)C \), that is, \( \overline{G} \) is also a subgenerator of \( T(\cdot) \). That \( C^{-1}\overline{G}C \) is the generator follows from Theorem 3.5(c).

**Remark 4.7.** It is seen from Proposition 4.6 (ii) and Theorem 3.5(c) that any nondegenerate local \( \alpha \)-times integrated \( C \)-semigroup has a unique generator \( A \), which is closed and...
satisfies $C^{-1}AC = A$, and that the generator $A$ is precisely the operator defined by

$$x \in D(A), \quad Ax = y \iff S(t)y = T(t)x - j_\alpha(t)Cx \quad \forall 0 \leq t < \tau. \quad (4.14)$$

**Example 4.8.** If $G$ is a $(C, \alpha)$-subgenerator of a strongly continuous function $T(\cdot)$ and $C_1 \in B(X)$ is such that $CC_1 = C_1C$ and $C_1G \subset GC_1$, then $G$ is $(CC_1, \alpha)$-subgenerator of $C_1T(\cdot)$.

**Example 4.9.** Let $T_0 : C_0[0, \infty) \to C_0[0, \infty)$ be the translation semigroup. Then, $T_0(\cdot)$ is not a $(C_0, \alpha)$-semigroup but $\{j_\alpha \ast T_0(t)\}_{t \geq 0}$ is an $\alpha$-times integrated semigroup on $[0, \infty)$ for all $\alpha > 0$.

**Example 4.10.** Let $C \in B(X)$. $T(t) := j_\alpha(t)C, t \geq 0$, is an $\alpha$-times integrated $C$-semigroup. It is easily seen from (3.1) and (3.2) that an operator $G \in B(X)$ is a subgenerator of $T(\cdot)$ if and only if $CG = GC = 0$. For example, for any $2 \times 2$ matrix $H$ the matrix $\left( \begin{array}{cc} 0 & 0 \\ 0 & H \end{array} \right)$ is a maximal subgenerator of the $\alpha$-times integrated $\left( \begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array} \right)$-semigroup $T(t) := \left( \begin{array}{cc} 2j_\alpha(t) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right)$.

**Example 4.11.** More generally, let $T(\cdot)$ be a nondegenerate local $\alpha$-times integrated $C_X$-semigroup on a Banach space $X$ with generator $G$. If $Y \neq \{0\}$ is another Banach space and $C_Y \in B(Y)$, then

$$\tilde{T}(\cdot) := \left( \begin{array}{cc} T(\cdot) & 0 \\ 0 & j_\alpha(\cdot)C_Y \end{array} \right) \quad (4.15)$$

is a local $\alpha$-times integrated $\left( \begin{array}{cc} C_X & 0 \\ 0 & C_Y \end{array} \right)$-semigroup on $X \oplus Y$. $\tilde{T}(\cdot)$ is nondegenerate if and only if $C_Y$ is injective. If $C_Y$ is not injective, then for any $H \in B(Y)$ which satisfies $C_YH = HC_Y = 0$, the operator $\left( \begin{array}{cc} G & 0 \\ 0 & H \end{array} \right)$ is a maximal subgenerator of $\tilde{T}(\cdot)$. If $C_Y$ is injective, then $\left( \begin{array}{cc} G & 0 \\ 0 & 0 \end{array} \right)$ is the generator of $\tilde{T}(\cdot)$.

Thus a degenerate local $\alpha$-times integrated $C$-semigroup may have more than one maximal subgenerator, and hence has no generator. This is in contrast to the nondegenerate case (Proposition 4.6(ii)).

**Example 4.12.** Let $T(\cdot)$ be the family of operators on $c_0$ (or $\ell^1$) defined by $T(t)x := ((n - k)e^{-n}j_\alpha-1(t-s)e^{n}ds)x_n$, for $x = (x_n) \in c_0$ (or $\ell^1$) and for $t \in [0, 1]$. Let $C$ denote the operator defined by $Cx := ((n - k)e^{-n}x_n)$. $T(\cdot)$ is a local $\alpha$-times integrated $C$-semigroup which cannot be extended beyond 1. If $k = 0$, then $C$ is injective and the generator of $T(\cdot)$ is the operator $G : (x_n) \to (nx_n)$. If $k = 1$, $T(\cdot)$ is a degenerate local $\alpha$-times integrated $C$-semigroup and for each $a \in \mathbb{C}$ the operator $G_a$ defined by $G_a(x) := (ax_1, 2x_2, 3x_3, \ldots)$ is a maximal subgenerator of $T(\cdot)$.

