EXISTENCE OF POSITIVE SOLUTIONS FOR SOME POLYHARMONIC NONLINEAR EQUATIONS IN $\mathbb{R}^n$

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Received 25 December 2004; Accepted 1 March 2005

We will study the following polyharmonic nonlinear elliptic equation

$$(-\Delta)^m u + f(\cdot, u) = 0$$

in $\mathbb{R}^n$, $n > 2m$. Under appropriate conditions on the nonlinearity $f(x,t)$, related to a class of functions called $m$-Green-tight functions, we give some existence results for the above equation.

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1. Introduction

In this paper, we deal with the higher order elliptic equation

$$(-\Delta)^m u = f(\cdot, u), \quad \text{in } \mathbb{R}^n,$$

(1.1)

where $m$ is a positive integer such that $n > 2m$.

In the case $m = 1$, (1.1) contains several well-known types which have been studied extensively by many authors (see for example [1–3, 8, 9, 11, 12, 14] and the references therein). Their basic tools are essentially some properties of functions belonging to the classical Kato class $K_n(\mathbb{R}^n)$ and the subclass of Green-tight functions $K_{\infty}^n(\mathbb{R}^n)$ (some properties pertaining to these classes can be found in [1, 4, 14]).

In this paper, we are concerned with the high order. Our purpose is two folded. One is to extend the Kato class $K_n(\mathbb{R}^n)$ and the subclass $K_{\infty}^n(\mathbb{R}^n)$ to the order $m \geq 2$. The second purpose is to investigate the existence of positive solutions for (1.1). The outline of the paper is as follows. The existence results are given in Sections 3, 4 and 5. In Section 2, we give the explicit formula of the Green function $G_{m,n}(x,y)$ of $(-\Delta)^m$ in $\mathbb{R}^n$. Namely, for each $x, y$ in $\mathbb{R}^n$

$$G_{m,n}(x,y) = k_{m,n} \frac{1}{|x-y|^{n-2m}},$$

(1.2)
where \( k_{m,n} \) is a positive constant which will be precised later. The 3G-Theorem proved in [13] for the case \( m = 1 \), is also valid for every \( m \). Indeed, for each \( x, y, z \in \mathbb{R}^n \), we have

\[
\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq 2^{n-2m-1} [G_{m,n}(x,z) + G_{m,n}(z,y)].
\] (1.3)

This 3G-Theorem will be useful to state our existence results.

Next, we study the Kato class \( K_{m,n}(\mathbb{R}^n) \) defined as follows.

**Definition 1.1.** A Borel measurable function \( \varphi \) in \( \mathbb{R}^n \) (\( n > 2m \)), belongs to the Kato class \( K_{m,n}(\mathbb{R}^n) \) if

\[
\lim_{\alpha \to 0} \left( \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \right) = 0.
\] (1.4)

Indeed, first we prove some properties of functions belonging to this class similar to those established in [1, 4]. In particular, we have the following characterization

\[
\varphi \in K_{m,n}(\mathbb{R}^n) \iff \lim_{t \to 0} \left( \sup_{x \in \mathbb{R}^n} \int_0^t s^{n-1} \int_{\mathbb{R}^n} p(s,x,y) |\varphi(y)| dy ds \right) = 0,
\] (1.5)

where \( p(t,x,y) = (1/(4\pi t)^{n/2}) \exp(-|x-y|^2/4t) \), for \( t \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \), is the density of the Gauss semi-group on \( \mathbb{R}^n \).

Secondly, we study a subclass of \( K_{m,n}(\mathbb{R}^n) \) denoted by \( K_{m,n}^\infty(\mathbb{R}^n) \) and defined by the following.

**Definition 1.2.** A Borel measurable function \( \varphi \) belongs to the class \( K_{m,n}^\infty(\mathbb{R}^n) \) and it is called \( m \)-Green-tight function if \( \varphi \in K_{m,n}(\mathbb{R}^n) \) and satisfies

\[
\lim_{M \to \infty} \left( \sup_{x \in \mathbb{R}^n} \int_{|y| \geq M} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \right) = 0.
\] (1.6)

In particular, we characterize the class \( K_{m,n}^\infty(\mathbb{R}^n) \) as follows.

**Theorem 1.3.** Let \( \varphi \in \mathcal{B}^+(\mathbb{R}^n) \), (\( n > 2m \)). Then the following assertions are equivalent

1. \( \varphi \in K_{m,n}^\infty(\mathbb{R}^n) \).
2. The \( m \)-potential of \( \varphi \), \( V\varphi(x) := \int_{\mathbb{R}^n} G_{m,n}(x,y) \varphi(y) dy \) is in \( G_+^1(\mathbb{R}^n) \).

This Theorem improves the result of Zhao in [14], for the case \( m = 1 \). A more fine characterization will be given in the radial case.

One can easily check that \( L^1(\mathbb{R}^n) \cap K_{m,n}(\mathbb{R}^n) \subset K_{m,n}^\infty(\mathbb{R}^n) \). Also we show that for \( p > n/2m \) and \( \lambda < 2m - n/p < \mu \), we have

\[
L^p(\mathbb{R}^n) \left( (1 + |\cdot|)^{\mu-\lambda} |\cdot|^{\lambda} \right) \subset K_{m,n}^\infty(\mathbb{R}^n),
\] (1.7)

and we precise the behaviour of the \( m \)-potential of functions in this class.
In Section 3, we are interested in the following polyharmonic problem

\[-\triangle^m u + u\varphi(\cdot, u) = 0, \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)}\]

\[\lim_{|x| \to \infty} u(x) = c > 0.\]  

(1.8)

The function \(\varphi\) is required to verify the following assumptions.

(H1) \(\varphi\) is a nonnegative measurable function on \(\mathbb{R}^n \times (0, \infty)\).

(H2) For each \(\lambda > 0\), there exists a nonnegative function \(q_\lambda \in K^\infty_{m,n}(\mathbb{R}^n)\) with \(\alpha q_\lambda \leq 1/2\) (see (1.24)) and such that for each \(x \in \mathbb{R}^n\), the mapping \(t \to t(q_\lambda(x) - \varphi(x, t))\) is continuous and nondecreasing on \([0, \lambda]\).

Under these hypotheses, we give an existence result for the problem (1.8). In fact, we will prove the following theorem.

**Theorem 1.4.** Assume (H1) and (H2). Then the problem (1.8) has a positive continuous solution \(u\) in \(\mathbb{R}^n\) satisfying for each \(x \in \mathbb{R}^n\), \(c/2 \leq u(x) \leq c\).

To establish this result, we use a potential theory approach. In particular, we prove that if the function \(q \in K^\infty_{m,n}(\mathbb{R}^n)\) is sufficiently small and \(f\) is a nonnegative function on \(\mathbb{R}^n\), then the equation

\[-\triangle^m u + qu = f,\]  

(1.9)

has a positive solution on \(\mathbb{R}^n\). In [6], Grunau and Sweers gave a similar result in the unit ball of \(\mathbb{R}^n\), with operators perturbed by small lower order terms:

\[-\triangle^m u + \sum_{|k|<2m} a_k(u)D^k u = f.\]  

(1.10)

In the case \(m = 1\), the problem (1.8) has been studied by Máagli and Masmoudi in [7, 8], where they gave an existence and an uniqueness result in both bounded and unbounded domain \(\Omega\).

In Section 4, we are concerned with the following polyharmonic problem

\[-\triangle^m u = f(\cdot, u), \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)}\]

\[\lim_{|x| \to \infty} u(x) = 0,\]  

(1.11)

Here \(f\) is required to satisfy the following assumptions.

(H3) \(f\) is a nonnegative measurable function on \(\mathbb{R}^n \times (0, \infty)\), continuous with respect to the second variable.

(H4) There exist a nonnegative function \(p\) in \(\mathbb{R}^n\) such that

\[0 < a_0 := \int_{\mathbb{R}^n} \frac{p(y)}{(|y| + 1)^{2(n-2m)}} dy < \infty\]  

(1.12)

and a nonnegative function \(q \in K^\infty_{m,n}(\mathbb{R}^n)\) such that for \(x \in \mathbb{R}^n\) and \(t > 0\)

\[p(x)h(t) \leq f(x, t) \leq q(x)g(t),\]  

(1.13)
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where $h$ is a nonnegative nondecreasing measurable function on $[0, \infty)$ satisfying

$$m_0 := \frac{1}{k_{m,n}} < h_0 := \liminf_{t \to 0^+} \frac{h(t)}{t} \leq \infty$$

and $g$ is a nonnegative measurable function locally bounded on $[0, \infty)$ satisfying

$$0 \leq g^\infty := \limsup_{t \to \infty} \frac{g(t)}{t} < M_0 := \frac{1}{\|Vq\|_\infty}.$$ 

By using a fixed point argument, we will state the following existence result.

