EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN BOUNDED DOMAINS OF $\mathbb{R}^n$

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Let $D$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). We consider the following nonlinear elliptic problem: $\Delta u = f(\cdot, u)$ in $D$ (in the sense of distributions), $u_{|\partial D} = \varphi$, where $\varphi$ is a nonnegative continuous function on $\partial D$ and $f$ is a nonnegative function satisfying some appropriate conditions related to some Kato class of functions $K(D)$. Our aim is to prove that the above problem has a continuous positive solution bounded below by a fixed harmonic function, which is continuous on $D$. Next, we will be interested in the Dirichlet problem $\Delta u = -\rho(\cdot, u)$ in $D$ (in the sense of distributions), $u_{|\partial D} = 0$, where $\rho$ is a nonnegative function satisfying some assumptions detailed below. Our approach is based on the Schauder fixed-point theorem.

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1. Introduction

Let $D$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ ($n \geq 2$), and let $G$ be the Green function for the Laplace operator with zero Dirichlet boundary condition on $\partial D$. In [4], Chung and Zhao have established interesting inequalities for the Green function $G$. In particular, they showed that there exists a constant $C > 0$ such that for each $x, y$ in $D$,

$$
\frac{1}{C}H(x, y) \leq G(x, y) \leq CH(x, y),
$$

(1.1)

where

$$
H(x, y) := \begin{cases} 
\frac{1}{|x-y|^{n-2}} \min \left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n \geq 3, \\
\log \left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n = 2,
\end{cases}
$$

(1.2)

and $\delta(x)$ denotes the Euclidean distance between $x$ and $\partial D$. 

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Another crucial inequality for the Green function $G$ called $3G$-theorem is given by Kalton and Verbitsky [7] for $n \geq 3$ and by Selmi [12] for $n = 2$, namely, there exists a constant $C_0 > 0$ depending only on $D$ such that for all $x, y, z \in D$,

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0 \left( \frac{\delta(z)}{\delta(x)} G(x,z) + \frac{\delta(z)}{\delta(y)} G(y,z) \right). \quad (1.3)$$

This $3G$-theorem was investigated by Mâagli and Zribi [10], Zeddini [13], and Mâagli and Máatoug [9] to introduce a new class of functions denoted by $K(D)$, (see Definition 1.1 below), which contains properly the classical Kato class introduced by Aizenman and Simon [1]. Moreover, they used the properties of functions belonging to this class $K(D)$ to study some nonlinear differential equations.

**Definition 1.1.** A Borel measurable function $q$ in $D$ belongs to the class $K(D)$ if $q$ satisfies

$$\lim_{\alpha \to 0} \left( \sup_{x \in D} \int_{D \cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)} G(x,y) | q(y) | \, dy \right) = 0. \quad (1.4)$$

In this paper, we will exploit the properties pertaining to $K(D)$ to give some results about the existence of positive solutions of nonlinear elliptic problems. Our plan is as follows.

In Section 2, we establish some estimates on the Green function $G$ and some properties of functions belonging to the Kato class $K(D)$.

In Section 3, we are concerned with the existence of positive continuous solutions of the nonlinear elliptic problem

$$\Delta u = f(\cdot, u) \quad \text{(in the sense of distributions)},$$

$$u > 0 \quad \text{in } D, \quad u|_{\partial D} = \varphi, \quad (1.5)$$

where $\varphi$ is a nontrivial nonnegative continuous function on $\partial D$. Then, we fix a nontrivial nonnegative harmonic function $h_0$ in $D$, which is continuous in $\overline{D}$, and we suppose that $f$ satisfies the following hypotheses.

(H1) $f : D \times (0, +\infty) \to [0, +\infty)$ is measurable, continuous with respect to the second variable and satisfies

$$f(x,t) \leq \theta(x,t), \quad \text{for } (x,t) \in D \times (0, +\infty), \quad (1.6)$$

where $\theta$ is a nonnegative measurable function on $D \times (0, +\infty)$ such that the function $t \to \theta(x,t)$ is nonincreasing on $(0, +\infty)$.

(H2) The function $\psi$ defined on $D$ by $\psi(x) = \theta(x,h_0(x))/h_0(x)$ belongs to the class $K(D)$.

**Remark 1.2.** Note that the condition “$\forall c > 0, \theta(\cdot, c\delta(\cdot))/\delta(\cdot) \in K(D)$” implies the hypothesis (H2). Indeed, from [14], there exists $c > 0$ such that for each $x \in D$, $h_0(x) \geq c\delta(x)$. So, using the fact that $t \to \theta(x,t)/t$ is nonincreasing function on $(0, +\infty)$, we obtain (H2).
Under the assumptions \((H_1)-(H_2)\), we aim at proving the following result: there exists a constant \(c > 1\) such that if \(\varphi \geq c h_0\) on \(\partial D\), then problem (1.5) has a positive continuous solution \(u\) satisfying for each \(x \in D\),

\[
h_0(x) \leq u(x) \leq H_D\varphi(x),
\]

where \(H_D\varphi\) is the harmonic continuous function having boundary value \(\varphi\) on \(\partial D\).

This result improves the one of Atherya [2], who considered the following problem:

\[
\Delta u = g(u) \quad \text{in } \Omega, \quad u|_{\partial \Omega} = \varphi, \tag{P}
\]

where \(\Omega\) is a simply connected bounded \(C^2\)-domain in \(\mathbb{R}^n (n \geq 3)\) and \(g(u) \leq \max(1, u^{-\alpha})\), for \(0 < \alpha < 1\). He proved the existence of a positive continuous solution bounded below by a fixed positive harmonic function \(h_0\) provided that there exists a positive constant \(c > 1\) such that \(\varphi \geq c h_0\) on \(\partial D\).