From Lemma 4.1, Proposition 4.4, and Theorem 3.5, we deduce the next corollary.

**Corollary 4.13.** If $T(\cdot)$ is a nondegenerate local $\alpha$-times integrated $C$-semigroup, then $T(s)T(t) = T(t)T(s)$ for all $0 \leq s, t < \tau$.

**Remark 4.14.** In the proof of Proposition 4.4 (i), we have used the commutativity: $T(s)T(t) = T(t)T(s)$ only for $0 \leq s, t < \tau$ with $s + t < \tau$, as given by (1.1). Now, Corollary 4.13 shows that the restriction $s + t < \tau$ can be removed, and consequently, one can show that the relation in Proposition 4.4 (i) actually holds for all $s, t \in [0, \tau)$.
We can deduce the following characterization theorem for nondegenerate local $\alpha$-times integrated C-semigroups.

**Theorem 4.15.** Let $C \in B(X)$ and let $T(\cdot) : [0, \tau] \to B(X)$ be a strongly continuous function. Then, $T(\cdot)$ is a nondegenerate local $\alpha$-times integrated C-semigroup if and only if $C$ is injective and there is a closed $(C, \alpha)$-subgenerator $G$ (i.e., satisfying (3.1) and (3.2)) of $T(\cdot)$. In this case, $G$ is a closed subgenerator and $A := C^{-1}GC$ is the generator of $T(\cdot)$.

**Proof.** The necessity follows from Lemma 4.1 and Proposition 4.4; the sufficiency follows from Theorem 3.5(a) and Lemma 4.1. \hfill \Box

5. Relation with abstract Cauchy problems

**Theorem 5.1.** Let $C \in B(X)$ be injective and $\alpha \geq 0$, and let $A$ be a closed linear operator on $X$. Then, the following statements are equivalent

(i) $A$ is a subgenerator of a local $\alpha$-times integrated C-semigroup $T(\cdot)$.

(ii) $CA \subset AC$ (i.e., $Cx \in D(A)$ and $CAx = ACx$ for $x \in D(A)$) and the equation: $v(t) = A(1 \ast v)(t) + j_{\alpha}(t)Cx$, $0 \leq t < \tau$, has a unique solution $v_x$ for every $x \in X$.

(iii) $CA \subset AC$ and the equation: $u'(t) = Au(t) + j_{\alpha}(t)Cx$, $0 \leq t < \tau$; $u(0) = 0$, has a unique solution $u_x$ for every $x \in X$.

Moreover, the solutions are given by $v_x = T(\cdot)x$ and $u_x(t) = \int_0^t T(s)xds$, $t \geq 0$.

**Proof.** (i) $\Rightarrow$ (ii). Since $T(\cdot)$ is an $\alpha$-times integrated C-semigroup with $A$ as a subgenerator and $C$ is injective, (3.1)–(3.3) hold. Thus (ii) can be deduced from (3.2), Lemmas 3.1 and 4.1, and Theorem 3.5(c).

(ii) $\Rightarrow$ (i). We define the operator $T(t)$ by $T(t)x := v_x(t)$ for $x \in X$. Then, $T(\cdot)x$ is strongly continuous on $[0, \tau)$ for every $x \in X$. Since both $A$ and $C$ are linear, the uniqueness of solution implies that $T(t)$ is a linear operator on $X$ for all $0 \leq t < \tau$.