**Theorem 1.5.** Assume $(H_3)$ and $(H_4)$. Then the problem (1.11) has a positive continuous solution $u$ in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$,

$$\frac{a}{(|x| + 1)^{n-2m}} \leq u(x) \leq bVq(x),$$

where $a, b$ are positive constants.

This result follows up the one of Dalmasso (see [5]), who studied the problem (1.11) in the unit ball $B$, with more restrictive conditions on the function $f$. Indeed, he assumed that $f$ is nondecreasing with respect to the second variable and satisfies

$$\lim \min_{t \to 0^+} \frac{f(x,t)}{t} = +\infty, \quad \lim \max_{t \to \infty} \frac{f(x,t)}{t} = 0.$$ 

He proved the existence of a positive solution and he gave also an uniqueness result for positive radial solution when $f(x,t) = f(|x|,t)$.

When $m = 1$, similar conditions, but more restrictive, on the nonlinearity $f$ have been adopted by Mäagli and Masmoudi in [8]. In fact in [8], the authors studied (1.11) in an unbounded domain $D$ of $\mathbb{R}^n$, $n \geq 3$, with compact nonempty boundary $\partial D$ and gave an existence result as Theorem 1.5.

On the other hand, Brezis and Kamin proved in [3], the existence and the uniqueness of a positive solution for the problem

$$-\Delta u = \rho(x)u^\alpha \text{ in } \mathbb{R}^n,$$

$$\liminf_{|x| \to \infty} u(x) = 0,$$ 

with $0 < \alpha < 1$ and $\rho$ is a nonnegative measurable function satisfying some appropriate conditions. We improve in this section the result of Brezis and Kamin in [3] and the one of Mäagli and Masmoudi in [8].

In Section 5, we will study the existence of solutions to the following polyharmonic problem

$$(-\triangle)^m u = f(\cdot, u), \text{ in } \mathbb{R}^n \text{ (in the sense of distributions)}$$

$$u(x) > 0, \text{ in } \mathbb{R}^n,$$ 

under the following assumptions on the nonlinearity $f$. 


(H5) \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \times (0, \infty) \), continuous with respect to the second variable on \((0, \infty)\).

(H6) \( f(x,t) \leq q(x,t) \), where \( q \) is a nonnegative measurable function on \( \mathbb{R}^n \times (0, \infty) \) such that the function \( t \to q(x,t) \) is nondecreasing on \((0, \infty)\).

(H7) There exists a constant \( c > 0 \) such that \( q(\cdot, c) \in \mathcal{K}_{m,n}(\mathbb{R}^n) \) and
\[
\| V(q(\cdot, c)) \|_\infty < c. \tag{1.20}
\]

Put \( c^* = c - \| V(q(\cdot, c)) \|_\infty \). We give in this section the following existence result.

**Theorem 1.6.** Assume (H5), (H6), and (H7). Then for each \( \delta \in (0, c^*] \), the problem (1.19) has a positive continuous solution \( u \) in \( \mathbb{R}^n \) satisfying for each \( x \in \mathbb{R}^n \)
\[
\delta \leq u(x) \leq c,
\lim_{|x| \to \infty} u(x) = \delta. \tag{1.21}
\]

If \( m = 1 \), Yin gave in [11] an existence result of the following problem
\[
\triangle u + f(x,u) = 0, \quad \text{in } G_B,
\]
\[
u(x) > 0, \quad \text{in } G_B, \tag{1.22}
\]
where \( G_B = \{ x \in \mathbb{R}^n, |x| > B \} \), for some \( B \geq 0 \). His method relies on the technique of radial super/subsolutions. Our approach is different, in fact we will use a fixed point argument. We improve the result of Yin under more general assumptions (see Remark 5.3).

In order to simplify our statements, we define some convenient notations.

**Notations.**

(i) \( \mathcal{B}(\mathbb{R}^n) \) denotes the set of Borel measurable functions in \( \mathbb{R}^n \) and \( \mathcal{B}^+(\mathbb{R}^n) \) the set of nonnegative ones.

(ii) \( C_0(\mathbb{R}^n) := \{ w \text{ continuous on } \mathbb{R}^n \text{ and } \lim_{|x| \to \infty} w(x) = 0 \} \) and \( C_0^+(\mathbb{R}^n) \) the set of nonnegative ones.

(iii) For \( \varphi \in \mathcal{B}^+(\mathbb{R}^n) \), we put the \( m \)-potential of \( \varphi \) on \( \mathbb{R}^n \) by
\[
V\varphi(x) := V_{m,n}\varphi(x) = \int_{\mathbb{R}^n} G_{m,n}(x,y)\varphi(y)dy = k_{m,n} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n-2m}}dy. \tag{1.23}
\]

(iv) For \( \varphi \in \mathcal{B}^+(\mathbb{R}^n) \), we put
\[
\alpha\varphi = \sup_{x,y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} |\varphi(z)| dz. \tag{1.24}
\]

(v) Let \( \lambda \in \mathbb{R} \), we denote by \( \lambda^+ = \max(\lambda, 0) \).

(vi) Let \( f \) and \( g \) be two positive functions on a set \( S \).

We call \( f \sim g \), if there is \( c > 0 \) such that
\[
\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S. \tag{1.25}
\]
We call $f \preceq g$, if there is $c > 0$ such that
\[ f(x) \leq cg(x) \quad \forall x \in S. \] (1.26)

The following properties will be used several times: for $s, t \geq 0$, we have
\[ \min(s, t) = s \land t \sim \frac{st}{s + t}, \]
\[ (s + t)^p \sim s^p + t^p, \quad p \in \mathbb{R}^+. \] (1.27)

2. Properties of the Kato class

In this section, we characterize functions belonging to the Kato class $K_{m,n}(\mathbb{R}^n)$ and the subclass $K_{m,n}^\infty(\mathbb{R}^n)$ of $m$-Green-tight functions and we prove Theorem 1.3. We recall that throughout this paper, we are concerned with $n > 2m$.

We set $p(t,x,y) = (1/(4\pi t)^{n/2}) \exp(-|x - y|^2/4t)$, for $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$, the density of the Gauss semi-group on $\mathbb{R}^n$. By a simple computation, we obtain that the Green function of $(-\Delta)^m$ in $\mathbb{R}^n$, for each $m \geq 1$, is given by
\[ G_{m,n}(x,y) = \frac{1}{(m-1)!} \int_0^\infty s^{m-1} p(s,x,y)ds, \quad \text{for } x, y \text{ in } \mathbb{R}^n. \] (2.1)

Then we have the following explicit expression
\[ G_{m,n}(x,y) = k_{m,n} \frac{1}{|x-y|^{n-2m}}, \quad \text{for } x, y \text{ in } \mathbb{R}^n, \] (2.2)
where $k_{m,n} = \Gamma(n/2 - m)/4^m \pi^{n/2} (m-1)!$.

2.1. The class $K_{m,n}(\mathbb{R}^n)$. We will study properties of functions belonging to $K_{m,n}(\mathbb{R}^n)$. First we remark the following comparison on the classes $K_{j,n}(\mathbb{R}^n)$, for $j \geq 1$.

Remark 2.1. Let $j, m \in \mathbb{N}$ such that $1 \leq j \leq m$, then we have for each $n > 2m$
\[ K_n(\mathbb{R}^n) := K_{1,n}(\mathbb{R}^n) \subseteq K_{j,n}(\mathbb{R}^n) \subseteq K_{m,n}(\mathbb{R}^n), \] (2.3)
where $K_n(\mathbb{R}^n)$ is the classical Kato class introduced in [1].

Example 2.2. Let $\varphi \in \mathcal{B}(\mathbb{R}^n)$. Suppose that for $p > n/2m$, we have
\[ \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq 1} |\varphi(y)|^p dy < \infty. \] (2.4)

Then by the Hölder inequality, we conclude that $\varphi \in K_{m,n}(\mathbb{R}^n)$.

