One of our main results is the following convergence theorem for one-parameter nonexpansive semigroups: let $C$ be a bounded closed convex subset of a Hilbert space $E$, and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $u \in C$ and $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by $x_n = \frac{(1 - \alpha_n)}{(t_2 - t_1)} \int_{t_1}^{t_2} T(s)x_n ds + \alpha_n u$ for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

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1. Introduction

Let $C$ be a closed convex subset of a Banach space $E$, and let $T$ be a nonexpansive mapping on $C$, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that $T$ has a fixed point in the case that $E$ is uniformly convex and $C$ is bounded; see Browder [4], Göhde [10], and Kirk [15]. We denote by $F(T)$ the set of fixed points of $T$.

Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings (nonexpansive semigroup, in short) on a closed convex subset $C$ of a Banach space $E$, that is,

(i) for each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on $C$;
(ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;
(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $\mathbb{R}_+$ into $C$ is strongly continuous.

We also know that $\{T(t) : t \in \mathbb{R}_+\}$ has a common fixed point in the case that $E$ is uniformly convex and $C$ is bounded; see Browder [4]. Bruck [7] prove the following theorem.

Theorem 1.1 (Bruck [7]). Suppose a closed convex subset $C$ of a Banach space has the fixed point property for nonexpansive mappings, and $C$ is either weakly compact, or bounded and separable. Then for any commuting family $S$ of nonexpansive mappings on $C$, the set of common fixed points of $S$ is a nonempty nonexpansive retract of $C$. 

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2006, Article ID 58684, Pages 1–10
DOI 10.1155/AAA/2006/58684
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This theorem yields that \( \{ T(t) : t \in \mathbb{R}_+ \} \) has a common fixed point in the case that \( C \) has the fixed point property, and that \( C \) is weakly compact, or bounded and separable.

Several authors have studied about convergence theorems for nonexpansive semigroups; see [1, 2, 13, 16, 19, 21, 22] and others. For example, the following theorem is a corollary of Theorem 8 in [19].

**Theorem 1.2 (Shioji and Takahashi [19]).** Let \( C \) be a bounded closed convex subset of a Hilbert space \( E \). Let \( \{ T(t) : t \in \mathbb{R}_+ \} \) be a strongly continuous semigroup of nonexpansive mappings on \( C \). Let \( \{ \alpha_n \} \) and \( \{ t_n \} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, \lim_n \alpha_n = 0, t_n > 0 \) and \( \lim_n t_n = \infty \). Fix \( u \in C \) and define a sequence \( \{ x_n \} \) in \( C \) by

\[
x_n = \frac{1 - \alpha_n}{t_n} \int_0^{t_n} T(s)x_n \, ds + \alpha_n u
\]

for \( n \in \mathbb{N} \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

Also, Suzuki [21] proved the following theorem.

**Theorem 1.3 (Suzuki [21]).** Let \( E, C, \{ T(t) : t \in \mathbb{R}_+ \} \) be as in Theorem 1.2. Let \( \{ \alpha_n \} \) and \( \{ t_n \} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, t_n > 0 \) and \( \lim_n t_n = \lim_n \alpha_n / t_n = 0 \). Fix \( u \in C \) and define a sequence \( \{ x_n \} \) in \( C \) by

\[
x_n = (1 - \alpha_n) T(t_n)x_n + \alpha_n u
\]

for \( n \in \mathbb{N} \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

We note that in these theorems, real sequences \( \{ t_n \} \) converge to 0 and \( \infty \). So, it is natural to study convergence theorems under the assumption that \( \{ t_n \} \) is a constant sequence. In this paper, motivated by Theorems 1.2 and 1.3, we consider such type of convergence theorems to a common fixed point of \( \{ T(t) : t \in \mathbb{R}_+ \} \).

2. Preliminaries

Throughout this paper we denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{R}_+ \) the set of nonnegative real numbers, and by \( \mathbb{N} \) the set of positive integers. For a Banach space \( E \), we also denote by \( E^* \) the dual space of \( E \).

We recall that a Banach space \( E \) is called strictly convex if \( \| x + y \| / 2 < 1 \) for all \( x, y \in E \) with \( \| x \| = \| y \| = 1 \) and \( x \neq y \). We know the following lemma.

