A THREE-POINT BOUNDARY VALUE PROBLEM
WITH AN INTEGRAL CONDITION FOR A
THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION

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We prove the existence and uniqueness of a strong solution for a linear third-order equa-
tion with integral boundary conditions. The proof uses energy inequalities and the den-
sity of the range of the operator generated.

1. Introduction

In the rectangle $\Omega = (0,1) \times (0,T)$, we consider the equation

$$f(x,t) = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right)$$  \hspace{1cm} (1.1)

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1),$$  \hspace{1cm} (1.2)

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1),$$  \hspace{1cm} (1.3)

the Dirichlet condition

$$u(0,t) = 0 \quad \forall t \in (0,T),$$  \hspace{1cm} (1.4)

and the integral condition

$$\int_0^1 u(x,t) dx = 0, \quad 0 \leq l < 1, \quad t \in (0,T).$$  \hspace{1cm} (1.5)
In addition, we assume that the function $a(x,t)$ and its derivatives satisfy the conditions

$$0 < a_0 < a(x,t) < a_1 \quad \forall x, t \in \Omega,$$

$$\left| \frac{\partial a}{\partial x} \right| \leq b \quad \forall x, t \in \Omega,$$  \hspace{1cm} (1.6)

$$c_k' < \frac{\partial^k u}{\partial t^k}(x,t) < c_k \quad \forall x, t \in \Omega, \quad k = 1, 3,$$ with $c'_1 > 0$.

Over the last few years, many physical phenomena were formulated into nonlocal mathematical models with integral boundary conditions [1, 9, 10, 11]. The reader should refer to [13, 14] and the references therein. The importance of these kinds of problems has also been pointed out by Samarskii [22]. This type of boundary value problems has been investigated in [2, 3, 4, 6, 7, 8, 12, 18, 19, 20, 23, 25] for parabolic equations, in [21, 24] for hyperbolic equations, and in [15, 16, 17] for mixed-type equations. The basic tool in [5, 15, 16, 17, 20, 25] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation.

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of the problem (1.1)–(1.5). For this, we consider the solution of problem (1.1)–(1.5) as a solution of the operator equation

$$Lu = \mathcal{F},$$  \hspace{1cm} (2.1)

where the operator $L$ has domain of definition $D(L)$ consisting of functions $u \in L^2(\Omega)$ such that $(\partial^{k+1} u/\partial t^k \partial x)(x,t) \in L^2(\Omega)$, $k = 1, 3$ and satisfying the conditions (1.4)-(1.5).

The operator $L$ is considered from $E$ to $F$, where $E$ is the Banach space consisting of function $u \in L^2(\Omega)$, with the finite norm

$$\|u\|_E^2 = \int_\Omega \Theta(x) \left[ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx dt$$

$$+ \int_\Omega \Theta(x) \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \right] dx dt$$

$$+ \int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx dt.$$  \hspace{1cm} (2.2)

$F$ is the Hilbert space of functions $\mathcal{F} = (f,0,0,0)$, $f \in L^2(\Omega)$, with the finite norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega \Theta(x) \left| f(x,t) \right|^2 dx dt,$$  \hspace{1cm} (2.3)
\[ \Theta(x) = \begin{cases} (1 - l)^2, & 0 < x \leq l, \\ (1 - x)^2, & l \leq x < 1, \end{cases} \]

\[ \Phi(x) = \begin{cases} 0, & 0 < x < l, \\ 1, & l \leq x < 1. \end{cases} \]

3. An energy inequality and its application

**Theorem 3.1.** For any function \( u \in D(L) \), the a priori estimate

\[ \|u\|_E \leq k \|Lu\|_F \quad \text{for } u \in D(L), \quad (3.1) \]

where \( k^2 = 40\exp(cT)/k_1 \) with \( k_1 = \inf \{1/4, (c_3' - 3cc_1' + 3c^2c_1 - c^3a_1 - b^2)/2, a_0^2/2, (3/2)(ca_0 - c_1)\} \). The constant \( c \) satisfies

\[ \sup_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} \right) < c < \inf_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} + 1 \right), \]

\[ c_3' - 3cc_1' + 3c^2c_1 - c^3a_1 - b^2 > 0, \]

\[ c_2' - 2cc_1' + c^2a_1^2 + ca_0 - c_1 > 0. \]

**Proof.** Let

\[ Mu = \begin{cases} (1 - l)^2 \frac{\partial^3 u}{\partial t^3}, & 0 < x < l, \\ (1 - x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1 - x)J_x \frac{\partial^3 u}{\partial t^3}, & l \leq x < 1, \end{cases} \]

where \( J_x u = \int_x^1 u(x,t) \, dx \).