In particular, we have that for $p > n/2m$, $L^p(\mathbb{R}^n) \subset K_{m,n}(\mathbb{R}^n)$.

To establish the characterization (1.5) of the Kato class $K_{m,n}(\mathbb{R}^n)$, we need the following lemmas.
Lemma 2.3. For each \( t > 0 \) and \( x, y \in \mathbb{R}^n \), we have

\[
\int_0^t s^{n-1} p(s, x, y) \, ds \leq G_{m,n}(x, y). \tag{2.5}
\]

Moreover, for \( |x - y| \leq 2\sqrt{t} \), we have that

\[
G_{m,n}(x, y) \leq \int_0^t s^{m-1} p(s, x, y) \, ds. \tag{2.6}
\]

**Proof.** Let \( t > 0 \) and \( x, y \in \mathbb{R}^n \). Then (2.5) follows immediately from (2.1).

If we suppose further that \( |x - y| \leq 2\sqrt{t} \), then we have

\[
\int_0^t s^{m-1} p(s, x, y) \, ds = c \int_0^t s^{m-n/2-1} \exp \left( -\frac{|x - y|^2}{4s} \right) \, ds
\]

\[
= c \frac{|x - y|^{n-2m}}{|x - y|^{n/2} \sqrt{4t}} \int_0^\infty r^{n/2-m-1} e^{-r} \, dr 
\]

\[
\geq c |x - y|^{n-2m} \int_1^\infty r^{n/2-m-1} e^{-r} \, dr
\]

\[
= c G_{m,n}(x, y),
\]

where the letter \( c \) is a positive constant which may vary from line to line. \( \square \)

Lemma 2.4. Let \( \varphi \in K_{m,n}(\mathbb{R}^n) \). Then for each compact \( L \subset \mathbb{R}^n \), we have

\[
\sup_{x \in \mathbb{R}^n} \int_{x+L} |\varphi(y)| \, dy < \infty. \tag{2.8}
\]

In particular, we have \( K_{m,n}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n) \).

**Proof.** Let \( \varphi \in K_{m,n}(\mathbb{R}^n) \), then by (1.4) there exists \( \alpha > 0 \) such that

\[
\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} \, dy \leq 1. \tag{2.9}
\]

Let \( a_1, \ldots, a_p \in L \) such that \( L \subseteq \bigcup_{1 \leq i \leq p} B(a_i, \alpha) \). Hence for each \( x \in \mathbb{R}^n \), we have

\[
\int_{x+L} |\varphi(y)| \, dy \leq \sum_{i=1}^p \int_{B(x+a_i, \alpha)} |\varphi(y)| \, dy
\]

\[
\leq \sum_{i=1}^p \alpha^{n-2m} \int_{B(x+a_i, \alpha)} \frac{|\varphi(y)|}{|x+a_i-y|^{n-2m}} \, dy
\]

\[
\leq p \alpha^{n-2m}.
\]

So, \( \sup_{x \in \mathbb{R}^n} \int_{x+L} |\varphi(y)| \, dy < \infty. \) \( \square \)
Proposition 2.5. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$. Then for each fixed $\alpha > 0$, we have

$$
\sup_{0 \leq t \leq 1} \left( \sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq \alpha} t^{m-1} p(t,x,y) \left| \varphi(y) \right| \, dy \right) := M(\alpha) < \infty. \tag{2.11}
$$

Proof. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$, $0 < t \leq 1$. Let $\alpha > 0$, then we have that

$$
\sup_{x \in \mathbb{R}^n} \int_{|x-y| \geq \alpha} t^{m-1} p(t,x,y) \left| \varphi(y) \right| \, dy 
\leq \frac{\exp\left(-\alpha^2/8t\right)}{t^{n/2-m+1}} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) \left| \varphi(y) \right| \, dy. \tag{2.12}
$$

So to prove (2.11), we need to show that

$$
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) \left| \varphi(y) \right| \, dy < \infty. \tag{2.13}
$$

Indeed, using Lemma 2.4, we denote by

$$
c := \sup_{x \in \mathbb{R}^n} \int_{x+B(0,1)} \left| \varphi(y) \right| \, dy < \infty. \tag{2.14}
$$

On the other hand, since any ball $B(0,k)$ of radius $k \geq 1$ in $\mathbb{R}^n$ can be covered by $a(n) := A_n k^n$ balls of radius 1, where $A_n$ is a constant depending only on $n$ (see [4, page 67]), then there exist $a_1, a_2, \ldots, a_{a(n)} \subset B(0,k)$ such that

$$
B(0,k) \subset \bigcup_{1 \leq i \leq a(n)} B(a_i,1). \tag{2.15}
$$

Hence for each $x \in \mathbb{R}^n$, we have

$$
\int_{x+B(0,k)} \left| \varphi(y) \right| \, dy \leq \sum_{i=1}^{a(n)} \int_{B(x+a_i,1)} \left| \varphi(y) \right| \, dy \leq c A_n k^n, \tag{2.16}
$$

which implies that for each $x \in \mathbb{R}^n$,

$$
\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{8}\right) \left| \varphi(y) \right| \, dy 
\leq \sum_{k=0}^{\infty} \exp\left(-\frac{k^2}{8}\right) \int_{k \leq |x-y| \leq k+1} \left| \varphi(y) \right| \, dy 
\leq c A_n \sum_{k=0}^{\infty} \exp\left(-\frac{k^2}{8}\right)(k+1)^n \tag{2.17}
$$

< \infty.

Thus (2.13) holds. This ends the proof. \qed
Proposition 2.6. Let $\varphi \in B(\mathbb{R}^n)$. Then $\varphi \in K_{m,n}(\mathbb{R}^n)$ if and only if

$$\lim_{t \to 0} \left( \sup_{x \in \mathbb{R}^n} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) |\varphi(y)| \, dy \, ds \right) = 0. \quad (2.18)$$

Proof. Suppose $\varphi$ verifies (2.18), then from (2.6) we have that

$$\int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} \, dy \leq \int_{\mathbb{R}^n} \int_0^{\alpha^2/4} s^{m-1} p(s,x,y) |\varphi(y)| \, ds \, dy, \quad (2.19)$$

which implies that the function $\varphi$ satisfies (1.4).

Conversely, suppose that $\varphi \in K_{m,n}(\mathbb{R}^n)$. Let $\varepsilon > 0$, then by (1.4), there exists $\alpha > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} \, dy \leq \varepsilon. \quad (2.20)$$

Thus from (2.5) and (2.11), we deduce that for each $x \in \mathbb{R}^n$ and $t \leq 1$, we have

$$\int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) |\varphi(y)| \, dy \, ds \leq \int_{|x-y| \leq \alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} \, dy + t M(\alpha) \quad (2.21)$$

This implies (2.18) and completes the proof. \(\square\)

2.2. The class $K_{m,n}^\infty(\mathbb{R}^n)$. We will characterize the subclass of $m$-Green-tight functions $K_{m,n}^\infty(\mathbb{R}^n)$. In fact, we will prove Theorem 1.3 and we give in particular a more precise characterization in the radial case.

Example 2.7. Let $p > n/2m$. Then $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset K_{m,n}^\infty(\mathbb{R}^n)$.

Proof of Theorem 1.3. Let $\varphi \in B_+(\mathbb{R}^n)$. First we suppose that $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$, then using similar arguments as in the proof [9, Proposition 6], we obtain easily that $V \varphi \in C_0^\infty(\mathbb{R}^n)$.

