We prove the generalized stability of $C^*$-ternary quadratic mappings in $C^*$-ternary rings for the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$.

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1. Introduction and preliminaries

A $C^*$-ternary ring is a complex Banach space $A$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $A^3$ into $A$, which is $C$-linear in the outer variables, conjugate $C$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [1]).

If a $C^*$-ternary ring $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital $C^*$-algebra. Conversely, if $(A, \circ)$ is a unital $C^*$-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes $A$ into a $C^*$-ternary ring (see [2]).

Ulam [3] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms. Hyers [4] proved the stability problem of additive mappings in Banach spaces. Rassias [5] provided a generalization of Hyers’ theorem which allows the Cauchy difference to be unbounded: let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Inequality (1.1) provided a lot of influence in the development of a generalization of the Hyers-Ulam stability
A square norm on an inner product space satisfies the important parallelogram equality
\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \]  
(1.2)

The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]  
(1.3)
is called the \textit{quadratic functional equation} whose solution is said to be a \textit{quadratic mapping}. A generalized stability problem for the quadratic functional equation was proved by Skof [9] for mappings \( f : E_1 \to E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group. Czerwik [11] proved the generalized stability of the quadratic functional equation, and Park [12] proved the generalized stability of the quadratic functional equation in Banach modules over a \( C^* \)-algebra. Jun and Lee [13] proved the further generalized stability of a Pexiderized quadratic functional equation
\[ f(x + y) + g(x - y) = 2h(x) + 2k(y). \]  
(1.4)

Recently, a fixed point approach to the stability of Pexiderized quadratic equation was established by Mirzavaziri and Moslehian [14].

Throughout this paper, assume that \( A \) is a \( C^* \)-ternary ring with norm \( \|\cdot\|_A \) and that \( B \) is a \( C^* \)-ternary ring with norm \( \|\cdot\|_B \).

A quadratic mapping \( Q : A \to B \) is called a \( C^* \)-ternary quadratic mapping if
\[ Q([x, y, z]) = [Q(x), Q(y), Q(z)] \]  
(1.5)
for all \( x, y, z \in A \).

Example 1.1. Let \((A, [\cdot, \cdot, \cdot])\) be a \( C^* \)-ternary ring derived from a unital commutative \( C^* \)-algebra \( A \), and let \( Q : A \to A \) satisfy \( Q(x) = x^2 \) for all \( x \in A \). It is easy to show that the mapping \( Q : A \to A \) is a \( C^* \)-ternary quadratic mapping.

In this paper, we prove the further generalized stability of \( C^* \)-ternary quadratic mappings in \( C^* \)-ternary rings.

2. Stability of \( C^* \)-ternary quadratic mappings

We prove the further generalized stability of \( C^* \)-ternary quadratic mappings in \( C^* \)-ternary rings for the quadratic functional equation
\[ Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y). \]  
(2.1)
Theorem 2.1. Let \( f : A \to B \) be a mapping for which there exists a function \( \varphi : A^3 \to [0, \infty) \) such that

\[
\sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{2.2}
\]

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|_B \leq \varphi(x, y, 0), \tag{2.3}
\]

\[
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \varphi(x, y, z) \tag{2.4}
\]

for all \( x, y, z \in A \). Then there exists a unique \( \ast \)-ternary quadratic mapping \( Q : A \to B \) such that

\[
\|f(x) - Q(x)\|_B \leq \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, 0\right) \tag{2.5}
\]

for all \( x \in A \). Here,

\[
\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \tag{2.6}
\]

for all \( x, y, z \in A \).

Proof. If follows from (2.3) that \( f(0) = 0 \). Letting \( y = x \) in (2.3), we get

\[
\|f(2x) - 4f(x)\|_B \leq \varphi(x, x, 0) \tag{2.7}
\]

for all \( x \in A \). So

\[
\|f(x) - 4f\left(\frac{x}{2}\right)\|_B \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \tag{2.8}
\]

for all \( x \in A \). Hence,

\[
\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|_B \leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \tag{2.9}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in A \). It follows from (2.9) that the sequence \( \{4^n f(x/2^n)\} \) is a Cauchy sequence for all \( x \in A \). Since \( B \) is complete, the sequence \( \{4^n f(x/2^n)\} \) converges. So one can define the mapping \( Q : A \to B \) by

\[
Q(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \tag{2.10}
\]

for all \( x \in A \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.9), we get (2.5).
It follows from (2.3) that
\[
\|Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y)\|_B
\]
\[
= \lim_{n \to \infty} 4^n \|f\left(\frac{x + y}{2^n}\right) + f\left(\frac{x - y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\|_B
\]
\[
\leq \lim_{n \to \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0
\]
for all \(x, y \in A\). So
\[
Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)
\]
(2.12)
for all \(x, y \in A\).