We consider the quadratic form obtained by multiplying (1.1) by \( \exp(-ct)\overline{Mu} \), with the constant \( c \) satisfying (3.2), integrating over \( \Omega = (0,1) \times (0,T) \), and taking the real part:

\[ \Phi(u,u) = \text{Re} \int_\Omega \exp(-ct) f(x,t) \overline{Mu} dx \, dt. \]
By substituting the expression of \( Mu \) in (3.4), integrating with respect to \( x \), and using the Dirichlet and integral conditions, we obtain

\[
\text{Re} \left[ \int_{\Omega} \exp(-ct) f(x,t) Mu \, dx \, dt \right]
\]

\[
= \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_0^1 \frac{\Theta(x)}{2} \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c \frac{\partial a}{\partial t} - c^3 a \right] \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_1^l \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial x^3} \right|^2 \, dx \, dt
\]

(3.5)

Integrating by parts \(-2 \text{Re} \int_0^T \int_0^1 \exp(-ct)a(x,t)u(\frac{\partial^3 u}{\partial t^3}) \, dx \, dt\) with respect to \( t \), and using the initial conditions, the final conditions, and the elementary inequalities, we obtain

\[
\int_0^T \int_0^1 \frac{\Theta(x)}{2} \exp(-ct) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_0^1 \frac{\Theta(x)}{2} \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c \frac{\partial a}{\partial t} - c^3 a \right] \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_0^1 \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial x^3} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_0^1 \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c \frac{\partial a}{\partial t} - c^3 a \right] u^2 \, dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_0^1 \frac{\Theta(x)}{2} \exp(-ct) \left[ a - \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt
\]

and using the initial conditions, the final conditions, and the elementary inequalities, we obtain
− \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right] \left| \frac{\partial u}{\partial x} \right|^2 dx|_{t=T} \\
+ \int_0^1 \Phi(x) \exp(-ct) \left[ a - \left| \frac{\partial a}{\partial t} - ca \right| \right] \left| \frac{\partial u}{\partial x} \right|^2 dx|_{t=T} \\
- \int_0^1 \Phi(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right] |u|^2 dx|_{t=T} \\
\leq 17 \int_0^T \int_1^0 \Theta(x) \exp(-ct) |f|^2 dx dt. 

(3.6)

From (1.1), we get

\int_\Omega \Theta(x) a^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx dt \\
\leq 2 \int_\Omega \Theta(x) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt + 2 \int_\Omega \Theta(x) \left( \frac{\partial a}{\partial x} \right)^2 \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\
+ 4 \int_\Omega \Theta(x) |f|^2 dx dt. 

(3.7)

Combining this last inequality with (3.6) and using the conditions (3.2) yield

\int_\Omega \Theta(x) \left[ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx dt \\
+ \int_\Omega \Theta(x) \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \right] dx dt + \int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx dt 

(3.8)

\leq k \int_\Omega \Theta(x) |f(x,t)|^2 dx dt,

which is the desired inequality.

It can be proved in a standard way that the operator \( L : E \to F \) is closable. Let \( \overline{L} \) be the closure of this operator, with the domain of definition \( D(\overline{L}) \).

Definition 3.2. A solution of the operator equation \( \overline{L}u = \overline{F} \) is called a strong solution of problem (1.1)–(1.5).

The a priori estimate (3.1) can be extended to strong solutions, that is, we have the estimate

\[ \|u\|_E \leq c \|\overline{L}u\|_F \quad \forall u \in D(\overline{L}). \]

(3.9)

This last inequality implies the following corollaries.

Corollary 3.3. A strong solution of (1.1)–(1.5) is unique and depends continuously on \( \overline{F} \).