Conversely we suppose that $V \varphi \in C_0^\infty(\mathbb{R}^n)$. Then, we aim at proving that $\varphi \in K_{m,n}^\infty(\mathbb{R}^n)$. So we divide the proof into two steps.
Step 1. We will prove that \( \varphi \) satisfies (2.18). Indeed it is clear from (2.1), that for each \( x \in \mathbb{R}^n \), we have that

\[
V \varphi (x) = \frac{1}{(m-1)!} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) \varphi (y) dy ds
+ \frac{1}{(m-1)!} \int_t^{\infty} s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) \varphi (y) dy ds
= I_1 (x) + I_2 (x).
\] (2.22)

From the properties of the density \( p(s,x,y) \), we deduce that \( x \to I_1 (x) \) and \( x \to I_2 (x) \) are nonnegative lower semi-continuous functions in \( \mathbb{R}^n \). Then using the fact that \( V \varphi \in C_0^+ (\mathbb{R}^n) \), we get that the function \( x \to I_1 (x) \) is also in \( C_0^+ (\mathbb{R}^n) \). So, for each \( x \in \mathbb{R}^n \), the family \( \{ \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) \varphi (y) dy ds, t > 0 \} \) is decreasing in \( C_0^+ (\mathbb{R}^n) \), which together with the fact that for each \( x \in \mathbb{R}^n \),

\[
\lim_{t \to 0} \int_0^t s^{m-1} \int_{\mathbb{R}^n} p(s,x,y) \varphi (y) dy ds = 0 \quad (2.23)
\]

imply by Dini Lemma, that (2.18) is satisfied.

Step 2. We will prove that \( \varphi \) satisfies (1.6). Let \( \varepsilon > 0 \), then since \( V \varphi \in C_0^+ (\mathbb{R}^n) \), there exists \( a > 0 \) such that for \( |x| \geq a \), we have that \( V \varphi (x) \leq \varepsilon \).

Let \( M \geq 2a \), then

\[
\sup_{x \in \mathbb{R}^n} \frac{\varphi (y)}{|x-y|^{n-2m}} dy \leq \sup_{|x| \geq a} \frac{\varphi (y)}{|x-y|^{n-2m}} dy + \sup_{|x| \leq a} \frac{\varphi (y)}{|x-y|^{n-2m}} dy
\leq \varepsilon + \int_{|y| \geq M} \frac{\varphi (y)}{|y|^{n-2m}} dy.
\] (2.24)

Now, since \( V \varphi (0) < \infty \), we deduce that

\[
\lim_{M \to \infty} \int_{|y| \geq M} \frac{\varphi (y)}{|y|^{n-2m}} dy = 0. \quad (2.25)
\]

Then (1.6) holds and this ends the proof.

For a nonnegative function \( \rho \) in \( K_{m,n}^\infty (\mathbb{R}^n) \), we denote by

\[
M_\rho := \{ \varphi \in B (\mathbb{R}^n), \ |\varphi| \leq \rho \}.
\] (2.26)

**Proposition 2.8.** For a nonnegative function \( \rho \) in \( K_{m,n}^\infty (\mathbb{R}^n) \), the family of functions

\[
V (M_\rho) := \{ V \varphi, \ \varphi \in M_\rho \}
\] (2.27)

is uniformly bounded and equicontinuous in \( C_0 (\mathbb{R}^n) \) and consequently it is relatively compact in \( C_0 (\mathbb{R}^n) \).
Proof. Let $\rho \in K_{m,n}^{\infty}(\mathbb{R}^n)$. Obviously, since each function $\phi$ in $M_{\rho}$ is in $K_{m,n}^{\infty}(\mathbb{R}^n)$, we obtain by Theorem 1.3 that the family $V(M_{\rho}) \subset C_{0}(\mathbb{R}^n)$ and is uniformly bounded. Next, we prove the equicontinuity of functions in $V(M_{\rho})$ on $\mathbb{R}^n \cup \{\infty\}$ by same arguments as in the proof of [9, Proposition 6]. Thus by Ascoli’s Theorem the family $V(M_{\rho})$ is relatively compact in $C_{0}(\mathbb{R}^n)$. This ends the proof. □

Remark 2.9. We recall (see [12, 14]) that for $m = 1$ and $n \geq 3$, a radial function is in $K_{n}^{\infty}(\mathbb{R}^n)$ if and only if

$$\int_{0}^{\infty} r^{n-1} |\phi(r)| dr < \infty.$$  

Similarly, we will give in the sequel a characterization of radial functions belonging to $K_{m,n}^{\infty}(\mathbb{R}^n)$.

Proposition 2.10. Let $\phi$ be a radial function in $\mathbb{R}^n$, then \( \phi \in K_{m,n}^{\infty}(\mathbb{R}^n) \) if and only if

$$\int_{0}^{\infty} r^{2m-1} |\phi(r)| dr < \infty. \quad (2.28)$$

In order to prove Proposition 2.10, we will use the following behaviour of the $m$-potential of radial functions on $\mathbb{R}^n$.

Proposition 2.11. Let $\phi \in \mathcal{B}^{+}(\mathbb{R}^n)$ be a radial function on $\mathbb{R}^n$, then for $x \in \mathbb{R}^n$, we have

$$V\phi(x) \sim \int_{0}^{\infty} \frac{r^{n-1}}{(|x| \vee r)^{n-2m}} \phi(r) dr. \quad (2.29)$$

Proof. Let $\phi \in \mathcal{B}^{+}(\mathbb{R}^n)$. First, we recall the well known results for $x, y \in \mathbb{R}^n$,

\[
(n-2)k_{1,n} \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2}} dy = \int_{0}^{\infty} \frac{r^{n-1}}{(|x| \vee r)^{n-2}} \phi(r) dr,
\]

\[
\int_{\mathbb{R}^n} \frac{dz}{|x-z|^{n-2} |y-z|^{n-2}} = \frac{c_n}{|x-y|^{n-4}}. \quad (2.30)
\]

This implies that there exists a constant $c > 0$ such that

\[
\int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-4}} dy = c \int_{0}^{\infty} r^{n-1} \phi(r) \int_{0}^{\infty} t^{n-1} \frac{1}{(|x| \vee t)^{n-2} (t \vee r)^{n-2}} dtdr
\]

\[
\geq c \int_{0}^{\infty} r^{n-1} \phi(r) \int_{|x| \vee r}^{\infty} \frac{1}{t^{n-3}} dtdr
\]

\[
\geq \frac{c}{n-4} \int_{0}^{\infty} r^{n-1} \phi(r) \frac{1}{(|x| \vee r)^{n-4}} dr. \quad (2.31)
\]

Hence, we obtain by recurrence that

\[
\int_{0}^{\infty} \frac{r^{n-1}}{(|x| \vee r)^{n-2m}} \phi(r) dr \leq \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-2m}} dy. \quad (2.32)
\]
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On the other hand, there exists a constant $\tilde{c} > 0$ such that for each $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n-2m}} dy = \tilde{c} \int_0^\infty \int_0^\pi r^{n-1} \varphi(r)(\sin \theta)^{n-2} \frac{d\theta dr}{(|x|^2 + r^2 - 2r|x| \cos \theta)\left((n-2m)/2\right)}$$

$$\leq \tilde{c} \int_0^\infty \int_0^\pi r^{n-1} \varphi(r)(\sin \theta)^{n-2} \frac{d\theta dr}{(|x| \vee r)^n - 2m(\sin \theta)^{n-2m}}$$

$$= \tilde{c} \left( \int_0^\pi (\sin \theta)^{2m-2} d\theta \right) \left( \int_0^\infty \frac{r^{n-1} \varphi(r)}{(|x| \vee r)^{n-2m}} dr \right).$$

Thus (2.29) holds. □

Proof of Proposition 2.10. Suppose that $\varphi$ is a radial function in $K_{m,n}^\infty(\mathbb{R}^n)$, then by Theorem 1.3, $V \varphi(0) < \infty$ and so we deduce (2.28) from (2.29).

Conversely, suppose that $\varphi$ satisfies (2.28). Let $\alpha > 0$ and $t = |x|$, then by (2.29), we have

$$\int_{|x - y| \leq \alpha} \frac{\varphi(y)}{|x - y|^{n-2m}} dy \leq \int_{(t-\alpha)^+}^{t+\alpha} \frac{r^{n-1} \varphi(r)}{(t \vee r)^{n-2m}} \varphi(r) dr$$

$$\leq \int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} \varphi(r) dr.$$ 

Let $\phi(s) = \int_0^s r^{2m-1} \varphi(r) dr$, for $s \in [0, \infty]$. Using (2.28), we deduce that $\phi$ is a continuous function on $[0, \infty]$. This implies that

$$\int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} \varphi(r) dr = \phi(t + \alpha) - \phi((t - \alpha)^+),$$

converges to zero as $\alpha \to 0$ uniformly for $t \in [0, \infty]$. So $\varphi$ verifies (1.4).