**Lemma 2.1.** Let \( E \) be a Banach space. Then the following are equivalent:

(i) \( E \) is strictly convex;

(ii) \( \| \lambda x + (1 - \lambda) y \| < 1 \) for all \( \lambda \in (0, 1) \) and \( x, y \in E \) with \( \| x \| = \| y \| = 1 \) and \( x \neq y \);

(iii) if \( \| x \| = \| y \| = \| \lambda x + (1 - \lambda) y \| \) for some \( \lambda \in (0, 1) \), then \( x = y \).

A Banach space \( E \) is called uniformly convex if for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \| x + y \| / 2 < 1 - \delta \) for all \( x, y \in E \) with \( \| x \| = \| y \| = 1 \) and \( \| x - y \| \geq \epsilon \). It is clear that a uniformly convex Banach space is strictly convex. The norm of \( E \) is called Fréchet differentiable if for each \( x \in E \) with \( \| x \| = 1 \), \( \lim_{t \to 0^+} (\| x + ty \| - \| x \|) / t \) exists and is attained uniformly in \( y \in E \) with \( \| y \| = 1 \). A Banach space \( E \) is said to have the Opial property [17]
if for each weakly convergent sequence \( \{x_n\} \) in \( E \) with weak limit \( z \), \( \liminf_n \|x_n - z\| < \liminf_n \|x_n - y\| \) for all \( y \in E \) with \( y \neq z \). All Hilbert spaces, all finite dimensional Banach spaces and \( \ell^p (1 \leq p < \infty) \) have the Opial property. Gossez and Lami Dozo\[11\] prove that every weakly compact convex subset of a Banach space with the Opial property has normal structure. We also know that every separable Banach space can be equivalently renormed so that it has the Opial property; see \[23\].

3. Common fixed points

In this section, we give our main results. The following proposition plays an important role in this paper.

**Proposition 3.1.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \). Let \( \tau_\infty > 0 \) and let \( \{T(t) : t \in [0, \tau_\infty)\} \) be a family of mappings on \( C \) satisfying the following:

(i) for each \( t \in [0, \tau_\infty) \), \( T(t) \) is nonexpansive;

(ii) there exists a strictly increasing sequence \( \{\tau_n\} \) in \( [0, \tau_\infty) \) such that \( \tau_1 = 0 \), \( \{\tau_n\} \) converges to \( \tau_\infty \), and mappings \( t \mapsto T(t)x \) are weakly continuous on \( [\tau_n, \tau_{n+1}) \) for all \( x \in C \) and \( n \in \mathbb{N} \).

Suppose that

\[
\bigcap_{t \in [0, \tau_\infty)} F(T(t)) \neq \emptyset. \tag{3.1}
\]

Then

\[
\bigcap_{t \in [0, \tau_\infty)} F(T(t)) = F(S), \tag{3.2}
\]

where \( S \) is a nonexpansive mapping on \( C \) defined by

\[
Sx = \frac{1}{\tau_\infty} \int_0^{\tau_\infty} T(s)x \, ds \tag{3.3}
\]

for all \( x \in C \).

**Remark 3.2.** We do not assume \( \{T(\cdot)\} \) is a nonexpansive semigroup.

**Proof.** Fix \( f \in E^* \). Then the functions \( t \mapsto f(T(t)x) \) from \( [\tau_n, \tau_{n+1}) \) into \( \mathbb{R} \) are continuous on \( [\tau_n, \tau_{n+1}) \) for \( x \in C \) and \( n \in \mathbb{N} \). So, the functions \( t \mapsto f(T(t)x) \) from \( [0, \tau_\infty) \) into \( \mathbb{R} \) are measurable for \( x \in C \). We also have \( \{T(t)x : t \in [0, \tau_\infty)\} \) is separable for each \( x \in C \). Fix \( w \in \bigcap_{t \in [0, \tau_\infty)} F(T(t)) \). Since

\[
\|T(t)x\| = \|T(t)x\| - \|T(t)w\| + \|w\| \leq \|T(t)x - T(t)w\| + \|w\| \\
\leq \|x - w\| + \|w\|, \tag{3.4}
\]

