We study the existence of solutions an $H$-system for a revolution surface without boundary for $H$ depending on the radius $f$. Under suitable conditions we prove that the existence of a solution is equivalent to the solvability of a scalar equation $N(a) = L/\sqrt{2}$, where $N: \mathcal{A} \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function depending on $H$. Moreover, using the method of upper and lower solutions we prove existence results for some particular examples. In particular, applying a diagonal argument we prove the existence of unbounded surfaces with prescribed $H$.

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1. Introduction

The prescribed mean curvature equation for a vector function $X: \Omega \rightarrow \mathbb{R}^3$ is given by the following nonlinear system of partial differential equations:

$$\Delta X = 2H(X)X_u \wedge X_v \quad (u, v) \in \Omega. \quad (1.1)$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\wedge$ denotes the exterior product in $\mathbb{R}^3$ and $H: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a given function. It is well known that if $X$ is isothermal, namely

$$|X_u| - |X_v| = X_uX_v = 0 \quad (1.2)$$

then $H$ is the mean curvature of the surface parameterized by $X$ (see, e.g., [8]). Equation (1.1) is also known in the literature as an $H$-system.

The parametric Plateau and Dirichlet problems for (1.1) have been extensively studied by different authors (see [3–5, 8–10]). Nonparametric and more general quasilinear equations are considered in [1, 2, 6, 7].
An $H$-system for a revolution surface without boundary

We will consider the particular case of a revolution surface

$$X(u,v) = (f(u) \cos v, f(u) \sin v, g(u))$$  \hspace{1cm} (1.3)

with $f, g \in C^2(I) \cap C(\bar{I})$ such that $f > 0$ over the open interval $I \subset \mathbb{R}$. Then (1.1) reads

$$\begin{align*}
f'' - f &= -2H(f,g)fg' \quad \text{in } I \\
g'' &= 2H(f,g)ff' \quad \text{in } I,
\end{align*}$$

(1.4)

where $H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is given.

It is easy to see that any solution of (1.4) verifies the equality

$$(f')^2 + (g')^2 = f^2 + c.$$  \hspace{1cm} (1.5)

Hence, the isothermal condition (1.2) holds if and only if $c = 0$.

We will study (1.4) for a compact surface without boundary. Without loss of generality we may assume that $I = (0,L)$, and hence the problem reads

$$\begin{align*}
f'' - f &= -2H(f,g)fg' \quad \text{in } I \\
g'' &= 2H(f,g)ff' \quad \text{in } I \\
f(0) &= f(L) = 0, \quad f > 0 \text{ in } I \\
g'(0) &= g'(L) = 0.
\end{align*}$$

(1.6)

In particular, when $H$ depends only on the radius $f$, from the equality

$$g'' = 2H(f)f f', \quad g'(0) = 0,$$

(1.7)

we easily reduce problem (1.6) to a single equation: indeed, if $\tilde{H}(t) = \int_0^t sH(s)ds$, it holds that $g'(t) = 2\tilde{H}(f(t))$, and $g'(L) = 2\tilde{H}(f(L))$. Thus, solving (1.6) is equivalent to obtain a positive solution of the problem

$$\begin{align*}
f'' - f &= -4H(f)\tilde{H}(f) \quad \text{in } (0, +\infty) \\
f(0) &= f(L) = 0
\end{align*}$$

(1.8)

with $H : \mathbb{R}^+ \to \mathbb{R}$. We remark that if $\tilde{H} > 0$ then $g' > 0$, and if $f$ is a positive solution of (1.8) the parametrization $X$ given in (1.3) defines a regular revolution surface. For example, this holds when $H$ is positive.

We will also consider the case $L = +\infty$, namely the problem

$$\begin{align*}
f'' - f &= -4H(f)\tilde{H}(f) \quad \text{in } I \\
f(0) &= 0, \quad f(+\infty) = r,
\end{align*}$$

(1.9)
where \( r > 0 \) is a constant. Note that if \( f \) is a positive solution of (1.9) then \( g'(+\infty) = 2\tilde{H}(r) \). Thus, if \( \tilde{H}(r) > 0 \) it follows that the surface parameterized by \( X \) is unbounded in the direction \( z \to +\infty \) of the upper halfspace \( \mathbb{R}^2 \times \mathbb{R}^+ \).