Corollary 3.4. The range \( R(\overline{L}) \) of \( \overline{L} \) is closed in \( F \) and \( R(\overline{L}) = R(L) \).
Corollary 3.4 shows that to prove that problem (1.1)–(1.5) has a strong solution for arbitrary \( \mathfrak{F} \), it suffices to prove that set \( R(L) \) is dense in \( F \).

4. Solvability of problem (1.1)–(1.5)

To prove the solvability of problem (1.1)–(1.5) it is sufficient to show that \( R(L) \) is dense in \( F \). The proof is based on the following lemma.

**Lemma 4.1.** Suppose that the function \( a(x, t) \) and its derivatives are bounded. Let \( u \in D_0(L) = \{ u \in D(L), u(x, 0) = 0, \frac{\partial u}{\partial t}(x, 0) = 0, (\partial^2 u/\partial t^2)(x, T) = 0 \} \). If for \( u \in D_0(L) \) and some functions \( w(x, t) \in L^2(\Omega) \),

\[
\int_{\Omega} h(x) f \, w \, dx \, dt = 0, \tag{4.1}
\]

where

\[
h(x) = \begin{cases} 
1 - l, & 0 < x < l, \\
1 - x, & l < x < 1,
\end{cases}
\]

holds, for arbitrary \( u \in D_0(L) \), and then \( w = 0 \).

**Proof.** The equality (4.1) can be written as follows:

\[
\int_{\Omega} h(x) \frac{\partial^3 u}{\partial t^3} \, \bar{w} \, dx \, dt = \int_{\Omega} A(t)u \bar{v} \, dx \, dt, \tag{4.3}
\]

for a given \( v(x, t) \), where

\[
v = \begin{cases} 
(1 - l)w, & 0 < x < l, \\
w - \int_l^x \frac{w}{1 - \zeta} d\zeta, & l < x < 1,
\end{cases}
\]

\[
A(t)u = \frac{\partial}{\partial x} \left( h(x) a(x, t) \frac{\partial u}{\partial x} \right),
\]

\[
Nv = \begin{cases} 
(1 - l)v, & 0 < x < l, \\
(1 - x)v + J_x v, & l < x < 1.
\end{cases}
\]

For \( v = w - \int_l^x w/(1 - \zeta) d\zeta, l < x < 1 \) we deduce \( \int_l^x v(\zeta, t) d\zeta = (1 - x) \int_l^x w/(1 - \zeta) d\zeta \), then \( \int_l^1 v(\zeta, t) d\zeta = 0 \).

Following [25], we introduce the smoothing operators with respect to \( t \), \( (J^{-1}_\epsilon) = (I - \epsilon(\partial^3/\partial t^3))^{-1} \), and \( (J^{-1}_\epsilon)^* = (I + \epsilon(\partial^3/\partial t^3))^{-1} \) which provide the solution of the respective problems:

\[
\begin{align*}
\frac{\partial^3 u_\epsilon}{\partial t^3} = u, \\
\frac{\partial u_\epsilon}{\partial t}(x, 0) = 0, \\
\frac{\partial^2 u_\epsilon}{\partial t^2}(x, 0) = 0, \\
\frac{\partial^2 u_\epsilon}{\partial t^2}(x, T) = 0, \\
\frac{\partial^3 v_\epsilon^*}{\partial t^3} = v, \\
\frac{\partial v_\epsilon^*}{\partial t}(x, 0) = 0, \\
\frac{\partial^2 v_\epsilon^*}{\partial t^2}(x, 0) = 0, \\
\frac{\partial^2 v_\epsilon^*}{\partial t^2}(x, T) = 0.
\end{align*}
\]

(4.5)
And also, we have the following properties: for any \( u \in L^2(0, T) \), the function \( J_{\epsilon}^{-1}u \in W^2_2(0, T), (J_{\epsilon}^{-1})^*u \in W^2_2(0, T) \). If \( u \in D(L), J_{\epsilon}^{-1}u \in D(L) \).

\[
\lim_{\epsilon \to 0} ||J_{\epsilon}^{-1}u - u||_{L^2(0, T)} = 0, \quad \lim_{\epsilon \to 0} ||(J_{\epsilon}^{-1})^*u - u||_{L^2(0, T)} = 0. \tag{4.6}
\]