In the last section, we will study the following nonlinear problem:

\[
\Delta u = -\rho(\cdot, u) \quad \text{in } D \quad \text{(in the sense of distributions)}, \quad u|_{\partial D} = 0, \tag{1.8}
\]

where \(\rho\) is required to verify the following hypotheses.

(H_3) \(\rho\) is nonnegative Borel measurable function on \(D \times (0, \infty)\), continuous with respect to the second variable.

(H_4) There exist \(p, q : D \to (0, \infty)\) nontrivial Borel measurable functions and \(h, k : (0, \infty) \to (0, \infty)\) nontrivial and nondecreasing Borel measurable functions satisfying

\[
p(x)h(t) \leq \rho(x, t) \leq q(x)k(t), \quad \text{for } (x, t) \in D \times (0, \infty),
\]

such that

\[(A_1) \ p \in L^1_{\text{loc}}(D), \quad (A_2) \ q \in K(D), \quad (A_3) \ \lim_{t \to 0^+} (h(t)/t) = +\infty, \quad (A_4) \ \lim_{t \to +\infty} (k(t)/t) = 0.
\]

Under these hypotheses, we will prove that (1.8) has a positive continuous solution \(u\) satisfying on \(D\),

\[
a \delta(x) \leq u(x) \leq b,
\]

where \(a, b\) are positive constants.

Problem (1.8) has been studied by Dalmasso [5] on the unit ball with more restrictive conditions on \(\rho\). Indeed, Dalmasso proved the existence of positive solutions provided that \(\rho\) is nondecreasing with respect to the second variable and satisfies

\[
\lim_{t \to 0^+} \left( \min_{x \in B} \frac{\rho(x, t)}{t} \right) = +\infty, \quad \lim_{t \to +\infty} \left( \max_{x \in B} \frac{\rho(x, t)}{t} \right) = 0. \tag{1.11}
\]

When \(\rho(x, t) = \rho(|x|, t)\), he showed the uniqueness of positive radial solution of (1.8).
4 Nonlinear elliptic problems

On the other hand, problem (1.8) has been studied on the entire space $\mathbb{R}^n$ by Brezis and Kamin [3] for the special nonlinearity $\rho(x,t) = \nu(x) t^\alpha$, $0 < \alpha < 1$. More precisely they proved the existence and the uniqueness of positive solution for the problem below:

$$\Delta u = -\nu(x) u^\alpha \quad \text{in } \mathbb{R}^n \quad \lim_{|x| \to \infty} \inf u = 0.$$  \hfill (1.12)

Notations and preliminaries. In order to simplify our statement, we adopt the following notations.

(i) $C_0(D) := \{ f \in C(D) : \lim_{x \to D} f(x) = 0 \}$.

We note that $C_0(D)$ is a Banach space endowed with the uniform norm

$$\| f \|_\infty = \sup_{x \in D} | f(x) |.$$  \hfill (1.13)

(ii) Let $f$ and $g$ be two nonnegative functions on a set $S$.

We call $f \sim g$, if there exists a constant $c > 0$ such that

$$\frac{1}{c} g(x) \leq f(x) \leq c g(x) \quad \forall x \in S.$$  \hfill (1.14)

We call $f \preceq g$, if there exists a constant $c > 0$ such that

$$f(x) \leq c g(x) \quad \forall x \in S.$$  \hfill (1.15)

(iii) Let $f$ be a nonnegative function in $D$, then we denote by $Vf$ the potential of $f$ defined on $D$ by

$$Vf(x) = \int_D G(x,y) f(y) dy.$$  \hfill (1.16)

We recall that if $f \in L^1_{\text{loc}}(D)$ and $Vf \in L^1_{\text{loc}}(D)$, then we have $\Delta(Vf) = -f$ in $D$ (in the sense of distributions) (see [4, page 52]).

(iv) We denote by $d$ the diameter of $D$.

(v) For $x, y \in D$, we denote $[x, y]^2 = |x - y|^2 + \delta(x) \delta(y)$.

2. Properties of the Green function and the class $K(D)$

In this section, we establish some results concerning the Green function $G(x,y)$ and the Kato class $K(D)$.

Proposition 2.1 (see [9, 10]). Let $q$ be a nonnegative function in $K(D)$. Then

(i) the potential $Vq \in C_0(D)$,

(ii) the function $x \to \delta(x) q(x)$ is in $L^1(D)$. 

In the sequel, we put
\[
\|q\|_D = \sup_{x \in D} \int_D \delta(y) G(x, y) |q(y)| \, dy,
\]
\[
\alpha_q = \sup_{x, y \in D} \int_D \frac{G(x, z) G(z, y)}{G(x, y)} |q(z)| \, dz
\]
(2.1)
(2.2)

We recall that if \(q \in K(D)\), then \(\|q\|_D < \infty\).

Now, it is obvious to see that by (1.3), we have
\[
\alpha_q \leq 2C_0 \|q\|_D, \quad (2.3)
\]
where \(C_0\) is the constant given by (1.3).

Next, we will prove that \(\alpha_q \sim \|q\|_D\).

**Proposition 2.2.** Let \(q\) be a function in \(K(D)\). Then
(i) for any nonnegative superharmonic function \(h\) in \(D\), we have
\[
\int_D G(x, y) |q(y)| h(y) \, dy \leq \alpha_q h(x), \quad \forall x \in D,
\]
(2.4)
(ii) there exists a constant \(C > 0\) such that
\[
C \|q\|_D \leq \alpha_q.
\]
(2.5)

**Proof.** (i) Let \(h\) be a nonnegative superharmonic function in \(D\), then from \([11, \text{Theorem 2.1, page 164}]\), there exists a sequence \((f_k)\) of nonnegative measurable functions on \(D\) such that for all \(y \in D\),
\[
h_k(y) = \int_D G(x, z) f_k(z) \, dz
\]
(2.6)
increases to \(h(y)\).
Since for each \(x, y \in D\), we have
\[
\int_D G(x, y) |q(y)| h_k(y) \, dy \leq \alpha_q h_k(x).
\]
(2.7)
Thus, from the monotone convergence theorem, we deduce the result.