The paper is organized as follows. In Section 2 we prove that under suitable conditions the existence of a positive solution of (1.8) is equivalent to the solvability of the scalar equation \( N(a) = L/\sqrt{2} \), where \( N \) is defined by

\[
N(a) = \int_0^a \frac{dz}{\sqrt{\phi(a) - \phi(z)}} \quad (1.10)
\]

with

\[
\phi(u) := 2\tilde{H}^2(u) - \frac{u^2}{2} \quad (1.11)
\]

Moreover, we prove existence and uniqueness of solutions for some particular examples.

In Section 3 we apply the method of upper and lower solutions and a diagonal argument in order to prove the existence of solutions of problem (1.9).

2. A scalar equation for (1.8)

In this section we study the existence of positive solutions of (1.8). Let us first note that if \( \phi \) is defined as in (1.11), the problem may be written as

\[
\begin{align*}
f'' + \phi'(f) &= 0 \quad \text{in } I \\
f(0) &= f(L) = 0 \quad (2.1)
\end{align*}
\]

Then we have the following theorem.

**Theorem 2.1.** Let \( a \in \mathcal{A} \), where

\[
\mathcal{A} = \{ a \in \mathbb{R}^+ : \phi(a) > \phi(u) \text{ for } 0 < u < a \}, \quad (2.2)
\]

and let \( N \) be defined by (1.10).

Then (2.1) admits at most one positive solution \( f \) with \( a = \| f \|_{C([0,1])} \). Furthermore, (2.1) admits a positive solution \( f \) with \( a = \| f \|_{C([0,1])} \) if and only if \( N(a) = L/\sqrt{2} \).

**Proof.** Let \( f \) be a positive solution of (2.1) with \( a = \| f \|_{C([0,1])} \), and fix \( x_0 \in (0,L) \) such that \( f(x_0) = a \). Multiplying the equation by \( f' \) it follows by integration that

\[
E := (f')^2 + 2\phi(f) = 2\phi(a). \quad (2.3)
\]

Note that if \( f'(x) = 0 \) for some \( x \in (0,L) \) then \( \phi(f(x)) = \phi(a) \), and hence \( f(x) = a \). We conclude that \( x_0 \) is the only critical point of \( f \). Thus,

\[
\begin{align*}
f' &= \sqrt{2(\phi(a) - \phi(f))} \quad 0 < x < x_0, \\
f' &= -\sqrt{2(\phi(a) - \phi(f))} \quad x_0 < x < L.
\end{align*}
\]
This implies that

\[
x_0 - x = \int_x^{x_0} \frac{f'}{\sqrt{2(\phi(a) - \phi(f))}} \, dz = \int_{f(x)}^{a} \frac{dz}{\sqrt{2(\phi(a) - \phi(z))}} \quad \text{for } 0 < x \leq x_0,
\]

\[
x - x_0 = -\int_{x_0}^{x} \frac{f'}{\sqrt{2(\phi(a) - \phi(f))}} \, dz = \int_{f(x)}^{a} \frac{dz}{\sqrt{2(\phi(a) - \phi(z))}} \quad \text{for } x_0 \leq x < L.
\]

In particular, \(x_0 = L - x_0\), and then \(x_0 = L/2\). Furthermore, for \(x = 0\) we obtain

\[
\frac{L}{2} = \int_0^{a} \frac{dz}{\sqrt{2(\phi(a) - \phi(z))}} = \frac{N(a)}{\sqrt{2}}.
\]

Conversely, if \(N(a) = L/\sqrt{2}\) for some \(a \in \mathbb{R}^+\), define \(f\) implicitly by

\[
\int_{f(x)}^{a} \frac{dz}{\sqrt{2(\phi(a) - \phi(z))}} = \sqrt{2} \left( x - \frac{L}{2} \right) \quad \text{for } x \geq \frac{L}{2}
\]

and extend it by symmetry for \(x < L/2\). It is immediate to verify that \(f\) is a positive solution of problem (2.1). Moreover, from the above computations it is clear that if \(\tilde{f}\) is a positive solution with \(\|\tilde{f}\|_{C([0,1])} = a\), then \(\tilde{f} = f\).

Remark 2.2. The proof of existence of a solution in the previous theorem holds for any \(a \in \text{Dom}(N) \subset \mathbb{R}^+\) such that \(N(a) = L/\sqrt{2}\).

Remark 2.3. If \(H\) is bounded in a neighborhood of 0, then \(\phi'(0^+) < 0\) and hence \(0 \notin \mathbb{R}^+\).