for \( x \in C \) and \( t \in [0, \tau_\infty) \), we have that the mappings \( t \mapsto T(t)x \) are Bochner integrable for all \( x \in C \) and hence \( S \) is well-defined. Using the separation theorem, we can easily prove
that $S$ is a mapping on $C$. Since
\[
\|Sx - Sy\| = \left| \frac{1}{\tau_\infty} \int_0^{\tau_\infty} (T(s)x - T(s)y) \, ds \right| \\
\leq \frac{1}{\tau_\infty} \int_0^{\tau_\infty} \| T(s)x - T(s)y \| \, ds \\
\leq \frac{1}{\tau_\infty} \int_0^{\tau_\infty} \| x - y \| \, ds = \| x - y \|
\]
(3.5)
for $x,y \in C$, $S$ is nonexpansive. Therefore $S$ is a nonexpansive mapping on $C$. It is obvious that $\bigcap_{t \in [0,\tau_\infty]} F(T(t)) \subset F(S)$. We assume that $z \in F(S) \setminus \bigcap_{t \in [0,\tau_\infty]} F(T(t))$. Then there exists $t_1 \in [0,\tau_\infty)$ such that $T(t_1)z \neq z$. Fix $g \in E^*$ with
\[
\|g\| = 1, \quad g(T(t_1)z - z) = \| T(t_1)z - z \|.
\]
(3.6)
For some $m \in \mathbb{N}$, $t_1$ belongs to $[\tau_m, \tau_{m+1})$. From the assumption (ii), there exists $t_2 \in (t_1, \tau_{m+1})$ such that
\[
g(T(t)z - z) > \frac{1}{2}\| T(t_1)z - z \|
\]
(3.7)
for all $t \in [t_1, t_2)$. Define nonexpansive mappings $S_1$ and $S_2$ on $C$ by
\[
S_1x = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x \, ds,
\]
\[
S_2x = \frac{1}{\tau_\infty - t_2 + t_1} \left( \int_{t_1}^{t_2} T(s)x \, ds + \int_{t_1}^{\tau_\infty} T(s)x \, ds \right)
\]
(3.8)
for all $x \in C$. We note that
\[
Sx = \frac{t_2 - t_1}{\tau_\infty} S_1x + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} S_2x
\]
(3.9)
for all $x \in C$. We have
\[
g(S_1z - Sz) = g \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)z \, ds - z \right)
\]
\[
= g \left( \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (T(s)z - z) \, ds \right)
\]
\[
= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(T(s)z - z) \, ds
\]
\[
\geq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{2}\| T(t_1)z - z \| \, ds
\]
\[
= \frac{1}{2}\| T(t_1)z - z \| > 0.
\]
(3.10)
Hence
\[
g(S_2z - Sz) = \frac{t_2 - t_1}{\tau_\infty - t_2 + t_1} g(Sz - S_1z) < 0.
\]
(3.11)
Therefore $S_1 z \neq S_2 z$. Fix $w \in \bigcap_{t \in [0, \tau_n)} F(T(t))$. Then we note that $S_1 w = S_2 w = w$. We have

\[
\|z - w\| = \|S z - w\| = \left\| \frac{t_2 - t_1}{\tau_\infty} S_1 z + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} S_2 z - w \right\|
\leq \frac{t_2 - t_1}{\tau_\infty} \|S_1 z - w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|S_2 z - w\|
= \frac{t_2 - t_1}{\tau_\infty} \|z - w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|z - w\| = \|z - w\|
\tag{3.12}
\]

and hence

\[
\|S z - w\| = \|S_1 z - w\| = \|S_2 z - w\|. \tag{3.13}
\]

This contradicts the strict convexity of $E$. Therefore, $F(S) \subset \bigcap_{t \in [0, \tau_n)} F(T(t))$. This completes the proof. \qed

As a direct consequence of Proposition 3.1, we can prove the following, which was proved by Bruck [6]; see also [20].