(ii) Let \(\varphi_1\) be a positive eigenfunction corresponding to the first eigenvalue of the Dirichlet problem \(\Delta u + \lambda u = 0, \ u_{|_{\partial D}} = 0\). Then, from \([8, \text{Proposition 2.6}]\) we have for \(x \in D\)
\[
\varphi_1(x) \sim \delta(x).
\]
(2.8)
Since, \(\varphi_1\) is a superharmonic function in \(D\), then by applying (i) to \(\varphi_1\), we deduce (ii). \(\square\)

**Proposition 2.3.** Let \(p > n/2\). Then for each \(\lambda < 2 - n/p\), we have
\[
\frac{1}{(\delta(\cdot))^2} L^p(D) \subset K(D).
\]
(2.9)
6 Nonlinear elliptic problems

To prove Proposition 2.3, we need the two next lemmas.

**Lemma 2.4.** On $D^2$, we have

(i) for $n \geq 3$, $G(x, y) \sim \delta(x)\delta(y)/|x - y|^{n-2}|x, y|^2$,

(ii) for $n = 2$, $G(x, y) \sim (\delta(x)\delta(y)/|x, y|^2) \log(1 + |x, y|^2/|x - y|^2)$.

**Proof.** (i) For each $a, b \geq 0$, we have

$$\min(a, b) \sim \frac{ab}{a + b}. \quad (2.10)$$

So, by (1.1) we deduce (i).

(ii) Using (1.1), the fact that for each $t \geq 0$, $\log(1 + t) \sim \min(1, t) \log(2 + t)$, and (2.10) we obtain that

$$G(x, y) \sim \min\left(1, \frac{\delta(x)\delta(y)}{|x - y|^2}\right) \log\left(1 + \frac{|x, y|^2}{|x - y|^2}\right). \quad (2.11)$$

**Lemma 2.5.** Let $\lambda \in \mathbb{R}$. Then on $D^2$, we have

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \begin{cases} \frac{1}{|x - y|^{n-2+\lambda}}, & \text{if } n \geq 3, \\ \frac{1}{|x - y|^\lambda} \log\left(\frac{2d}{|x - y|}\right), & \text{if } n = 2 \end{cases} \quad (2.12)$$

where $\lambda^+ = \max(0, \lambda)$.

**Proof.** By Lemma 2.4, we have on $D^2$

$$\frac{1}{(\delta(y))^\lambda} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \begin{cases} \frac{1}{|x - y|^{n-2}} \frac{(\delta(y))^{2-\lambda}}{|x, y|^2}, & \text{if } n \geq 3, \\ \frac{(\delta(y))^{2-\lambda}}{|x, y|^2} \log\left(1 + \frac{|x, y|^2}{|x - y|^2}\right), & \text{if } n = 2 \end{cases} \quad (2.13)$$

Now, we remark that

$$|x, y|^2 \sim |x - y|^2 + 4\delta(x)\delta(y). \quad (2.14)$$

So, we have

$$|x, y|^2 \geq \max\left(\delta(x) - \delta(y), |x - y|^2 + 4\delta(x)\delta(y), |x - y|^2\right) \geq \max\left((\delta(y))^2, |x - y|^2\right). \quad (2.15)$$
Therefore by (2.15) we have
\[
\frac{1}{[x, y]^2} \leq \frac{1}{|x - y|^{2-\lambda}}.
\] (2.16)

Hence, it follows that
\[
\frac{(\delta(y))^{2-\lambda}}{|x, y|^2} \leq \frac{1}{|x - y|^{2-\lambda}}.
\] (2.17)

Thus, for \( n \geq 3 \), we obtain
\[
\frac{1}{(\delta(y))^2} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \frac{1}{|x - y|^{n-2+\lambda}}.
\] (2.18)

Next, it is obvious to see that
\[
\log \left( 1 + \frac{[x, y]^2}{|x - y|^2} \right) \leq \log \left( 2 \frac{[x, y]^2}{|x - y|^2} \right) \leq \log \left( \frac{4d^2}{|x - y|^2} \right).
\] (2.19)

Then, for \( n = 2 \), we obtain by (2.17) and (2.19) that
\[
\frac{1}{(\delta(y))^{1}} \frac{\delta(y)}{\delta(x)} G(x, y) \leq \frac{1}{|x - y|^{2+\lambda}} \log \left( \frac{2d}{|x - y|} \right).
\] (2.20)

This completes the proof. \( \square \)

**Proof of Proposition 2.3.** Let \( \alpha > 0 \), \( p > n/2 \) and \( q \geq 1 \) such that \((1/p) + (1/q) = 1\). To show the claim, we use Lemma 2.5 and the Hölder inequality. We distinguish two cases.

**Case 1 (\( n \geq 3 \)).** Let \( f \in L^p(D) \) and \( \lambda < 2 - n/p \). Then, for \( x \in D \), we have
\[
\int_{D \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{|f(y)|}{(\delta(y))^{\lambda}} \, dy \\
\leq \int_{D \cap B(x, \alpha)} \frac{|f(y)|}{|x - y|^{n-2+\lambda}} \, dy \leq \|f\|_p \left( \int_0^\alpha r^{n(1-q)+(2-\lambda)q-1} \, dr \right)^{1/q} \leq \|f\|_p \alpha^{2-n/p-\lambda},
\] (2.21)

which tends to zero as \( \alpha \to 0 \).