Example 2.4. As an application, we may consider \(H : \mathbb{R}^+ \to \mathbb{R}^+\) given by \(H(u) = cu^\sigma\) for some \(\sigma > -2\) and \(c \neq 0\). In this case \(\tilde{H}(u) = (c/(\sigma + 2))u^{\sigma+2}\), and \(\phi(u) = (2c^2/(\sigma + 2)^2)u^{2\sigma+4} - u^2/2\). For \(\sigma > -1\) a simple computation shows that \(\mathcal{A} = [\alpha, +\infty)\), with \(\alpha = ((\sigma + 2)/2|c|)^{1/(\sigma+1)}\). Moreover, \(N\) is strictly non-increasing, with

\[
\lim_{a \to -\alpha^+} N(a) = +\infty \quad \lim_{a \to +\infty} N(a) = 0.
\]

On the other hand, if \(-2 < \sigma < -1\), it holds that \(\mathcal{A} = (0, \alpha]\), with \(\alpha = ((\sigma + 2)/4c^2)^{1/(2\sigma+2)}\). Moreover, \(N\) is strictly non-decreasing, with

\[
\lim_{a \to 0} N(a) = 0 \quad \lim_{a \to -\alpha} N(a) = +\infty.
\]

Thus, in both cases it follows that the problem admits a unique solution. The case \(\sigma = -1\)
corresponds to the well known linear problem \(-f'' = (4c^2 - 1)f\). Here
\[
\mathcal{A} = \begin{cases} 
\emptyset & \text{if } 4c^2 \leq 1 \\
\mathbb{R}^+ & \text{if } 4c^2 > 1.
\end{cases} \tag{2.10}
\]
Moreover, if \(4c^2 > 1\) then
\[
N(a) \equiv \frac{\pi}{\sqrt{2(4c^2 - 1)}}, \tag{2.11}
\]
and hence \(N(a) = L/\sqrt{2}\) if and only if \(4c^2 - 1 = (\pi/L)^2\).

3. Upper and lower solutions and unbounded revolution surfaces

In this section we apply the method of upper and lower solutions in order to solve a nonhomogeneous Dirichlet problem associated to (1.4). In particular, applying a diagonal argument we prove the existence of solutions of (1.9).

We recall that \((\alpha, \beta) \in (C^2([0, +\infty]))^2\) is an ordered couple of a lower and an upper solution of the problem if \(\alpha \leq \beta\) and
\[
\alpha'' - \alpha \geq -4H(\alpha)\alpha\tilde{H}(\alpha) \quad \text{in } (0, +\infty) \\
\alpha(0) \leq 0, \quad \alpha(+) \leq r,
\]
\[
\beta'' - \beta \leq -4H(\beta)\beta\tilde{H}(\beta) \quad \text{in } (0, +\infty) \\
\beta(0) \geq 0, \quad \beta(+) \geq r. \tag{3.1}
\]
For simplicity we will assume that \(H\) is continuously differentiable.

Remark 3.1. If \(f\) is a solution of (1.9), then \(f''(+) = r - 4r\tilde{H}(r)H(r)\). As \(f(+) < \infty\), it follows that
\[
4\tilde{H}(r)H(r) = 1. \tag{3.2}
\]
In particular, if (3.2) holds we may take \(\beta \equiv r\) as an upper solution.

Theorem 3.2. Let \((\alpha, \beta)\) be an ordered couple of a lower and an upper solution of (1.9), let \(N > 0\) and let \(c_N\) be any constant with \(\alpha(N) \leq c_N \leq \beta(N)\). Then the Dirichlet problem
\[
f'' - f = -4H(f)f\tilde{H}(f) \quad \text{in } (0, N) \\
f(0) = 0, \quad f(N) = c_N \tag{3.3}
\]
admits at least one solution \(f\) with \(\alpha_{|[0,N]} \leq f \leq \beta_{|[0,N]}\).

Proof. Fix a constant \(\lambda \geq -1\) such that
\[
\lambda \geq -2(\tilde{H}^2)''(u) \tag{3.4}
\]
This choice of \(\lambda\) implies that the function \(\xi(x) := -4H(x)x\tilde{H}(x) - \lambda x\) is non-increasing.