**Corollary 3.3 (Bruck [6]).** Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on $C$. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \alpha_n = 1$. Define a nonexpansive mapping $S$ on $C$ by

\[
S x = \sum_{n=1}^{\infty} \alpha_n T_n x \tag{3.14}
\]

for $x \in C$. Then $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

**Proof.** Define a strictly increasing sequence $\{\tau_n\}$ in $[0, 1)$ by $\tau_1 = 0$ and

\[
\tau_n = \sum_{k=1}^{n-1} \alpha_k \tag{3.15}
\]

for $n \in \mathbb{N}$ with $n \geq 2$. We note that $\lim_n \tau_n = 1$. Define a family $\{T(t) : t \in [0, 1)\}$ of nonexpansive mappings as follows: If $\tau_n \leq t < \tau_{n+1}$, then

\[
T(t)x = T_n x \tag{3.16}
\]

for all $x \in C$. Then we note that

\[
S x = \sum_{n=1}^{\infty} \alpha_n T_n x = \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} T(s)x \; ds = \int_{0}^{1} T(s)x \; ds = \frac{1}{\tau_\infty} \int_{0}^{1} T(s)x \; ds \tag{3.17}
\]
for $x \in C$ and

$$\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{t \in [0,1)} F(T(t)). \quad (3.18)$$

So, by Proposition 3.1, we obtain the desired result.

As another direct consequence of Proposition 3.1, we obtain the following proposition.

**Proposition 3.4.** Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let $	au > 0$ and let $\{T(t) : t \in [0, \tau)\}$ be a family of mappings on $C$ satisfying the following:

(i) for each $t \in [0, \tau)$, $T(t)$ is nonexpansive;

(ii) mappings $t \mapsto T(t)x$ are weakly continuous on $[0, \tau)$ for all $x \in C$.

Suppose that

$$\bigcap_{t \in [0,\tau)} F(T(t)) \neq \emptyset. \quad (3.19)$$

Then

$$\bigcap_{t \in [0,\tau)} F(T(t)) = F(S), \quad (3.20)$$

where $S$ is a nonexpansive mapping on $C$ defined by

$$Sx = \frac{1}{\tau} \int_{0}^{\tau} T(s)x \, ds \quad (3.21)$$

for all $x \in C$.

Now, we prove one of our main results.

**Theorem 3.5.** Let $C$ be a closed convex subset of a strictly convex Banach space $E$ and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Suppose that

$$\bigcap_{t \in \mathbb{R}_+} F(T(t)) \neq \emptyset. \quad (3.22)$$

Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, and define a nonexpansive mapping $S$ on $C$ by

$$Sx = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x \, ds \quad (3.23)$$

for all $x \in C$. Then

$$\bigcap_{t \in \mathbb{R}_+} F(T(t)) = F(S) \quad (3.24)$$

holds.
Proof. It is clear that \( \bigcap_{t \in R^+} F(T(t)) \subset F(S) \). By Proposition 3.4, we have
\[
\bigcap_{t \in [t_1, t_2]} F(T(t)) = F(S). \tag{3.25}
\]
So, \( T(t)w = w \) for \( t \in [t_1, t_2] \). Hence, for every \( t \in [0, (t_2 - t_1)/2] \), we have
\[
T(t)w = T(t) \circ T(t_1)w = T(t + t_1)w = w. \tag{3.26}
\]
Let \( t \in R^+ \) be fixed. Then there exist \( m \in N \cup \{0\} \) and \( u \in [0, (t_2 - t_1)/2] \) such that \( t = u + m(t_2 - t_1)/2 \). We have
\[
T(t)w = T(u + m(t_2 - t_1)/2)w = T(u) \circ T((t_2 - t_1)/2)^m w = T(u)w = w, \tag{3.27}
\]
where \( T((t_2 - t_1)/2)^0 \) is the identity mapping on \( C \). Therefore \( w \) is a common fixed point of \( \{T(t) : t \in R^+\} \). This completes the proof. \( \square \)

Similarly we can prove the following theorem.

**Theorem 3.6.** Let \( C \) be a closed convex subset of a strictly convex Banach space \( E \) and let \( \{T_n(t) : t \in R^+ : n \in N\} \) be a sequence of strongly continuous semigroups of nonexpansive mappings on \( C \). Let \( \{U_n : n \in N\} \) be a sequence of nonexpansive mappings on \( C \). Suppose that
\[
\bigcap_{n=1}^{\infty} \bigcap_{t \in R^+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) \neq \emptyset. \tag{3.28}
\]
Let \( \{t_n\}, \{u_n\}, \{\alpha_n\} \) and \( \{\beta_n\} \) be real sequences such that \( 0 \leq t_n < u_n, \alpha_n > 0 \) and \( \beta_n > 0 \) for all \( n \in N \), and \( \sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \beta_n = 1 \). Define a nonexpansive mapping \( S \) on \( C \) by
\[
Sx = \sum_{n=1}^{\infty} \frac{\alpha_n}{u_n - t_n} \int_{t_n}^{u_n} T_n(s)x \, ds + \sum_{n=1}^{\infty} \beta_n U_n x \tag{3.29}
\]
for all \( x \in C \). Then
\[
\bigcap_{n=1}^{\infty} \bigcap_{t \in R^+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S). \tag{3.30}
\]
holds.