**Case 2 (\( n = 2 \)).** Let \( f \in L^p(D) \) and \( \lambda < 2/q \). Then, for \( x \in D \), we have
\[
\int_{D \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y) \frac{|f(y)|}{(\delta(y))^{\lambda}} \, dy \\
\leq \int_{D \cap B(x, \alpha)} \frac{|f(y)|}{|x - y|^{1+\lambda}} \log \left( \frac{2d}{|x - y|} \right) \, dy \leq \|f\|_p \left( \int_0^\alpha r^{n-1-\lambda} q \left( \log \left( \frac{2d}{r} \right) \right)^q \, dr \right)^{1/q},
\] (2.22)

which tends to zero as \( \alpha \to 0 \). This completes the proof. \( \square \)
8 Nonlinear elliptic problems

In the sequel, we put for \( f \in \mathcal{B}(D) \) and \( x \in D \),

\[
v(x) = \int_D G(x,y) \frac{|f(y)|}{(\delta(y))^\lambda} \, dy. \tag{2.23}
\]

**Remark 2.6.** From (1.1), we remark that for \( x, y \in D \), we have \( \delta(x)\delta(y) \geq G(x,y) \). This implies that there exists a constant \( C > 0 \) such that for each \( f \in \mathcal{B}(D) \) and \( x \in D \),

\[
C \delta(x) \int_D (\delta(y))^{1-\lambda} |f(y)| \, dy \leq v(x). \tag{2.24}
\]

In the next proposition, we will give upper estimates on the function \( v \).

**Proposition 2.7.** Let \( p > n/2 \) and \( \lambda < 2 - n/p \). Then there exists a constant \( c > 0 \), such that for each \( f \in L^p(D) \) and \( x \in D \),

\[
v(x) \leq \begin{cases} 
  c \| f \|_p (\delta(x))^{2-n/p-\lambda}, & \text{if } 1 - \frac{n}{p} < \lambda < 2 - \frac{n}{p}, \\
  c \| f \|_p \delta(x) \left( \log \frac{2d}{\delta(x)} \right)^{1/q}, & \text{if } \lambda = 1 - \frac{n}{p}, \\
  c \| f \|_p \delta(x), & \text{if } \lambda < 1 - \frac{n}{p}.
\end{cases} \tag{2.25}
\]

To prove Proposition 2.7, we need the following lemma.

**Lemma 2.8** (see [8]). Let \( x, y \in D \). Then we have the following properties:

(i) if \( \delta(x)\delta(y) \leq |x-y|^2 \) then \( \min(\delta(x),\delta(y)) \leq ((\sqrt{5}+1)/2)|x-y| \),

(ii) if \( |x-y|^2 \leq \delta(x)\delta(y) \) then \( ((3-\sqrt{5})/2)\delta(x) \leq \delta(y) \leq ((3+\sqrt{5})/2)\delta(x) \).

**Proof of Proposition 2.7.** Let \( p > n/2 \), \( q \geq 1 \) such that \( (1/p) + (1/q) = 1 \) and \( \lambda < 2 - n/p \). Let \( f \in L^p(D) \), then for each \( x \in D \), we have

\[
v(x) = \int_{D_1} G(x,y) \frac{|f(y)|}{(\delta(y))^\lambda} \, dy + \int_{D_2} G(x,y) \frac{|f(y)|}{(\delta(y))^\lambda} \, dy = I_1 + I_2, \tag{2.26}
\]

where

\[
D_1 = \{ y \in D : \delta(x)\delta(y) \geq |x-y|^2 \}, \\
D_2 = \{ y \in D : \delta(x)\delta(y) \leq |x-y|^2 \}. \tag{2.27}
\]

Now, we remark that for each \( x \in D \) and \( y \in D_1 \), we have by (1.1) and Lemma 2.8

\[
\frac{1}{(\delta(y))^\lambda} G(x,y) \leq \begin{cases} 
  \frac{1}{(\delta(x))^{\lambda}} |x-y|^{n-2}, & \text{for } n \geq 3, \\
  \frac{1}{(\delta(x))^{\lambda}} \log \left( 1 + \left( \frac{2\delta(x)}{|x-y|} \right)^2 \right), & \text{for } n = 2.
\end{cases} \tag{2.28}
\]
Then, by the Hölder inequality and Lemma 2.8, we obtain for \( n \geq 3 \)

\[
I_1 \leq \| f \|_p (\delta(x))^{-\lambda} \left( \int_{D_1} \frac{1}{|x - y|^{(n-2)q}} dy \right)^{1/q}
\]

\[
\leq \| f \|_p (\delta(x))^{-\lambda} \left( \int_0^{((\sqrt{5}+1)/2)\delta(x)} r^{n-1-(n-2)q} dr \right)^{1/q}
\]

\[
\leq \| f \|_p (\delta(x))^{2-\lambda-n/p}. \tag{2.29}
\]

Now, assume that \( n = 2 \), then since \( q > 1 \) and \( \log(1 + t) \preceq t^{1/2} \), for each \( t \geq 1 \), we obtain

\[
\frac{1}{(\delta(y))^\lambda} G(x, y) \leq \frac{(\delta(x))^{1/\lambda}}{|x - y|^{1/4}}. \tag{2.30}
\]

So, by the Hölder inequality and Lemma 2.8, it follows that

\[
I_1 \leq \| f \|_p (\delta(x))^{1/q-\lambda} \left( \int_{D_1} \frac{1}{|x - y|} dy \right)^{1/q}
\]

\[
\leq \| f \|_p (\delta(x))^{1/q-\lambda} \left( \int_0^{((\sqrt{5}+1)/2)\delta(x)} dr \right)^{1/q}
\]

\[
\leq \| f \|_p (\delta(x))^{2/\lambda - 1/2} = \| f \|_p (\delta(x))^{2-\lambda/2}. \tag{2.31}
\]