We will construct a sequence \(\{f_n\}\) given recursively by \(f_0 = \alpha\) and \(f_{n+1}\) the unique solution of the linear problem

\[
\begin{align*}
\frac{d^2}{dx^2} f_{n+1} + (1 + \lambda) f_{n+1} &= -4H(f_n) f_n\tilde{H}(f_n) - \lambda f_n, &\text{in } (0,N), \\
\frac{d}{dx} f_{n+1}(0) &= 0, & f_{n+1}(N) = c_N.
\end{align*}
\]

We claim that \(\{f_n\}\) is non-decreasing, with \(\alpha \leq f_n \leq \beta\). Indeed, as

\[
\frac{d^2}{dx^2} f_1 + (1 + \lambda) f_1 = -4H(\alpha)\alpha\tilde{H}(\alpha) - \lambda \alpha \leq \alpha'' - (1 + \lambda)\alpha,
\]

by the comparison principle we deduce that \(f_1 \geq \alpha\). Now assume that \(f_n \geq f_{n-1}\) then

\[
\begin{align*}
\frac{d^2}{dx^2} f_{n+1} + (1 + \lambda) f_{n+1} &= -4H(f_n) f_n\tilde{H}(f_n) - \lambda f_n \\
&\leq -4H(f_{n-1}) f_{n-1}\tilde{H}(f_{n-1}) - \lambda f_{n-1} \\
&= \frac{d^2}{dx^2} f_n'' + (1 + \lambda) f_n
\end{align*}
\]

and we deduce that \(f_{n+1} \geq f_n\).

On the other hand, \(f_0 = \alpha \leq \beta\), and if \(f_n \leq \beta\) we have that

\[
\begin{align*}
\frac{d^2}{dx^2} f_{n+1} + (1 + \lambda) f_{n+1} &= -4H(f_n) f_n\tilde{H}(f_n) - \lambda f_n \geq -4H(\beta)\beta\tilde{H}(\beta) - \lambda \beta \leq \beta'' - (1 + \lambda)\beta.
\end{align*}
\]

As

\[
f_{n+1}(0) \leq \beta(0), \quad f_{n+1}(N) \leq \beta(N),
\]

using again the comparison principle we deduce that \(f_{n+1} \leq \beta\).

It follows that \(\{f_n\}\) converges pointwise to some function \(f\). By the standard a priori bounds and using the fact that \(\alpha \leq f_n \leq \beta\) for each \(n\) we have that

\[
\|f_n\|_{H^2} \leq c_0 + c_1\|4H(f_{n-1}) f_{n-1}\tilde{H}(f_{n-1}) - \lambda f_{n-1}\|_{L^2} \leq C
\]

for some constant \(C\). Thus, if we suppose that \(f_n \rightharpoonup f\) uniformly, taking a subsequence we may assume that \(\|f_n - f\|_{C([0,N])} \geq \epsilon\) for some \(\epsilon > 0\). By the Sobolev imbedding \(H^2(0,N) \hookrightarrow C^1([0,N])\), taking a subsequence we may assume that \(f_n\) converges to some function \(g \neq f\)
for the $C^1$-norm, a contradiction. Hence $f_n \to f$ uniformly, and $f_n'' \to f - 4H(f)\hat{f}$.

It follows that $f$ is a solution of the problem.

\[ \square \]

Remark 3.3. In the previous proof, it is easy to see that the convergence is more accurate for smaller values of $\lambda$. Indeed, if $\lambda \geq \lambda$, with $\lambda$ as before, the corresponding sequence \{\(f_n\)\} given recursively by $f_0 = \alpha$ and $f_{n+1}$ the unique solution of the linear problem

\[ f''_{n+1} - (1 + \lambda) f_{n+1} = -4H(f_n) \tilde{f}_n \hat{f}(f_n) - \lambda f_n \quad \text{in} \ (0,N) \]

\[ f_{n+1}(0) = 0, \quad f_{n+1}(N) = c_N \]  

(3.12)

is non-decreasing and converges to a solution of the problem. We claim that $f_n \leq f_n$ for every $n$: indeed, this is trivial for $n = 0$, and if the claim is true for $n$ we have that

\[ f''_{n+1} - (1 + \lambda) f_{n+1} = -4H(f_n) f_n \hat{H}(f_n) - \lambda f_n \leq -4H(f_n) \tilde{f}_n \hat{f}(f_n) - \lambda f_n \]

\[ = f''_{n+1} - (1 + \lambda) f_{n+1} + (\lambda - \lambda) f_n \]

\[ = f''_{n+1} - (1 + \lambda) f_{n+1} + (\lambda - \lambda) (f_n - f_{n+1}). \]  

(3.13)

Using the inductive hypothesis and the fact that \{\(f_n\)\} is nondecreasing, it follows that $f_{n+1} \geq f_{n+1}$.

To conclude this remark, note that \{\(f_n\)\} and \{\(f_{n+1}\)\} converge to the same solution. Indeed, it suffices to replace $\beta$ by

\[ \tilde{\beta} = \lim_{n \to \infty} f_n \]  

(3.14)

in the proof of Theorem 3.2. As $\tilde{\beta} \leq \beta$, the definition of \{\(f_n\)\} coincides with the previous one, and $f_n \leq \tilde{\beta}$ for every $n$.