Substituting the function \( u \) in (4.3) by the smoothing function \( u_\epsilon \) and using the relation \( A(t)u_\epsilon = J_{\epsilon}^{-1}A(t)u + \epsilon J_{\epsilon}^{-1}B_\epsilon(t)u \), where \( B_\epsilon(t) = (3\partial/\partial t)((\partial A(t)/\partial t)(\partial u_\epsilon/\partial t)) + (3^3 A(t)/\partial t^3)u_\epsilon \), we obtain

\[
\int_\Omega uN \frac{\partial^3 \nu^\epsilon}{\partial t^3} \, dx \, dt = \int_\Omega A(t)u \nu^\epsilon \, dx \, dt - \epsilon \int_\Omega B_\epsilon(t)u \nu^\epsilon \, dx \, dt. \tag{4.7}
\]

The operator \( A(t) \) has a continuous inverse in \( L^2(0, 1) \) defined by

\[
A^{-1}(t)g = \begin{cases} 
- \frac{1}{1-l} \int_0^x \int_0^\zeta \frac{d\zeta}{a(\zeta, t)} \int_0^\eta \frac{d\eta}{a(\zeta, t)} , & 0 < \xi < l, \\
\int_1^x \int_1^\eta \frac{-d\zeta}{(1-\zeta)a(\zeta, t)} \int_0^\eta \frac{d\eta}{a(\zeta, t)} + u(l), & l < \xi < 1,
\end{cases} \tag{4.8}
\]

where

\[
C_1(t) = \frac{(1-l)u(l) + \int_0^l (d\zeta/a(\zeta, t)) \int_0^\zeta g(\eta) \, d\eta}{\int_0^l (d\zeta/a(\zeta, t))}, \quad C_2(t) = \frac{-u(l) + \int_1^l (d\zeta/a(\zeta, t)) \int_0^\zeta g(\eta) \, d\eta}{\int_1^1 (d\zeta/a(\zeta, t))}. \tag{4.9}
\]

Then we have \( \int_1^l A^{-1}(t)u = 0 \), hence, the function \( J_{\epsilon}^{-1}u = u_\epsilon \) can be represented in the form

\[
u_\epsilon = J_{\epsilon}^{-1}A^{-1}(t)A(t)u. \tag{4.10} \]

The adjoint of \( B_\epsilon(t) \) has the form

\[
B_\epsilon^*(t)v = \frac{1}{a} (J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial t^3} v + \frac{3}{a} (J_{\epsilon}^{-1})^* \frac{\partial (\partial a \partial v)}{\partial t} - G_\epsilon(v)(x)
\]

\[
+ \frac{\int_0^l (d\zeta/a(\zeta, t))}{\int_0^l (d\zeta/a(\zeta, t))} G_\epsilon(v)(1), \tag{4.11}
\]

where

\[
G_\epsilon(v)(x) = \int_0^x \left[ \frac{3}{a} (J_{\epsilon}^{-1})^* \frac{\partial (\partial^2 a \partial v)}{\partial t (\partial a \partial v)} - \frac{3}{a^2} \frac{\partial a}{\partial \zeta} (J_{\epsilon}^{-1})^* \frac{\partial (\partial a \partial v)}{\partial t} - \frac{1}{a^2} \frac{\partial a}{\partial \zeta} (J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial \zeta^3} v \right] \, d\zeta. \tag{4.12}
\]
Consequently, equality (4.7) becomes
\[
\int_{\Omega} u N \frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt = \int_{\Omega} A(t) u h^*_\epsilon \, dx \, dt, \tag{4.13}
\]
where \( h^*_\epsilon = v^*_\epsilon - \epsilon B^*_\epsilon(t) v^*_\epsilon \).

The left-hand side of (4.13) is a continuous linear functional of \( u \), hence the function \( h^*_\epsilon \) has the derivatives \( \partial h^*_\epsilon / \partial x \), \((1-x)(\partial h^*_\epsilon / \partial x) \in L^2(\Omega)\), and the condition \( h^*_\epsilon(0,t) = 0 \) is satisfied.