Next, we show that $T(t)$ is a bounded operator for each $0 \leq t < \tau$. Let $C([0, \tau), X)$ be the Fréchet space with the quasinorm $\|v\| := \sum_{k=1}^{\infty} \|v\|_k/(2^k(1 + \|v\|_k))$ for $v \in C([0, \tau), X)$, where $\|v\|_k := \max_{t \in [0, p_k]} \|v(t)\|$, $k = 1, 2, \ldots$, and $0 < p_k < \tau$. Consider the map $\eta : X \to C([0, \tau), X)$ defined by $\eta(x) := T(\cdot)x = v_x$. We show that $\eta$ is a closed linear operator. Let $\{x_n\}$ be a sequence in $X$ such that $(x_n, \eta(x_n)) \to (x, \eta(x))$ strongly as $n \to \infty$ for some $x \in X$ and $v \in C([0, \tau), X)$. Since $A$ is closed and $v_{x_n} = A(1 \ast v_{x_n}) + j_{\alpha}Cx_n$, we obtain $v = A(1 \ast v) + j_{\alpha}Cx$. It follows from the uniqueness of solutions that $v = v_x = T(\cdot)x = \eta(x)$. Hence $\eta$ is closed. It follows from the closed graph theorem that $\eta$ is continuous. This shows that $T(\cdot)$ is a strongly continuous function of bounded linear operators on $X$ and it satisfies (3.2).

If $A$ is shown to be a $(C, \alpha)$-subgenerator of $T(\cdot)$, then by Theorem 3.5(c) we conclude that $T(\cdot)$ is a local $\alpha$-times integrated C-semigroup with subgenerator $A$. This will be done if we can show $S(t)Ax = AS(t)x$ for all $x \in D(A)$ and $0 \leq t < \tau$. Since $A$ is closed, we obtain from (3.2) that $AS(\cdot)x \in C([0, \tau), X)$ for all $x \in X$. Since (ii) implies that condition (3.3) holds for $G = A$ and Proposition 3.4 (a) holds for $B = A$, applying Proposition 3.4 we obtain $S(t)A \subset AS(t) \ (0 \leq t < \tau)$ as desired. Thus, $A$ is a subgenerator of $T(\cdot)$.

Clearly, (ii) and (iii) are equivalent. This completes the proof. \hfill \Box
Lemma 5.2. Let $C \in B(X)$ be injective and $\alpha \geq 0$, and let $A$ be a closed subgenerator of a local $\alpha$-times integrated $C$-semigroup $S(\cdot)$ on $X$, and let $1 \leq k \leq \lfloor \alpha \rfloor + 1$. Then, for every $x \in D(A^k)$, the problem $ACP(A; j_{a-k}Cx, \delta_{a,[\alpha]}Cx)$ has a unique solution, which is given by

$$u_k(t) := S^{(k-1)}(t)x = S(t)A^{k-1}x + \sum_{j=0}^{k-2} j_{a-1-j}(t)CA^{k-2-j}x, \quad 0 \leq t < \tau.$$  \hfill (5.1)

Proof. Let $X_k = D(A^k)$ be equipped with the norm $\|x\|_k$ by $\|x\|_k = \sum_{i=0}^{k} \|A^i x\|_k$ for $x \in X_k$, $k = 1, 2, \ldots$. If $y \in D(A)$, then (3.1) and (3.2) imply that $S(\cdot)y \in C^1([0, \infty), X) \cap C([0, \infty), X_1)$ and

$$S'(t)y = S(t)Ay + j_{a-1}(t)Cy, \quad 0 \leq t < \tau.$$ \hfill (5.2)

If $x \in D(A^k)$, then $x, Ax, A^2x, \ldots, A^{k-1}x \in D(A)$, so that by applying (5.2) repeatedly, we obtain that $S(\cdot)x \in C^k((0, \tau), X) \cap C([0, \tau), X_k)$ (where $X_k = D(A^k)$ with $\|x\|_k = \sum_{i=0}^{k} \|A^i x\|$ for $x \in X_k$) and

$$S^{(k)}(t)x = S(t)A^kx + \sum_{j=0}^{k-1} j_{a-1-j}(t)CA^{k-1-j}x, \quad 0 \leq t < \tau.$$ \hfill (5.3)

Let $u_k(t)$ be defined as in (5.1). Then, $u_k(0) = \delta_{a,k-1}Cx$ and

$$u_k'(t) = S^{(k)}(t)x = A \left( S(t)A^{k-1}x + \sum_{j=0}^{k-2} j_{a-1-j}(t)CA^{k-2-j}x \right) + j_{a-k}(t)Cx$$

$$= Au_k(t) + j_{a-k}(t)Cx.$$ \hfill (5.4)