Next, we have by (2.29)

$$\int_{|y| \leq M} \frac{|\varphi(y)|}{|x - y|^{n-2m}} dy \leq \int_M^\infty \frac{r^{n-1} \varphi(r)}{(t \vee r)^{n-2m}} \varphi(r) dr \leq \int_M^\infty r^{2m-1} \varphi(r) dr,$$

which, using (2.28), tends to zero as $M \to \infty$ and so $\varphi$ verifies (1.6). This completes the proof. □

We close this section by giving a class of functions included in $K_{m,n}^\infty(\mathbb{R}^n)$ and we precise the behaviour of the $m$-potential of functions in this class. We need the following lemma.
Lemma 2.12. Let $\alpha > 0$ and $a, b > 0$ such that $a + b < n$. Then
\[
\int_{|x - y| \leq \alpha} \frac{dy}{|y|^a |x - y|^b} \leq \alpha^{n-(a+b)}.
\]  
(2.37)

Proof. Let $\alpha > 0$ and $a, b$ be nonnegative real numbers such that $a + b < n$. Then
\[
\int_{|x - y| \leq \alpha} \frac{dy}{|y|^a |x - y|^b} \leq \int_{(|x - y| \leq \alpha) \cap (|x - y| \leq |y|)} \frac{dy}{|x - y|^{a+b}} + \int_{(|y| \leq |x - y| \leq \alpha)} \frac{dy}{|y|^{a+b}} 
\leq \int_0^\alpha \rho^{n-1-(a+b)} \, dr 
\leq \alpha^{n-(a+b)}.
\]  
(2.38)

Proposition 2.13. Let $p > n/2m$. Then for $\lambda < 2m - n/p < \mu$, we have
\[
\frac{L^p(\mathbb{R}^n)}{(1 + |\cdot|)^{\mu - \lambda}} \subset K_{m,n}^\infty(\mathbb{R}^n).
\]  
(2.39)

Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let $a$ be a function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. First, we will prove that the function $\varphi(x) := a(x)/(1 + |x|)^{\mu - \lambda} |x|^\lambda$ satisfies (1.4). Let $\alpha > 0$, then by the Hölder inequality and Lemma 2.12, we have for $x \in \mathbb{R}^n$
\[
\int_{|x - y| \leq \alpha} \frac{|\varphi(y)|}{|x - y|^{n-2m}} \, dy \leq \|a\|_p \left( \int_{|x - y| \leq \alpha} \frac{dy}{(1 + |y|)^{(\mu - \lambda)q} |y|^\lambda |x - y|^{(n-2m)q}} \right)^{1/q} 
\leq \|a\|_p \left( \int_{|x - y| \leq \alpha} \frac{dy}{|y|^{q\lambda^*} |x - y|^{(n-2m)q}} \right)^{1/q} 
\leq \|a\|_p \alpha^{2m-n/p-\lambda^*},
\]  
(2.40)

which converges to zero as $\alpha \to 0$.

Secondly, we claim that $\varphi$ satisfies (1.6). To show the claim we use the Hölder inequality. Let $M > 1$, then we have
\[
\int_{|y| \geq M} \frac{|\varphi(y)|}{|x - y|^{n-2m}} \, dy \leq \|a\|_p \left( \int_{|y| \geq M} \frac{dy}{(1 + |y|)^{(\mu - \lambda)q} |y|^\lambda |x - y|^{(n-2m)q}} \right)^{1/q} 
\sim \|a\|_p \left( \int_{|y| \geq M} \frac{dy}{|y|^{aq} |x - y|^{(n-2m)q}} \right)^{1/q} 
= \|a\|_p (A(x))^{1/q}.
\]  
(2.41)
Furthermore

\[ A(x) \leq \sup_{|x| \leq M/2} \int_{|y| \geq M} \frac{dy}{y^{(n-2m+\mu)q}} \]

\[ + \sup_{|x| \geq M/2} \frac{1}{|x|^{\mu q}} \int_{|y| \geq M \cap (|x-y| \leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}} \]

\[ + \sup_{|x| \geq M/2} \int_{(|y| \geq M \cap (|x-y| \leq 2|x|))} dy \]

\[ \leq \frac{1}{M^{(n-2m+\mu)q-n}} + \sup_{|z| \geq M/2} \frac{\log(3|z|/M)}{|z|^{(n-2m)q}}, \]  

which converges to zero as \( M \to \infty \). This ends the proof.

**Remark 2.14.** It is obvious to see that for each \( \varphi \in \mathbb{B}^+(\mathbb{R}^n) \), we have

\[ \frac{k_{m,n}}{(|x|+1)^{n-2m}} \int_{\mathbb{R}^n} \frac{\varphi(y)}{(1+|y|)^{\mu - \lambda} |y|^{\lambda}} dy \leq V \varphi(x). \]  

(2.43)

We precise in the following, some upper estimates on the \( m \)-potential of functions in the class \( L^p(\mathbb{R}^n)/(1 + |\cdot|)^{\mu - \lambda} \cdot |\cdot|^{\lambda} \). Indeed, put for a nonnegative function \( a \in L^p(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \)

\[ W(a) := V \left( \frac{a}{(1 + |\cdot|)^{\mu - \lambda} |\cdot|^{\lambda}} \right)(x) = \int_{\mathbb{R}^n} G_{m,n}(x,y) \frac{a(y)}{(1+|y|)^{\mu - \lambda} |y|^{\lambda}} dy. \]  

(2.44)

Then we have the following.

**Proposition 2.15.** Let \( p > n/2m \) and \( \lambda < 2m - n/p < \mu \). Then there exists \( c > 0 \) such that for each nonnegative function \( a \in L^p(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we have the following estimates

\[ W(a) \leq c ||a||_p \begin{cases} \frac{1}{(1+|x|)^{n-2m}} \log(|x|+1)^{p/(p-1)}, & \text{if } \mu + \frac{n}{p} = n \\ \frac{1}{(1+|x|)^{(n-2m)\land(\mu+n/p-2m)}}, & \text{if } \mu + \frac{n}{p} \neq n. \end{cases} \]  

(2.45)
Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let $a$ be a nonnegative function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. Put $\varphi(x) = a(x)/(1 + |x|)^{\mu-\lambda}|x|^\lambda$, then by the Hölder inequality, we have for each $x \in \mathbb{R}^n$

$$V \varphi(x) \leq \|a\|_p \left( \int_{\mathbb{R}^n} \frac{dy}{|x - y|^{(n-2m)q}(1 + |y|)^{(\mu-\lambda)q}|y|^\lambda} \right)^{1/q}$$

$$= \|a\|_p (I(x))^{1/q}. \quad (2.46)$$

Furthermore,

(i) if $|x| \leq 1$, we have by Lemma 2.12, that

$$I(x) \leq \int_{B(x,2)} \frac{dy}{|x - y|^{(n-2m)q}|y|^{\mu q}} + \int_{B'(x,2)} \frac{dy}{|x - y|^{(n-2m)q}|y|^\mu}$$

$$\leq \int_{B(x,2)} \frac{dy}{|x - y|^{(n-2m)q}|y|^{\lambda q}} + \int_{B'(x,2)} \frac{dy}{|x - y|^{(n-2m+\mu)q}} \quad (2.47)$$

$$\leq 1,$$

(ii) if $|x| \geq 1$, we have

$$I(x) \leq \int_{(|y| \leq 1/2)} \frac{dy}{|x - y|^{(n-2m)q}|y|^\lambda} + \int_{(|y| \geq 1/2) \cap (|x - y| \leq |x|/2)} \frac{dy}{|x - y|^{(n-2m)q}|y|^\mu}$$

$$+ \int_{(|y| \geq 1/2) \cap (|x - y| \leq 2|x|)} \frac{dy}{|x - y|^{(n-2m)q}|y|^\mu}$$

$$+ \int_{(|y| \geq 1/2) \cap (|x - y| \geq 2|x|)} \frac{dy}{|x - y|^{(n-2m)q}|y|^\mu}$$

$$\leq \frac{1}{|x|^{(n-2m)q}} \left( \int_{(|y| \leq 1/2)} \frac{dy}{|y|^\lambda} + \int_{(|x - y| \leq |x|/2)} \frac{dy}{|y|^{\mu q}} \right)$$

$$+ \frac{1}{|x|^{(n-2m)q}} \left( \int_{(1/2 \leq |y| \leq 3|x|)} \frac{dy}{|y|^\mu} + \int_{(|x - y| > 2|x|)} \frac{dy}{|y|^{(n-2m+\mu)q}} \right)$$

$$\leq \frac{1}{|x|^{(n-2m)q}} \begin{cases} 
\log (|x| + 1), & \text{if } \frac{\mu + \frac{n}{p}}{p} = n \\
|x|^{n-\mu q}, & \text{if } \frac{\mu + \frac{n}{p}}{p} < n \\
1, & \text{if } \frac{\mu + \frac{n}{p}}{p} > n.
\end{cases} \quad (2.48)$$

By combining the above inequalities, we get the result. ☐

Corollary 2.16. The class of functions $L^\infty(\mathbb{R}^n)/(1 + | \cdot |)^{\mu-\lambda}| \cdot |^\lambda$ is included in $K^\infty_{m,n}(\mathbb{R}^n)$ if and only if $\lambda < 2m < \mu$. 
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Proof. “$\Leftarrow$” follows from Proposition 2.13.