Next, by (1.1), we have for each \( x \in D \) and \( y \in D_2 \)

\[
\frac{1}{(\delta(y))^\lambda} G(x, y) \sim \frac{\delta(x)(\delta(y))^{1-\lambda}}{|x - y|^n}. \tag{2.32}
\]

Then, using the Hölder inequality and Lemma 2.8, we obtain

\[
I_2 \leq \| f \|_p \left( \int_{D_2} \left( \frac{\delta(x)(\delta(y))^{1-\lambda}}{|x - y|^n} \right)^q dy \right)^{1/q}. \tag{2.33}
\]

For each \( y \in D_2 \), it follows from Lemma 2.8 that \( \delta(y) \preceq |x - y| \). So, we will discuss two cases.

Case 3. If \( \lambda \leq 1 \), it follows that

\[
I_2 \leq \| f \|_p (\delta(x)) \left( \int_{D_2} \frac{1}{|x - y|^{(n-1+\lambda)q}} dy \right)^{1/q}
\]

\[
\leq \| f \|_p (\delta(x)) \left( \int_{((\sqrt{5} - 1)/2)\delta(x)}^{d} r^{n-1-(n-1+\lambda)q} dr \right)^{1/q}. \tag{2.34}
\]
Thus, we distinguish the following two subcases.

(a) If \( \lambda \leq 1 - n/p \), then from (2.35) it follows that

\[
I_2 \leq \| f \|_p \delta(x) \left( \int_{((\sqrt{5} - 1)/2)\delta(x)}^d r^{(1-n-\lambda p)/(p-1)} dr \right)^{1/q} \\
\leq \| f \|_p \delta(x) \left\{ \begin{array}{ll}
(\log \frac{2d}{\delta(x)})^{1/q} & \text{if } \lambda = 1 - \frac{n}{p}; \\
1 & \text{if } \lambda < 1 - \frac{n}{p}.
\end{array} \right.
\]  

(2.36)

(b) If \( 1 - n/p < \lambda \leq 1 \), then by (2.34) we obtain

\[
I_2 \leq \| f \|_p (\delta(x))^{2-\lambda-n/p} \left( \int_{D_2} \frac{(\delta(x))^{(\lambda+n/p-1)q}}{|x-y|^{(\lambda+n/p-1)q}} \, dy \right)^{1/q} \\
= \| f \|_p (\delta(x))^{2-\lambda-n/p} \left( \int_{ \{(x,y) : |x-y| \leq d, |x-y|^{n/p-1} \} } \frac{1}{|x-y|^{n/p-1}} \, dy \right)^{1/q} \\
\leq \| f \|_p (\delta(x))^{2-\lambda-n/p}.
\]

(2.37)

Case 4. If \( \lambda > 1 \), then from (2.33) it follows that

\[
I_2 \leq \| f \|_p (\delta(x))^{2-\lambda-n/p} \left( \int_{D_2} \frac{(\delta(x))^{(\lambda+n/p-1)q}}{|x-y|^{(\lambda+n/p-1)q}} \, dy \right)^{1/q} \\
\leq \| f \|_p (\delta(x))^{2-\lambda-n/p} \left( \int_{D_2} \frac{(\delta(x))^{(\lambda+n/p-1)q}}{|x-y|^{n/p-1}} \, dy \right)^{1/q}.
\]  

(2.38)

Since \((\lambda - 1)q \in [0, 1]\), it follows from [8, Corollary 2.8] that

\[
I_2 \leq \| f \|_p (\delta(x))^{2-\lambda-n/p}.
\]  

(2.39)

This completes the proof. \qed

Remark 2.9. By taking \( p = +\infty \) (i.e., \( q = 1 \)), in Propositions 2.3 and 2.7, we find again the results of Mâagli in [8].

3. First existence result

In this section, we are interested in the existence of positive solutions for problem (1.5). We recall that \( h_0 \) is a fixed nontrivial nonnegative harmonic function in \( D \), which is continuous in \( \overline{D} \). Let \( \varphi \) be a nontrivial nonnegative continuous function on \( \partial D \).

We denote by \( H_D \varphi \) the solution of the Dirichlet problem

\[
\Delta w = 0 \quad \text{in } D, \quad w|_{\partial D} = \varphi.
\]  

(3.1)

The main result of this section is the following.
Theorem 3.1. Assume \((H_1)-(H_2)\). Then there exists a constant \(c > 1\) such that if \(\varphi \geq c h_0\) on \(\partial D\), then problem (1.5) has a positive continuous solution satisfying for each \(x \in D\)

\[
h_0(x) \leq u(x) \leq H_D \varphi(x).
\]

To prove Theorem 3.1, we need the following lemma.

For a fixed \(q \in K^+(D)\), put

\[
\Gamma_q = \{ v \in K(D) : |v| \leq q \},
\]

then, we have

Lemma 3.2. Let \(q\) be a nonnegative function belonging to \(K(D)\), the family of functions

\[
\mathcal{F}_q = \left\{ \int_D G(\cdot, y)v(y)dy : v \in \Gamma_q \right\}
\]

is uniformly bounded and equicontinuous in \(D\), and consequently, it is relatively compact in \(C_0(D)\).

Proof. Let \(q \in K(D)\) and \(T\) be the operator defined on \(\mathcal{F}_q\) by

\[
Tv(x) = \int_D G(x, y)v(y)dy.
\]

By Proposition 2.1(i), we obtain

\[
\sup_{x \in D} |Tv(x)| \leq \sup_{x \in D} \int_D G(x, y)q(y)dy < \infty.
\]

Then the family \(T(\mathcal{F}_q)\) is uniformly bounded.