Theorem 3.4. Let $(\alpha, \beta)$ be an ordered couple of a lower and an upper solution of (1.9) with $\alpha (+\infty) = \beta (+\infty) = r$. Then (1.9) admits a solution $f$ with $\alpha \leq f \leq \beta$.

Proof. For any $N \in \mathbb{N}$, by the previous theorem we may choose a solution $f_n$ of (3.3) with $c_N = (\alpha(N) + \beta(N))/2$ such that $\alpha|_{[0,N]} \leq f_N \leq \beta|_{[0,N]}$. Moreover, if $\varphi_N(x) = (f_N(M)/M)x$, there exist constants $c_M, \tau_M$ independent of $N$ such that

\[ \|f_N - \varphi_N\|_{H^2([0,M] \to \mathbb{R})} \leq \tau_M \|H(f_n) f_N \hat{H}(f_n)\|_{L^2([0,M] \to \mathbb{R})} \leq c_M \]  

(3.15)

for any $N \geq M$. For $M = 1$ we may take a subsequence, still denoted \{\(f_N\)\}, which converges uniformly in $[0,1]$ to some function $f^1$. Repeating the procedure we may assume that $f_N$ converges uniformly in $[0,M]$ to a function $f^M$. Then $f : [0, +\infty) \to [0, +\infty)$ given by $f(x) = f^N(x)$ if $x \leq N$ solves (1.9). Indeed, it is clear that $f$ is well defined,
and that $f(0) = 0$, $f(+\infty) = r$. Moreover, as $f_N''$ converges uniformly in $[0,M]$ to $f - 4H(f)f\tilde{H}(f)$, for any test function $\xi \in C_0^\infty(0,M)$ we obtain that

$$\int_0^M (f - 4H(f)f\tilde{H}(f))\xi = \lim_{N \to \infty} \int_0^M f_N''\xi = \lim_{N \to \infty} \int_0^M f_N\xi'' = \int_0^M f\xi'',$$  

(3.16)

and the proof follows. \hfill \Box

**Example 3.5.** Assume that (3.2) holds, and that $(\tilde{H}^2)'' \leq 0$ on $[0,r]$. Then $(\alpha, \beta)$ given by

$$\alpha(x) = r(1 - e^{-x}), \quad \beta \equiv r$$  

(3.17)

is an ordered couple of a lower and an upper solution of (1.9). Indeed,

$$\alpha'' - \alpha = -r = -2(\tilde{H}^2)'(r) \geq -2(\tilde{H}^2)'(\alpha)$$  

(3.18)

since $0 \leq \alpha \leq r$. From the previous theorem we deduce that (1.9) admits at least one positive solution between $\alpha$ and $\beta$.

### 3.1. Some numerical experiments.

The method described in the proof of Theorem 3.2 can be implemented as a numerical method to compute the solution in an effective way. At each step of the iterative procedure, we have to solve a linear differential equation, with Dirichlet boundary conditions.

With that purpose, we use the standard finite difference method: We split the interval $[0,N]$ into $k$ small sub-intervals of length $h = k/N$, and we denote by $f_i$ the approximate value of $f_n(x_i)$. Then, we approximate the linear problem (3.6) by the linear system of equations

$$\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{h^2} - (1 + \lambda)f_i^n = -4H(f_i^n)f_i^n\tilde{H}(f_i^n) - \lambda f_i^n \quad (1 \leq i \leq k - 1)$$  

(3.19)

subject to the boundary conditions

$$f_{n+1}^0 = 0, \quad f_{n+1}^k = c.$$  

(3.20)

Let us recall that the energy introduced in (2.3) is constant, for any solution of the problem, therefore we can use the discrete quantity

$$E_i(h) = \left(\frac{f_i^n - f_{i-1}^n}{h}\right)^2 + 2\phi(f_i^n)$$  

(3.21)

as a test for the accuracy of the method. We stop the iteration when this quantity is close enough to a constant, for the desired precision $\varepsilon_0$, that is, when

$$\left|\frac{E_i(h) - E_{i-1}(h)}{h}\right| < \varepsilon_0 \quad \forall i.$$  

(3.22)
We have implemented this numerical scheme using GNU Octave for different choices of $H$. In Figure 3.1, we present the case $H(x) = x$, $N = 1$, $\lambda = 10$ and $\epsilon_0 = 0.1$.

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References


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