“$\Rightarrow$” Suppose that the function $\varphi$ defined on $\mathbb{R}^n$ by $\varphi(x) = 1/(1 + |x|)^{\mu - \lambda} |x|^\lambda$ is in $K_{m,n}^\infty(\mathbb{R}^n)$. Then by Proposition 2.10, we have $\int_0^\infty r^{2m-1} \varphi(r) dr < \infty$. This implies that $\lambda < 2m < \mu$. \qed

Remark 2.17. Let $\lambda < 2m < \mu$ and $\varphi(x) = 1/(1 + |x|)^{\mu - \lambda} |x|^\lambda$, for $x \in \mathbb{R}^n$, then by simple calculus, we obtain the following behaviour on the $m$-potential

$$V\varphi(x) \sim \begin{cases} 
\frac{1}{(1 + |x|)^{n-2m}} \log (|x| + 1), & \text{if } \mu = n \\
\frac{1}{(1 + |x|)^{(n-2m)/\lambda + (\mu - 2m)/\lambda}}, & \text{if } \mu \neq n.
\end{cases} \quad (2.49)$$

3. First existence result

In this section, we aim at proving Theorem 1.4. The following lemmas are useful.

Lemma 3.1. Let $\varphi$ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$. Then we have

$$\|V\varphi\|_\infty \leq \alpha_\varphi \leq 2^{n-2m} \|V\varphi\|_\infty. \quad (3.1)$$

Proof. By (1.3) we obtain easily that $\alpha_\varphi \leq 2^{n-2m} \|V\varphi\|_\infty$. On the other hand, by letting $|y| \to \infty$ in (1.24), we deduce from Fatou Lemma that $\|V\varphi\|_\infty \leq \alpha_\varphi$. \qed

Lemma 3.2. Let $\varphi$ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$, we have

$$V(\varphi G_{m,n}(\cdot, y))(x) \leq \alpha_\varphi G_{m,n}(x, y). \quad (3.2)$$

Proof. The result holds by (1.24). \qed

In the sequel, let $\varphi$ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$ such that $\alpha_\varphi \leq 1/2$. For $f \in B^+ (\mathbb{R}^n)$, we will define the potential kernel $V_\varphi f := V_{m,n,\varphi} f$ as a solution for the perturbed polyharmonic equation (1.9).

We put for $x, y \in \mathbb{R}^n$,

$$q_{m,n}(x, y) = \begin{cases} 
\sum_{k=0}^\infty (-1)^k (V(q\cdot))^k (G_{m,n}(\cdot, y))(x), & \text{if } x \neq y \\
\infty, & \text{if } x = y.
\end{cases} \quad (3.3)$$

Then we have the following comparison result.

Lemma 3.3. Let $\varphi$ be a nonnegative function in $K_{m,n}^\infty(\mathbb{R}^n)$ such that $\alpha_\varphi \leq 1/2$. Then for $x, y \in \mathbb{R}^n$, we have

$$(1 - \alpha_\varphi) G_{m,n}(x, y) \leq q_{m,n}(x, y) \leq G_{m,n}(x, y). \quad (3.4)$$
Proof. Since \( \alpha_q \leq 1/2 \), we deduce from (3.2), that
\[
|\mathcal{G}_{m,n}(x,y)| \leq \sum_{k \geq 0} (\alpha_q)^k G_{m,n}(x,y)
\]
\[
= \frac{1}{1 - \alpha_q} G_{m,n}(x,y).
\]
(3.5)
Furthermore, we have for \( x \neq y \) in \( \mathbb{R}^n \)
\[
\mathcal{G}_{m,n}(x,y) = G_{m,n}(x,y) - V(q\mathcal{G}_{m,n}(\cdot,y))(x),
\]
(3.6)
which together with (3.2), imply that
\[
\mathcal{G}_{m,n}(x,y) \geq G_{m,n}(x,y) - \frac{\alpha_q}{1 - \alpha_q} G_{m,n}(x,y)
\]
\[
= \frac{1 - 2\alpha_q}{1 - \alpha_q} G_{m,n}(x,y)
\]
\[
\geq 0.
\]
(3.7)
Hence the result follows from (3.6) and (3.2).

Let us define the operator \( V_q \) on \( \mathcal{B}^+(\mathbb{R}^n) \) by
\[
V_q f(x) = \int_B \mathcal{G}_{m,n}(x,y) f(y) dy, \quad x \in \mathbb{R}^n.
\]
(3.8)
Then we obtain the following.

**Lemma 3.4.** Let \( f \in \mathcal{B}^+(\mathbb{R}^n) \). Then \( V_q f \) satisfies the following resolvent equation
\[
V f = V_q f + V_q (qV f) = V_q f + V (qV_q f).
\]
(3.9)
Proof. From the expression of \( \mathcal{G}_{m,n} \), we deduce that for \( f \in \mathcal{B}^+(\mathbb{R}^n) \) such that \( V f < \infty \),
\[
V_q f = \sum_{k \geq 0} (-1)^k (V(q\cdot))^k V f.
\]
(3.10)
So we obtain that
\[
V_q (qV f) = \sum_{k \geq 0} (-1)^k (V(q\cdot))^k [V(qV f)]
\]
\[
= \sum_{k \geq 1} (-1)^k (V(q\cdot))^k V f
\]
\[
= V f - V_q f.
\]
(3.11)
The second equality holds by integrating (3.6).

**Proposition 3.5.** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) such that \( V f \in L^1_{\text{loc}}(\mathbb{R}^n) \). Then \( V_q f \) is a solution (in the sense of distributions) of the perturbed polyharmonic equation (1.9).
Proof. Using the resolvent equation (3.9), we have
\[ V_q f = V f - V(qV_q f). \]  
(3.12)

Applying the operator \((-\Delta)^m\) on both sides of the above equality, we obtain that
\[ (-\Delta)^m(V_q f) = f - qV_q f \quad \text{(in the sense of distributions)}. \]  
(3.13)

This completes the proof. \(\square\)

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let \(c > 0\). Then by (H2), there exists a nonnegative function \(q := q_c \in K_{m,n}(\mathbb{R}^n)\), such that \(\alpha_q \leq 1/2\) and for each \(x \in \mathbb{R}^n\), the map
\[ t \rightarrow t(q(x) - \varphi(x,t)) \]  
is continuous and nondecreasing on \([0,c]\),
(3.14)

which implies in particular that for each \(x \in \mathbb{R}^n\) and \(t \in [0,c]\),
\[ 0 \leq \varphi(x,t) \leq q(x), \]  
(3.15)

Let
\[ \Lambda := \{ u \in B^+(\mathbb{R}^n) : (1 - \alpha_q)c \leq u \leq c \}. \]  
(3.16)

We define the operator \(T\) on \(\Lambda\) by
\[ Tu(x) := c(1 - V_q(q(x))) + V_q[(q - \varphi(\cdot,u))u](x). \]  
(3.17)

First, we prove that \(\Lambda\) is invariant under \(T\). Indeed, for each \(u \in \Lambda\), we have
\[ Tu \leq c(1 - V_q(q(x))) + cV_q(q(x)) \leq c. \]  
(3.18)