Next, we propose to prove the equicontinuity of \(T(\mathcal{F}_q)\) in \(\overline{D}\).

Let \(v \in \mathcal{F}_q, x_0 \in D\), and \(\alpha > 0\). Let \(x, x' \in B(x_0, \alpha) \cap D\).

Then

\[
|Tv(x) - Tv(x')| \leq |Vq(x) - Vq(x')|.
\]

Since, by Proposition 2.1(i), \(Vq \in C_0(D)\), it follows that

\[
|Tv(x) - Tv(x')| \longrightarrow 0 \quad \text{as} \quad |x - x'| \longrightarrow 0.
\]

Similarly, we have \(\lim_{x \to \partial D} Tv(x) = 0\). Which implies that the family \(T(\mathcal{F}_q)\) is equicontinuous in \(\overline{D}\).

Finally, by Ascoli’s theorem, the family \(T(\mathcal{F}_q)\) is relatively compact in \(C_0(D)\). Which completes the proof. \(\square\)

Proof of Theorem 3.1. We will use a fixed-point argument.

Let \(c = 1 + \alpha_\psi\), where \(\alpha_\psi\) is the constant defined by (2.2) associated to the function \(\psi\) given in \((H_2)\). Let \(\varphi \in C^+(\partial D)\) such that \(\varphi \geq c h_0\) on \(\partial D\).
We consider the set $\Lambda$ given by

$$\Lambda = \{ u \in C(\bar{D}) : h_0 \leq u \leq H_D \varphi \}. \quad (3.9)$$

Since $\varphi \geq ch_0$ on $\partial D$, we obtain

$$H_D \varphi \geq ch_0 \quad \text{on } D. \quad (3.10)$$

So $\Lambda$ is a nonempty closed bounded and convex set in $C(\bar{D})$.

For each $u \in \Lambda$, define

$$Tu(x) = H_D \varphi(x) - \int_D G(x,y) f(y,u(y)) \, dy, \quad \forall x \in D. \quad (3.11)$$

Now, we will prove that the family $T\Lambda$ is relatively compact in $C(\bar{D})$.

For each $y \in D$ and $u \in \Lambda$, we have by (H2)

$$0 \leq f(y,u(y)) \leq \frac{\theta(y,h_0(y))}{h_0(y)} h_0(y) \leq c \psi(y). \quad (3.12)$$

with $c = \sup_{y \in D} h_0(y)$. Then, the function $y \to f(y,u(y)) \in \Gamma_c \psi$.

Hence the family

$$\left\{ \int_D G(\cdot, y) f(y,u(y)) \, dy : u \in \Lambda \right\} \subseteq \tilde{\Gamma}_{c \psi}. \quad (3.13)$$

So, using Lemma 3.2 and the fact that $H_D \varphi$ is continuous in $\bar{D}$, we conclude that $T\Lambda$ is a relatively compact set in $C(\bar{D})$.

Next, we intend to show that $T$ maps $\Lambda$ to itself.

It’s obvious to see that

$$Tu(x) \leq H_D \varphi(x), \quad \forall x \in D. \quad (3.14)$$

Moreover, from (H1), and by using (3.11), (2.4), and (3.10), we obtain that for each $x \in D$

$$Tu(x) \geq H_D \varphi(x) - \alpha \psi h_0(x) \geq h_0(x), \quad (3.15)$$

which proves that $T\Lambda \subset \Lambda$.

Now, let us prove the continuity of the operator $T$ in $\Lambda$ in the supremum norm. Let $(u_k)_k$ be a sequence in $\Lambda$ which converges uniformly to a function $u$ in $\Lambda$. Then, for each $x \in D$, we have

$$| Tu_k(x) - Tu(x) | \leq \int_D G(x,y) | f(y,u_k(y)) - f(y,u(y)) | \, dy. \quad (3.16)$$

On the other hand, by hypothesis (H1), we have

$$| f(y,u_k(y)) - f(y,u(y)) | \leq 2h_0(y) \psi(y) \leq \psi(y). \quad (3.17)$$
Since $V\psi \in C_0(D)$, we conclude by the continuity of $f$ with respect to the second variable and the dominated convergence theorem that
\[ \forall x \in \overline{D}, \quad T_{u_k}(x) \rightarrow T_u(x) \quad \text{as} \ k \rightarrow +\infty. \tag{3.18} \]
Since $T\Lambda$ is a relatively compact family in $C(\overline{D})$, therefore the pointwise convergence implies the uniform convergence, namely,
\[ \| T_{u_k} - T_u \|_{\infty} \rightarrow 0 \quad \text{as} \ k \rightarrow +\infty. \tag{3.19} \]
Thus, $T$ is a compact mapping on $\Lambda$.