From the equality
\[
(1-x) \frac{\partial h^*_\epsilon}{\partial x} = \left[ 1 - \epsilon \frac{1}{a} (J^{-1}_\epsilon)^* \left( \frac{\partial^3 a}{\partial t^3} \right) \right] (1-x) \frac{\partial v^*_\epsilon}{\partial x} - 3 \epsilon \frac{1}{a} (J^{-1}_\epsilon)^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial}{\partial x} (1-x) \frac{\partial v^*_\epsilon}{\partial x} \right), \tag{4.14}
\]
and since the operator \((J^{-1}_\epsilon)^*\) is bounded in \( L^2(\Omega) \), for sufficiently small \( \epsilon \), we have \( \| \epsilon (1/a)(J^{-1}_\epsilon)^* (\partial^3 a/\partial t^3) \| < 1 \). Hence, the operator \( I - \epsilon (1/a)(J^{-1}_\epsilon)^* (\partial^3 a/\partial t^3) \) has a bounded inverse in \( L^2(\Omega) \). We conclude that \((1-x)(\partial v^*_\epsilon / \partial x) \in L^2(\Omega)\). Similarly, we conclude that \((\partial / \partial x)((1-x)(\partial v^*_\epsilon / \partial x)) \) exists and belongs to \( L^2(\Omega) \), and the condition \( v^*_\epsilon(0,t) = 0 \) is satisfied.

Putting \( u = \int_0^T \int_0^T \int_\eta^T \exp(ct) v^*_\epsilon \, d\tau \, d\eta \, d\zeta \) in (4.3), where the constant \( c \) satisfies (3.2) and using the properties of smoothing operator, we obtain
\[
\int_{\Omega} \exp(ct) v^*_\epsilon \, \overline{Nv} \, dx \, dt = - \int_{\Omega} A(t) u \overline{v^*_\epsilon} \, dx \, dt - \epsilon \int_{\Omega} A(t) u \frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt, \tag{4.15}
\]
and from
\[
- \epsilon \int_{\Omega} A(t) u \frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt
= 3 \int_{\Omega} h(x) \exp(-ct) \left( \frac{\partial^3 a}{\partial t^3} - c \frac{\partial^2 a}{\partial t^2} \right) \left( \frac{\partial^3 u}{\partial t^3} \right) \left( \frac{\partial^3 u}{\partial t^2 \partial x} \right) \, dx \, dt
- 3 \int_{\Omega} h(x) \exp(-ct) \left( \frac{\partial^3 a}{\partial t^3} - c \frac{\partial^2 a}{\partial t^2} \right) \left( \frac{\partial^3 u}{\partial t^3} \right) \left( \frac{\partial^2 u}{\partial t \partial x} \right) \, dx \, dt
+ 3 \int_0^T h(x) \frac{1}{2} \exp(-ct) \frac{\partial a}{\partial t} \left( \frac{\partial^3 u}{\partial t^3} \right) \left( \frac{\partial^2 u}{\partial t^2 \partial x} \right) \, dx \bigg|_{t=T}
+ 3 \int_0^T h(x) \frac{1}{2} \exp(-ct) \frac{\partial a}{\partial t} \left( \frac{\partial^3 u}{\partial t^3} \right) \left( \frac{\partial^2 u}{\partial t \partial x} \right) \, dx \bigg|_{t=T}
- \int_{\Omega} h(x) \exp(-ct) a \left( \frac{\partial^3 v^*_\epsilon}{\partial t^3} \right) \, dx \, dt
- \int_{\Omega} h(x) \exp(-ct) \frac{\partial a}{\partial t} \left( \frac{\partial^3 u}{\partial t^3} \right) \left( \frac{\partial^2 u}{\partial t \partial x} \right) \, dx \, dt, \tag{4.16}
\]

...
By using the conditions (3.2), inequalities (4.17) and (4.19), we obtain

\[
- \varepsilon \text{Re} \int_{\Omega} A(t)u \frac{\partial^3 \nu^{\varepsilon}_t}{\partial t^3} \, dx \, dt 
\leq \varepsilon \left\{ \int_{\Omega} h(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} + \frac{1}{2} \left| \frac{\partial^3 a}{\partial t^3} - c \frac{\partial^2 a}{\partial t^2} \right| \right] \left| \frac{\partial^3 u}{\partial t^3 \partial x} \right|^2 \, dx \, dt 
+ \frac{3}{2} \int_{\Omega} h(x) \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - c \frac{\partial a}{\partial t} + \left| \frac{\partial^3 a}{\partial t^3} - c \frac{\partial^2 a}{\partial t^2} \right| \right] \left| \frac{\partial^2 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt 
\right.
\]