This shows that $u_k$ is a solution of $ACP(A; j_{a-k}Cx, \delta_{a,[\alpha]}Cx)$, or equivalently, $v_k = u_k'$ is a solution of $v = A(1 * v) + j_{a-k}Cx$. The uniqueness of solution follows from Lemma 3.1. \hfill $\square$

6. Extension of local $\alpha$-times integrated $C$-semigroups

Let $T(\cdot)$ be a local $\alpha$-times integrated $C$-semigroup on $[0, \tau)$ with generator $A$, and let $n$ be an integer greater than or equal to $\alpha$. We will show that $A$ also generates a local $(\alpha + n)$-times integrated $C^2$-semigroup on $[0, 2\tau)$. Let $H(t) := (j_{\alpha+n-1} * T)(t)$, $\tau > t \geq 0$. Then, $H(\cdot)$ is an $n$-times integrated $C$-semigroup. Fix any $\tau_0 \in (0, \tau)$. Define an operator-valued function $S_{\tau_0} : [0, 2\tau_0) \to B(X)$ by

$$S_{\tau_0}(t) := \begin{cases} (j_{\alpha-1} * T)(t)C & \text{for } 0 \leq t \leq \tau_0, \\ T(\tau_0)H(t-\tau_0) + \sum_{k=0}^{n-1} j_{\alpha-k-1}(\tau_0)(j_k * H)(t-\tau_0)C \\ + \sum_{k=0}^{n-1} j_{\alpha-k-1}(t-\tau_0)(j_k * T)(\tau_0)C & \text{for } \tau_0 \leq t < 2\tau_0, \end{cases} \hfill (6.1)$$
where the $k$ in the first summation runs over those nonnegative integers such that $k - \alpha$ is not a nonnegative integer, that is, $k$ runs from 0 to $\alpha - 1$ when $\alpha$ is an integer and runs over all nonnegative integers when $\alpha$ is not an integer.

Clearly, $S_{t_0}(\cdot)$ is a local $(\alpha + n)$-times integrated $C^2$-semigroup on $[0, t_0]$ with generator $A$. It is easy to see for every $x \in X$ that

$$\lim_{t \to t_0} S_{t_0}(t)x = (j_{n-1} \ast T)(t_0)Cx = S_{t_0}(t_0)x. \quad (6.2)$$

Therefore, $S_{t_0}(\cdot)$ is strongly continuous on $[0, 2t_0)$. Since $A$ is the generator of $T(\cdot)$, we see that $A$ and $S_{t_0}(\cdot)$ commute.

**Theorem 6.1.** Let $T(\cdot)$ be a local $\alpha$-times integrated $C$-semigroup on $[0, \tau)$ with generator $A$. For any $t_0 \in (0, \tau)$, the function $S_{t_0}(\cdot)$, defined in (6.1), is a local $\alpha + n$-times integrated $C^2$-semigroup on $[0, 2t_0)$ with generator $A$. Thus the function $S(\cdot) : [0, 2\tau) \to B(X)$, defined by $S(t) := S_{t_0}(t)$ for $0 \leq t < 2t_0 < 2\tau$, is a local $(\alpha + n)$-times integrated $C^2$-semigroup on $[0, 2\tau)$ with generator $A$.

**Proof.** Since $S_{t_0}(\cdot)$ is a local $(\alpha + n)$-times integrated $C^2$-semigroup on $[0, t_0]$ with generator $A$, by Theorem 4.15 we need only to show that $A$ and $S_{t_0}(\cdot)$ satisfy

$$R((1 \ast S_{t_0})(t)) \subset D(A), \quad A((1 \ast S_{t_0})(t)) = S_{t_0}(t)x - j_{\alpha+n}(t)Cx \quad (6.3)$$

for $x \in X$ and $t_0 \leq t < 2t_0$.