We recall that a closed convex subset \( C \) of a Banach space \( E \) is said to have the fixed point property for nonexpansive mappings (FPP, in short) if for every bounded closed convex subset \( D \) of \( C \), every nonexpansive mapping on \( D \) has a fixed point. So, by the results of Browder [4] and Göhde [10], every uniformly convex Banach space has FPP. Also, by Kirk’s fixed point theorem [15], every weakly compact convex subset with normal structure has FPP.

As a direct consequence of Theorem 3.6, we obtain the following corollary.
Corollary 3.7. Let $E$, $C$, \{\{T_n(t) : t \in \mathbb{R}_+ \} : n \in \mathbb{N}\}$, \{\{U_n : n \in \mathbb{N}\} \}, \{t_n\}, \{u_n\}, \{\alpha_n\}, and \{\beta_n\} be as in Theorem 3.6. Assume that $C$ is weakly compact and has FPP, and
\[
T_m(s) \circ T_n(t) = T_n(t) \circ T_m(s), \quad U_m \circ U_n = U_n \circ U_m, \quad U_m \circ T_n(t) = T_n(t) \circ U_m
\]
for all $s, t \in \mathbb{R}_+$ and $m, n \in \mathbb{N}$. Define a nonexpansive mapping $S$ on $C$ as in Theorem 3.6. Then
\[
\bigcap_{n=1}^{\infty} \bigcap_{t \in \mathbb{R}_+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S) \neq \emptyset.
\]
holds.

4. Convergence theorems

Using Theorem 3.5, we can prove many convergence theorems to a common fixed point of nonexpansive semigroups. In this section, we state some of them.

From the result of Ishikawa [14], we obtain the following theorem see also Edelstein [8].

Theorem 4.1. Let $E$ be a compact convex subset of a strictly convex Banach space $E$. Let \{\{T(t) : t \in \mathbb{R}_+\} : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and
\[
x_{n+1} = \frac{\alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + (1 - \alpha_n)x_n
\]
for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

From the results of Edelstein and O’Brien [9], and Reich [18], we obtain the following theorem.

Theorem 4.2. Let $E$ be a Banach space. Suppose either of the following holds:

(i) $E$ is strictly convex and has the Opial property; or
(ii) $E$ is uniformly convex and its norm is Fréchet differentiable.

Let $C$ be a weakly compact convex subset of $E$, and let \{\{T(t) : t \in \mathbb{R}_+\} : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and
\[
x_{n+1} = \frac{\alpha}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + (1 - \alpha)x_n
\]
for $n \in \mathbb{N}$, where $\alpha$ is a constant number in $(0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

We note that
\[
x \mapsto (1 - \alpha)Tx + \alpha u
\]
is a contractive mapping if $T$ is a nonexpansive mapping and $\alpha \in (0, 1)$. By the Banach contraction principle [3], such mappings have a unique fixed point. From the results of Browder [5], and Wittmann [24], we obtain the following theorem; see also [12]. Compare Theorem 4.3 with Theorems 1.2 and 1.3.

**Theorem 4.3.** Let $C$ be a bounded closed convex subset of a Hilbert space $E$, and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$. Fix $u \in C$ and $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in $C$ by

$$x_n = \frac{1 - \alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u \quad (4.4)$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

**Theorem 4.4.** Let $E, C, \{T(t) : t \in \mathbb{R}_+\}, u, t_1$ and $t_2$ be as in Theorem 4.3. Define a sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and

$$x_{n+1} = \frac{1 - \alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u \quad (4.5)$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying the following:

$$\lim_{n \to \infty} \alpha_n = 0; \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \quad (4.6)$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

**Acknowledgment**

The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

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