\[
- \int_{\Omega} h(x) \exp(-ct) a \left| \frac{\partial^3 \nu^{\varepsilon}_t}{\partial t^3} \right|^2 \, dx \, dt 
+ \frac{3}{2} \int_{\Omega} h(x) \exp(-ct) \left| \frac{\partial^3 a}{\partial t^3} \right|^2 \, dx \, dt 
+ \frac{1}{2} \int_{\Omega} h(x) \exp(-ct) \left| \frac{\partial^3 a}{\partial t^3} \right|^2 \, dx \, dt 
+ \frac{1}{2} \int_{\Omega} h(x) \exp(-ct) \frac{\partial a}{\partial t} \left| \frac{\partial^3 u}{\partial t^3 \partial x} \right|^2 \, dx \, dt \right\}.
\]

Integrating the first term on the right-hand side by parts in (4.15), we obtain

\[
- \varepsilon \text{Re} \int_{\Omega} A(t)u \nu^{\varepsilon}_t dx \, dt 
= \frac{3}{2} \int_{\Omega} h(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt 
- \int_{\Omega} h(x) \exp(-ct) \left\{ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt 
- \int_0^1 \int_{\Omega} \frac{1}{2} h(x) \exp(-ct) a \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx |_{t=T} 
+ \int_0^1 \int_{\Omega} \frac{1}{2} h(x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx |_{t=T} 
- \int_0^1 \int_{\Omega} h(x) \exp(-ct) \left\{ \frac{\partial a}{\partial t} - ca \right\} \frac{\partial u}{\partial x} \left| \frac{\partial^2 u}{\partial t^2 \partial x} \right| \, dx |_{t=T}.
\]

This last equality gives

\[
- \varepsilon \text{Re} \int_{\Omega} A(t)u \nu^{\varepsilon}_t dx \, dt 
\leq - \int_0^1 \int_{\Omega} h(x) \exp(-ct) \left| \frac{\partial a}{\partial t} + a - ca \right| \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx |_{t=T} 
+ \int_0^1 \int_{\Omega} \frac{1}{2} h(x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + ca \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx |_{t=T}.
\]

By using the conditions (3.2), inequalities (4.17) and (4.19), we obtain

\[
\text{Re} \int_{\Omega} \exp(ct) \nu^{\varepsilon}_t \overline{N} \, dx \, dt \leq 0 \quad \text{as } \varepsilon \to 0.
\]
Three-point boundary value problem

This implies \( \text{Re} \int_{\Omega} \exp(ct)(v^*_e - v)\bar{N}v \, dx \, dt + \text{Re} \int_{\Omega} \exp(ct)v\bar{N}v \, dx \, dt \leq 0 \), that is,

\[
\int_0^T \int_0^l \exp(-ct)(1-l)|v|^2 \, dx \, dt + \int_0^T \int_0^1 \exp(-ct)(1-x)|v|^2 \, dx \, dt + \int_0^T \int_1^l \exp(-ct)\left|J_xv\right|^2 \, dx \, dt + \int_0^T \int_0^1 \frac{1-l}{2l} \exp(-ct)\left|J_xv\right|^2 \, dx \, dt \leq 0.
\]

Then \( v = 0 \).

Finally from (4.4), we conclude \( w = 0 \). \( \square \)

**Theorem 4.2.** The range \( R(\mathcal{L}) \) of \( \mathcal{L} \) coincides with \( F \).

**Proof.** Since \( F \) is Hilbert space, then \( R(\mathcal{L}) = F \) if and only if the relation

\[
\int_{\Omega} \Theta(x)f \overline{g} \, dx \, dt = 0 \quad (4.22)
\]

holds.

Arbitrary \( u \in D_0(L) \) and \( \mathcal{F} = (f, 0, 0, 0) \in F \) implies \( f = 0 \). Taking in (4.22), \( u \in D_0(L) \), and using Lemma 4.1, we obtain

\[
w = \begin{cases} 
(1-l)g, & 0 < x < l, \\
(l-x)g, & l < x < 1,
\end{cases} \quad (4.23)
\]

then \( g = 0 \). \( \square \)

**References**


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