We need the following equations which follow from (4.14):

$$A(j_{k+1} \ast H)(t) = [(j_k \ast H)(t) - j_{n+k+1}(t)C],$$

$$A(j_k \ast T)(t) = (j_{k-1} \ast T)(t) - j_{k+n}(t)C \quad \text{for } k = -1, 0, 1, 2, \ldots. \quad (6.4)$$

From the Taylor expansion, we have the next identity:

$$j_{\alpha+n}(t + \tau) = \frac{\tau^{\alpha+n}}{\Gamma(\alpha + n + 1)} \sum_{k=0}^{\infty} \binom{\alpha + n}{k} \left( \frac{t}{\tau} \right)^k = \sum_{k=0}^{\infty} j_k(t) j_{\alpha+n-k}(\tau)$$

$$= j_{\alpha+n}(\tau) + \left( \sum_{k=n+1}^{\infty} + \sum_{k=1}^{n} \right) j_k(t) j_{\alpha+n-k}(\tau)$$

$$= j_{\alpha+n}(\tau) + \sum_{k=0}^{\infty} j_{\alpha-k-1}(\tau) j_{n+k+1}(t) + \sum_{k=0}^{n-1} j_{\alpha-k}(\tau) j_{\alpha+k}(\tau) \quad (6.5)$$

for $0 \leq t < \tau$. Note that when $\alpha$ is an integer, all those terms with $k > \alpha - 1$ in the first summation vanish.
It is easy to see that \((1 \ast S_{\tau_0})(t) = (j_n \ast T)(t)C\) for \(0 \leq t \leq \tau_0\), and
\[
(1 \ast S_{\tau_0})(t) = (1 \ast S_{\tau_0})(\tau_0) + \int_0^{t-\tau_0} S_{\tau_0}(r+\tau_0)dr \\
= (j_n \ast T)(\tau_0)C + T(\tau_0)(1 \ast H)(t-\tau_0) \\
+ \sum_{\alpha-k-1} j_{\alpha-k-1}(\tau_0) (j_{k+1} \ast H)(t-\tau_0)C \\
+ \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0) (j_k \ast T)(\tau_0)C \\
\text{(6.6)}
\]
for \(\tau_0 \leq t < 2\tau_0\). Then, using (6.4)-(6.5), we have for every \(\tau_0 \leq t < 2\tau_0\),
\[
A(1 \ast S_{\tau_0})(t) = A(j_n \ast T)(\tau_0)C + T(\tau_0)A(1 \ast H)(t-\tau_0) \\
+ \sum_{\alpha-k-1} j_{\alpha-k-1}(\tau_0) A(j_{k+1} \ast H)(t-\tau_0)C + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0) A(j_k \ast T)(\tau_0)C \\
= (j_{n-1} \ast T)(\tau_0)C - j_{\alpha+n}(\tau_0)C^2 + T(\tau_0)[H(t-\tau_0) - j_n(t-\tau_0)C] \\
+ \sum_{\alpha-k-1} j_{\alpha-k-1}(\tau_0) [(j_k \ast H)(t-\tau_0)C - j_{\alpha+k+1}(t-\tau_0)C^2] \\
+ \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0) [(j_{k-1} \ast T)(\tau_0)C - j_{\alpha+k}(\tau_0)C^2] \\
= T(\tau_0)H(t-\tau_0) + \sum_{\alpha-k-1} j_{\alpha-k-1}(\tau_0) (j_k \ast H)(t-\tau_0)C \\
+ \sum_{k=0}^{n-1} j_{n-k-1}(t-\tau_0) (j_k \ast T)(\tau_0)C \\
- \left[ j_{\alpha+n}(\tau_0) + \sum_{\alpha-k-1} j_{\alpha-k-1}(\tau_0) j_{\alpha+k+1}(t-\tau_0) + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0) j_{\alpha+k}(\tau_0) \right] C^2 \\
= S_{\tau_0}(t) - j_{\alpha+n}(t)C^2. \\
\text{(6.7)}
\]
Since \(S_{\tau_0}(\cdot)\) is a local \(\alpha + n\)-times integrated \(C^2\)-semigroup on \([0,\tau_0]\) generated by \(C^2AC^{-2} = A\), (6.7) implies that \(S_{\tau_0}(\cdot)\) is a local \((\alpha + n)\)-times integrated \(C^2\)-semigroup on \([0,2\tau_0]\) with generator \(A\), by Theorem 4.15. \(\square\)

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