Moreover, from (3.15), (3.4) and Lemma 3.1 we deduce that for each \(u \in \Lambda\), we have
\[ Tu \geq c(1 - V_q(q(x))) \geq c(1 - V(q(x))) \geq c(1 - \alpha_q). \]  
(3.19)

Next, we prove that the operator \(T\) is nondecreasing on \(\Lambda\). Indeed, let \(u, v \in \Lambda\) such that \(u \leq v\), then from (3.14) we obtain that
\[ Tv - Tu = V_q([q - \varphi(\cdot,u)]v - [q - \varphi(\cdot,v)u]) \geq 0. \]  
(3.20)

Now, consider the sequence \((u_k)\) defined by \(u_0 = (1 - \alpha_q)c\) and \(u_{k+1} = Tu_k\), for \(k \in \mathbb{N}\). Then since \(\Lambda\) is invariant under \(T\), we obtain obviously that \(u_1 = Tu_0 \geq u_0\) and so from the monotonicity of \(T\), we have
\[ u_0 \leq u_1 \leq \cdots \leq u_k \leq c. \]  
(3.21)
So from (3.14) and the dominated convergence theorem we deduce that the sequence \((u_k)\) converges to a function \(u \in \Lambda\) which satisfies

\[
u = c(1 - V_q(q)(x)) + V_q[ (q - \varphi(\cdot, u)) u](x). \tag{3.22}
\]

That is

\[
u - V_q(qu) = c(1 - V_q(q)(x)) - V_q(u\varphi(\cdot, u)). \tag{3.23}
\]

Applying the operator \((I + V(q \cdot))\) on both sides of the above equality and using (3.9) we deduce that \(u\) satisfies

\[
u = c - V(u\varphi(\cdot, u)). \tag{3.24}
\]

Finally, we claim that \(u\) is a positive continuous solution for the Problem (1.6). To prove the claim, we use Lemma 2.4. Indeed, since \(u \sim c\) on \(\mathbb{R}^n\) and

\[
0 \leq u\varphi(\cdot, u) \leq cq, \tag{3.25}
\]

we deduce that either \(u\) and \(u\varphi(\cdot, u)\) are in \(L^1_{\text{loc}}(\mathbb{R}^n)\).

Now, from (3.24) we can easily see that \(V(u\varphi(\cdot, u)) \in L^1_{\text{loc}}(\mathbb{R}^n)\). Hence \(u\) satisfies (in the sense of distributions) the elliptic differential equation

\[
(-\Delta)^mu + u\varphi(\cdot, u) = f \quad \text{in} \, \mathbb{R}^n. \tag{3.26}
\]

On the other hand, it follows from (3.25) that \(u\varphi(\cdot, u) \in M_q\) and so by Proposition 2.8, we obtain that \(V(u\varphi(\cdot, u)) \in C^+_0(\mathbb{R}^n)\).

This implies by (3.24) that \(\lim_{|x| \to \infty} u(x) = c\), which completes the proof. \(\square\)

**Remark 3.6.** Let \(c > 0\) and \(u\) be a solution of (1.8). Then we have by Theorem 1.4 that for each \(x \in \mathbb{R}^n\), \(0 \leq u(x) \leq c\). Let \(q\) be the nonnegative function in \(K^\infty_{m,n}(\mathbb{R}^n)\) given in the proof of Theorem 1.4. Then we deduce from (3.24) and (3.25), that

\[
0 \leq c - u(x) = V(u\varphi(\cdot, u))(x) \leq cV(q)(x). \tag{3.27}
\]

**Example 3.7.** Let \(p > n/2m\) and \(a\) be a nonnegative function in \(L^p(\mathbb{R}^n)\). Let \(\lambda < 2m - n/p < \mu\) and \(\alpha, \beta\) be two nonnegative constants.

Put \(q(x) = a(x)/(1 + |x|)^{\mu-\lambda}|x|^{\lambda}\). Then, for each \(c > 0\), the following polyharmonic problem

\[
(-\Delta)^mu + \beta u^{\alpha+1}q = 0, \quad \text{in} \, \mathbb{R}^n \, (\text{in the sense of distributions})
\]

\[
\lim_{|x| \to \infty} u(x) = c,
\]

has a positive continuous solution satisfying \(c/2 \leq u(x) \leq c\), provided that \(\beta\) is sufficiently small.
Moreover, by Remark 3.6 and Proposition 2.15, we have

\[
0 \leq c - u(x) \leq c\|a\|_p \begin{cases} \frac{1}{(1 + |x|)^{n-2m}} \log ((|x| + 1)^{p/(p-1)}, & \text{if } \mu + \frac{n}{p} = n \\ \frac{1}{(1 + |x|)^{(n-2m)/(\mu + n/p - 2m)}}, & \text{if } \mu + \frac{n}{p} \neq n. \end{cases} \tag{3.29}
\]

**Remark 3.8.** It is interesting to compare the asymptotics (3.29) with the results of Trubek [10], for the case \( m = 1 \).

### 4. Second existence result

In this section, we aim at proving Theorem 1.5.

**Proof of Theorem 1.5.** Assuming \((H3)\) and \((H4)\), we will use the Schauder fixed point theorem. From (1.14), there exists \( \eta > 0 \) such that

\[
h(t) \geq m_0 t, \quad \text{for each } t \in [0, \eta].\tag{4.1}
\]

On the other hand, let \( \alpha \in (g^\infty, M_0) \), then by (1.15), there exists \( \rho > 0 \) such that for \( t \geq \rho \), we have \( g(t) \leq \alpha t \). Put \( \beta = \sup_{0 \leq t \leq \rho} g(t) \). So we deduce that

\[
0 \leq g(t) \leq \alpha t + \beta, \quad \text{for each } t \geq 0.\tag{4.2}
\]

By Remark 2.14, we note that there exists a constant \( \alpha_1 > 0 \) such that

\[
\frac{\alpha_1}{(1 + |x|)^{n-2m}} \leq Vq(x).\tag{4.3}
\]

Let \( a \in (0, \eta) \) and \( b = \max\{a/\alpha_1, \beta/(1 - a\|Vq\|_{\infty})\} \). So we consider the closed convex set

\[
\Lambda = \left\{ u \in C_0(\mathbb{R}^n), \ a \frac{u}{(1 + |x|)^{n-2m}} \leq u(x) \leq b Vq(x), \ \forall x \in \mathbb{R}^n \right\}.\tag{4.4}
\]

Obviously by (4.3) we have that the set \( \Lambda \) is nonempty. Next we define the operator \( T \) on \( \Lambda \) by

\[
Tu(x) = \int_{\mathbb{R}^n} G_{m,n}(x, y) f(y, u(y)) \, dy.\tag{4.5}
\]

Let us prove that \( T\Lambda \subset \Lambda \). Let \( u \in \Lambda \), then by (4.2) we have

\[
Tu(x) \leq \int_{\mathbb{R}^n} G_{m,n}(x, y) q(y) g(u(y)) \, dy \\
\leq \int_{\mathbb{R}^n} G_{m,n}(x, y) q(y) [\alpha u(y) + \beta] \, dy \\
\leq (\alpha b \|Vq\|_{\infty} + \beta) Vq(x) \\
\leq b Vq(x).\tag{4.6}
\]
Moreover, since $h$ is nondecreasing, we deduce by (4.1) and (1.14) that

\[ Tu(x) \geq \int_{\mathbb{R}^n} G_{m,n}(x,y) p(y) h(u(y)) dy \]
\[ \geq \int_{\mathbb{R}^n} G_{m,n}(x,y) \frac{a}{(1 + |y|)^{n-2m}} dy \]
\[ \geq m_0 a \int_{\mathbb{R}^n} G_{m,n}(x,y) \frac{p(y)}{(1 + |y|)^{n-2m}} dy \]
\[ \geq m_0 a k_{m,n} \int_{\mathbb{R}^n} \frac{p(y)}{(1 + |y|)^{2(n-2m)}} dy \]
\[ = \frac{a}{(1 + |x|)^{n-2m}}. \]  

(4.7)

On the other hand, by (1.13), we have that for each $u \in \Lambda$

\[ f(\cdot,u) \leq g(b \|Vq\|_{\infty}) q. \]  

(4.8)

This implies by Proposition 2.8 that $Tu \in V(M_q) \subset C_0(\mathbb{R}^n)$. So $T \Lambda \subset \Lambda$.