Finally the Schauder fixed-point theorem implies the existence of $u \in \Lambda$ such that $T_u = u$, that is, for each $x \in D$
\[ u(x) = H_D\varphi(x) - \int_D G(x, y) f(y, u(y)) \, dy. \tag{3.20} \]
Now, let us verify that $u$ is a solution of problem (1.5).
Since $\psi \in K(D)$, it follows from Proposition 2.1(ii), that $\psi \in L^1_{\text{loc}}(D)$.
Furthermore, we have $f(\cdot, u) \leq c\psi$, then $f(\cdot, u) \in L^1_{\text{loc}}(D)$ and $V(f(\cdot, u)) \in \mathcal{F}_{c\psi}$. So by Lemma 3.2, we have
\[ V(f(\cdot, u)) \in C_0(D) \subset L^1_{\text{loc}}(D). \tag{3.21} \]
Thus, applying $\Delta$ to both sides of (3.20) and using the fact that $\Delta(Vf) = -f$, we obtain, that $u$ satisfies the elliptic differential equation
\[ \Delta u = f(\cdot, u) \quad \text{in} \ D \ (\text{in the sense of distributions}). \tag{3.22} \]
Moreover, since $H_D\varphi = \varphi$ in $\partial D$ and $V(f(\cdot, u)) \in C_0(D)$, we conclude that $u|_{\partial D} = \varphi$. So $u$ is a positive continuous solution of problem (1.5). \hfill \square

Now, let us state another comparison result for the solution $u$ of problem (1.5), in the case of the special nonlinearity $f(x, t) = q(x)\Phi(t)$.
For this aim, suppose that the following hypotheses on $q$ and $\Phi$ are adopted.
(i) $\Phi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable nonincreasing function.
(ii) $q$ is a nontrivial nonnegative function on $D$ such that
\[ q \in C^\alpha_{\text{loc}}(D), \quad 0 < \alpha < 1, \quad \forall c > 0, \quad x \rightarrow \frac{q(x)}{\delta(x)} \Phi(c\delta(x)) \in K(D). \tag{3.23} \]
Moreover, let $F$ be the function defined on $[0, \infty)$ by
\[ F(t) = \int_0^t \frac{1}{\Phi(s)} \, ds. \tag{3.24} \]
It is obviously seen, from hypotheses adopted on $\Phi$, that the function $F$ is a bijection from $[0, \infty)$ to itself. Then, we have the following.
14 Nonlinear elliptic problems

**Theorem 3.3.** Let $u$ be the solution given by (3.20) of the following problem:

$$\Delta u + q\Phi(u) = 0, \quad \text{in } D, \quad u_{|\partial D} = \varphi. \quad (3.25)$$

Then, we have $u \in C^{2+a}(D) \cap C(\overline{D})$. Further, $u$ satisfies on $D$

$$u(x) \leq \min \left( H_D\varphi(x), F^{-1}(H_D(F \circ \varphi)(x) - Vq(x)) \right). \quad (3.26)$$

**Proof.** Let $v$ be the function defined on $D$ by

$$v = F(u) - H_D(F \circ \varphi) + Vq. \quad (3.27)$$

Then $v \in C^2(D)$ and we have

$$\Delta v = \frac{1}{\Phi(u)} \Delta u - \frac{\Phi'(u)}{(\Phi(u))^2} |\nabla u|^2 - q = - \frac{\Phi'(u)}{(\Phi(u))^2} |\nabla u|^2. \quad (3.28)$$

Thus, $\Delta v \geq 0$. In addition, since $Vq \in C_0(D)$, it follows that $v \in C_0(D)$. Then, the maximum principle (see [6, pages 465-466]) implies that $v \leq 0$, in $D$. This completes the proof. \hspace{1cm} \Box

**Remark 3.4.** (1) Let $\lambda > 0$ and $\varphi(x) = \lambda, \ \forall x \in \partial D$. Then, we have for each $x \in D$,

$$H_D(F \circ \varphi)(x) - Vq(x) = F(\lambda)(x) - Vq(x) \leq F(\lambda). \quad (3.29)$$

Thus for each $x \in D$,

$$F^{-1}(H_D(F \circ \varphi)(x) - Vq(x)) \leq \lambda = H_D\varphi(x). \quad (3.30)$$

Therefore, from (3.26) we have for each $x \in D$,

$$h_0(x) \leq u(x) \leq F^{-1}(H_D(F \circ \varphi)(x) - Vq(x)). \quad (3.31)$$

(2) By hypothesis (i), we have

$$\Phi(\|u\|_{\infty}) \geq \Phi(\|\varphi\|_{\infty}). \quad (3.32)$$

Therefore,

$$h_0(x) \leq u(x) \leq H_D\varphi(x) - \Phi(\|\varphi\|_{\infty}) Vq(x). \quad (3.33)$$

Then we have

$$h_0 \leq u \leq \min \left( H_D\varphi - \Phi(\|\varphi\|_{\infty}), F^{-1}(H_D(F \circ \varphi) - Vq) \right). \quad (3.34)$$

**Example 3.5.** Let $h_0$ be a nontrivial nonnegative harmonic function, which is continuous on $\overline{D}$. Then, from [14], there exists $c_1$ such that for each $x \in D$

$$h_0(x) \geq c_1 \delta(x). \quad (3.35)$$
Let $\alpha > 0$, and $f$ be a nonnegative measurable function on $D \times (0, \infty)$, continuous with respect to the second variable satisfying
\[ f(x,t) \leq t^{-\alpha}(\delta(x))^{\alpha+1}q(x), \quad (3.36) \]
where the function $q$ belongs to $K^+(D)$.

Then, there exists $c > 0$ such that if $\varphi \geq (1 + c)h_0$ on $\partial D$, the problem
\[ \Delta u = f(\cdot,u) \quad \text{(in the sense of distributions)} \]
\[ u > 0 \quad \text{in } D, \quad u|_{\partial D} = \varphi, \quad (3.37) \]
has a positive continuous solution in $\overline{D}$ satisfying
\[ h_0(x) \leq u(x) \leq H_D\varphi(x). \quad (3.38) \]

4. Second existence result

In this section, we prove the following result for problem (1.8).

**Theorem 4.1.** Assume (H$_3$)-(H$_4$). Then problem (1.8) has a positive solution $u \in C_0(D)$. Moreover there exist positive constants $a$ and $b$, such that
\[ a\delta(x) \leq u(x) \leq b. \quad (4.1) \]

**Proof.** By (A$_2$) and (H$_4$), the function $q \in K^+(D)$. Then, from Proposition 2.1(i), we have $Vq \in C_0(D)$. So, $M := \sup_{x \in D}(Vq(x)) < \infty$.