It follows from (2.4) and the continuity of the ternary product that
\[
\|Q([x, y, z]) - [Q(x), Q(y), Q(z)]\|_B
\]
\[
= \lim_{n \to \infty} 4^n \|f\left(\frac{x + y + z}{2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\|_B
\]
\[
\leq \lim_{n \to \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\]
for all \(x, y, z \in A\). So
\[
Q([x, y, z]) = [Q(x), Q(y), Q(z)]
\]
(2.14)
for all \(x, y, z \in A\).

Now, let \(T : A \to B\) be another quadratic mapping satisfying (2.5). Then we have
\[
\|Q(x) - T(x)\|_B = 4^n \|Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\|_B
\]
\[
\leq 4^n \left(\|Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\|_B + \|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\|_B\right)
\]
\[
\leq 2 \cdot 4^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, 0\right),
\]
which tends to zero as \(n \to \infty\) for all \(x \in A\). So we can conclude that \(Q(x) = T(x)\) for all \(x \in A\). This proves the uniqueness of \(Q\). Thus, the mapping \(Q : A \to B\) is a unique \(C^*\)-ternary quadratic mapping satisfying (2.5). □

**Theorem 2.2.** Let \(f : A \to B\) be a mapping for which there exists a function \(\varphi : A^3 \to [0, \infty)\) satisfying (2.3) and (2.4) such that
\[
\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]
(2.16)
for all \( x, y, z \in A \). Then there exists a unique \( C^* \)-ternary quadratic mapping \( Q: A \to B \) such that

\[ \| f(x) - Q(x) \|_B \leq \frac{1}{4} \phi(x, x, 0) \]  

(2.17)

for all \( x \in A \).

**Proof.** It follows from (2.7) that

\[ \left\| f(x) - \frac{1}{4} f(2x) \right\|_B \leq \frac{1}{4} \phi(x, x, 0) \]  

(2.18)

for all \( x \in A \). So

\[ \left\| \frac{1}{4^n} f(2^n x) - \frac{1}{4^m} f(2^m x) \right\|_B \leq \sum_{j=1}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\|_B \leq \sum_{j=1}^{m-1} \frac{1}{4^{j+1}} \phi(2^j x, 2^j x, 0) \]  

(2.19)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in A \). It follows from (2.19) that the sequence \( \{ (1/4^n) f(2^n x) \} \) is a Cauchy sequence for all \( x \in A \). Since \( B \) is complete, the sequence \( \{ (1/4^n) f(2^n x) \} \) converges. So one can define the mapping \( Q: A \to B \) by

\[ Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x) \]  

(2.20)

for all \( x \in A \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.19), we get (2.17).

It follows from (2.4) and the continuity of the ternary product that

\[ \| Q([x, y, z]) - [Q(x), Q(y), Q(z)] \|_B \]

\[ = \lim_{n \to \infty} \frac{1}{4^n} \| f(2^n [x, y, z]) - [f(2^n x), f(2^n y), f(2^n z)] \|_B \]

\[ \leq \lim_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y, 2^n z) \]

\[ \leq \lim_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y, 2^n z) = 0 \]  

(2.21)

for all \( x, y, z \in A \). So

\[ Q([x, y, z]) = [Q(x), Q(y), Q(z)] \]  

(2.22)

for all \( x, y, z \in A \).

The rest of the proof is similar to the proof of Theorem 2.1. \[ \square \]

**Remark 2.3.** For a Pexiderized quadratic functional equation

\[ f(x + y) + g(x - y) = 2h(x) + 2k(y), \]  

(2.23)

one can obtain similar results to Theorems 2.1 and 2.2.
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