Next, we prove the continuity of $T$ in $\Lambda$. Let $(u_k)$ be a sequence in $\Lambda$, which converges uniformly to a function $u \in \Lambda$. Then using (4.8) and (H3), we deduce by Theorem 1.3 and the dominated convergence Theorem that for $x \in \mathbb{R}^n$,

\[ Tu_k(x) \to Tu(x) \quad \text{as} \quad k \to \infty. \]  

(4.9)

Now, since $T \Lambda \subset V(M_q)$, we deduce by Proposition 2.8 that $T \Lambda$ is relatively compact in $C_0(\mathbb{R}^n)$, which implies that

\[ \| Tu_k - Tu \|_{\infty} \to 0 \quad \text{as} \quad k \to \infty. \]  

(4.10)

Hence $T$ is a compact map from $\Lambda$ to itself. So the Schauder fixed point theorem leads to the existence of $u \in \Lambda$ such that

\[ u = V(f(\cdot,u)). \]  

(4.11)

Finally by (4.8) and Lemma 2.4, we conclude that $y \to f(y, u(y))$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, which together with (4.11) imply that $u$ satisfies (in the sense of distributions) the elliptic differential equation

\[ (-\Delta)^m u = f(\cdot,u) \quad \text{in} \quad \mathbb{R}^n. \]  

(4.12)

This ends the proof.

Example 4.1. Let $p$ be a nonnegative function in $K_{m,n}(\mathbb{R}^n)$ and $0 \leq \alpha < 1$. Then the following problem

\[ (-\Delta)^m u + p(x)u^\alpha = 0, \quad x \in \mathbb{R}^n, \]
\[ \lim_{|x| \to \infty} u(x) = 0, \]  

(4.13)
has a positive solution \( u \in C_0(\mathbb{R}^n) \) satisfying for each \( x \in \mathbb{R}^n \)

\[
\frac{1}{(1 + |x|)^{n-2m}} \leq u(x) \leq V p(x).
\]

(4.14)

5. Third existence result

In this section, we aim at proving Theorem 1.6.

Proof of Theorem 1.6. Let \( c > 0 \) be the constant given by (H7) and \( c^* = c - \| V(q(\cdot, c)) \|_\infty \). Let \( \delta \in (0, c^*] \). We will use the Schauder fixed point theorem, so we consider the closed convex set

\[
\Lambda = \{ u \in C(\mathbb{R}^n \cup \{ \infty \}) : \delta \leq u(x) \leq c, \forall x \in \mathbb{R}^n \}
\]

(5.1)

and we define the integral operator \( T \) on \( \Lambda \) by

\[
Tu(x) = \delta + V(f(\cdot, u))(x).
\]

(5.2)

First, we prove that \( T\Lambda \subset \Lambda \). Let \( u \in \Lambda \), then since \( f \) is a nonnegative function, we have that \( Tu(x) \geq \delta \), for each \( x \in \mathbb{R}^n \). Moreover by (H6), we have for \( x \in \mathbb{R}^n \),

\[
Tu(x) \leq \delta + V(q(\cdot, u))(x) \leq c^* + V(q(\cdot, c))(x) \leq c.
\]

(5.3)

Furthermore by (H7), since for all \( u \in \Lambda \), \( f(\cdot, u) \in M_q(c) \), then it follows from Proposition 2.8 that \( V(f(\cdot, u)) \in C_0(\mathbb{R}^n) \) and more precisely \( T\Lambda \) is relatively compact in \( C(\mathbb{R}^n \cup \{ \infty \}) \). Therefore \( T\Lambda \subset \Lambda \).

Next, let us prove the continuity of \( T \) in \( \Lambda \). Let \( (u_k) \) be a sequence in \( \Lambda \), which converges uniformly to a function \( u \in \Lambda \). Since \( f \) is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for each \( x \in \mathbb{R}^n \cup \{ \infty \} \),

\[
Tu_k(x) \longrightarrow Tu(x) \quad \text{as} \quad k \longrightarrow \infty.
\]

(5.4)

Now, since \( T\Lambda \) is relatively compact in \( C(\mathbb{R}^n \cup \{ \infty \}) \), then

\[
\| Tu_k - Tu \|_\infty \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.
\]

(5.5)

Finally the Schauder fixed point theorem implies the existence of \( u \in \Lambda \) such that

\[
u(x) = \delta + V(f(\cdot, u))(x), \quad \forall x \in \mathbb{R}^n.
\]

(5.6)

Using (H6), (H7) and Lemma 2.4, we deduce that the function \( y \rightarrow f(y, u(y)) \) is in \( L^1_{loc}(\mathbb{R}^n) \). So \( u \) satisfies (in the sense of distributions) the elliptic differential equation

\[
(-\triangle)^m u = f(\cdot, u) \quad \text{in} \quad \mathbb{R}^n.
\]

(5.7)

Moreover since \( V(f(\cdot, u)) \in C_0(\mathbb{R}^n) \), then by (5.6) it follows that \( \lim_{|x| \rightarrow \infty} u(x) = \delta \). This ends the proof.
Corollary 5.1. Assume that \( q(x,t) = p(x)g(t) \), where \( g \) is a nonnegative nondecreasing measurable function and \( p \) is a nonnegative function in \( K_{m,n}^{∞}(\mathbb{R}^n) \). If the function \( g \) satisfies either \( g(t) = o(t) \) as \( t \to 0 \) or \( g(t) = o(t) \) as \( t \to ∞ \), then the problem (1.19) has a positive solution \( u \in C(\mathbb{R}^n \cup \{∞\}) \).

Example 5.2. Among the equations of form (1.1), we have the Emden-Fowler equation of order \( m \)

\[
(-\triangle)^m u + p(x) u^α = 0, \quad α > 0, \quad x \in \mathbb{R}^n, \quad n > 2m, \quad (5.8)
\]

where \( p \in K_{m,n}^{∞}(\mathbb{R}^n) \).

(i) For the sublinear (\( 0 < α < 1 \)) or the superlinear (\( α > 1 \)) case, let \( c > 0 \) such that

\[
\|Vp\|_{∞} c^{α-1} < 1. \quad (5.9)
\]

Then applying Theorem 1.6, we deduce that for each \( δ \in (0, c(1 - c^{α-1}\|Vp\|_{∞})) \), (5.8) with \( α \neq 1 \) has a continuous positive solution \( u \) in \( \mathbb{R}^n \) with \( δ \leq u(x) \leq c \), for all \( x \in \mathbb{R}^n \) and \( \lim_{|x|→∞} u(x) = δ \).

(ii) For the linear case (\( α = 1 \)). If \( \|Vp\|_{∞} < 1 \), then applying Theorem 1.6, we deduce that for each \( c > 0 \) and \( δ \in (0, c(1 - ∥Vp∥_{∞})) \), (5.8) has a continuous positive solution \( u \) in \( \mathbb{R}^n \) with \( δ \leq u(x) \leq c \), for all \( x \in \mathbb{R}^n \) and \( \lim_{|x|→∞} u(x) = δ \).

Remark 5.3. We improve in this section the Yin’s result in [11]. Indeed, Yin proved in particular the existence of bounded positive solutions for the Emden-Fowler equation

\[
\triangle u + p(x) u^α = 0, \quad 0 < α \neq 1, \quad x \in \mathbb{R}^n, \quad n ≥ 3, \quad (5.10)
\]

provided that the function \( p \) satisfies

\[
\int_0^∞ s \max_{|x|=s} \{ p(x) \} \ ds < ∞. \quad (5.11)
\]

However by taking \( λ > (n - 1)/2 \) and

\[
p(x) = p(x', x_n) = \frac{1}{(1 + x_n^2) (1 + ∑_{i=1}^{n-1} x_i^2)^λ}, \quad x \in \mathbb{R}^n, \quad (5.12)
\]

then we have

\[
\max_{|x|=s} p(x) ≥ p(0,s) = \frac{1}{1 + s^2} \quad (5.13)
\]

which implies that (5.11) is not satisfied. On the other hand, we have that \( p \in L^∞(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset K_{m,n}^{∞}(\mathbb{R}^n) \). This implies by Corollary 5.1 that the Emden-Fowler equation (5.8) has a positive solution \( u \in C(\mathbb{R}^n \cup \{∞\}) \), for each \( m ≥ 1. \)
References


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