From (A$_4$), there exists $b > 0$ such that $Mk(b) \leq b$.

On the other hand, by (A$_1$), there exists a compact $K \subset D$ such that
\[ 0 < \int_K \delta(y)p(y)dy < \infty. \quad (4.2) \]

Furthermore, by (1.1), there exists $\alpha > 0$ such that for each $x, y$ in $D$
\[ G(x,y) \geq a\delta(x)\delta(y). \quad (4.3) \]

Next, let $r$ be the constant given by
\[ r := \inf_{y \in K} \delta(y). \quad (4.4) \]

Then, from (H$_4$), there exists $a > 0$ such that
\[ ah(ar) \int_K \delta(y)p(y)dy \geq a. \quad (4.5) \]

Now, let $\Omega$ be the convex set
\[ \Omega := \{ u \in C_0(D) : a\delta(x) \leq u(x) \leq b \} \quad (4.6) \]
and $S$ be the operator defined on $\Omega$ by

$$Su(x) = \int_D G(x, y) \rho(y, u(y)) dy.$$  \hfill (4.7)

We will prove that $S$ is a compact mapping on $\Omega$.

By (H$_4$), we have for each $u \in \Omega$

$$\rho(\cdot, u) \leq k(b)q = \tilde{q}. \hfill (4.8)$$

Since $q \in K^+(D)$, it follows that the function $y \to \rho(y, u(y)) \in \Gamma_{\tilde{q}}$.

Hence, the family

$$\left\{ \int_\Omega G(\cdot, y) \rho(y, u(y)) dy : u \in \Omega \right\} \subseteq \mathcal{F}_{\tilde{q}}. \hfill (4.9)$$

Consequently, by Lemma 3.2, the family $S(\Omega)$ is relatively compact in $C_0(D)$. Next, we need to verify that for $u \in \Omega$ and $x \in D$, we have

$$a\delta(x) \leq Su(x) \leq b. \hfill (4.10)$$

Let $u \in \Omega$ and $x \in D$, then by (H$_4$), we have

$$Su(x) \leq \int_D G(x, y)q(y)k(u(y))dy$$

$$\leq k(b)\int_D G(x, y)q(y)dy$$

$$\leq Mk(b) \leq b. \hfill (4.11)$$

On the other hand, from (H$_4$) and using (1.1) and (4.5), we have

$$Su(x) \geq a\delta(x)\int_D \delta(y)p(y)h(u(y))dy$$

$$\geq a\delta(x)\int_K \delta(y)p(y)h(a\delta(y))dy$$

$$\geq \delta(x)\left[ ah(ar)\int_K \delta(y)p(y)dy \right] \geq a\delta(x). \hfill (4.12)$$

Thus, it follows that $S(\Omega) \subset \Omega$.

Now, we consider a sequence $(u_k)_k$ in $\Omega$ which converges uniformly to $u$ in $\Omega$. Since $\rho$ is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for all $x \in D$,

$$Su_k(x) \longrightarrow Su(x) \quad \text{as} \quad k \longrightarrow +\infty. \hfill (4.13)$$

Therefore, using the fact that $S(\Omega)$ is relatively compact in $C_0(D)$, we conclude that $\|Su_k - Su\|_{\infty}$ as $k \to +\infty$. Hence $S$ is a compact mapping from $\Omega$ to itself. Then by the
Schauder fixed-point theorem, there exists a function \( u \in \Omega \) such that
\[
  u(x) = \int_D G(x,y) \rho(y,u(y)) \, dy. \tag{4.14}
\]

Now, since \( q \in K^+(D) \) then by Proposition 2.1(ii), we have \( \rho(\cdot,u) \in L^1_{\text{loc}}(D) \) and \( V(\rho(\cdot,u)) \in C_0(D) \subset L^1_{\text{loc}}(D) \).

So, \( u \) satisfies (in the sense of distributions) \( \Delta u = -\rho(\cdot,u) \) in \( D \). Moreover, \( \lim_{x \to \partial D} u(x) = \lim_{x \to \partial D} V(\rho(\cdot,u)(x)) = 0 \). So \( u \) is a solution of problem (1.8). \( \square \)

**Example 4.2.** Let \( p > n/2 \) and \( f \in L^p(D) \). Assume that the function \( g : (0, \infty) \to [0, \infty) \) is a nontrivial continuous and nondecreasing function satisfying
\[
  \lim_{t \to 0^+} \frac{g(t)}{t} = +\infty, \quad \lim_{t \to +\infty} \frac{g(t)}{t} = 0. \tag{4.15}
\]

Then for each \( \lambda < 2 - n/p \) the problem
\[
  \Delta u = -\left( \delta(x) \right)^{-\lambda} f(x) g(u) \quad \text{in } D, \quad u|_{\partial D} = 0, \tag{4.16}
\]

has a positive solution \( u \in C_0(D) \). Moreover, from Proposition 2.7, we have for each \( x \in D \),
\[
  u(x) \leq \begin{cases} 
    c\|f\|_p \left( \delta(x) \right)^{2-n/p-\lambda}, & \text{if } 1 - \frac{n}{p} < \lambda < 2 - \frac{n}{p}, \\
    c\|f\|_p \left( \frac{2d}{\delta(x)} \right)^{(p-1)/p}, & \text{if } \lambda = 1 - \frac{n}{p}, \\
    c\|f\|_p \delta(x), & \text{if } \lambda < 1 - \frac{n}{p}.
  \end{cases} \tag{4.17}
\]

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**References**


18 Nonlinear elliptic problems


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