LOGISTIC EQUATION WITH THE $p$-LAPLACIAN AND CONSTANT YIELD HARVESTING

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We consider the positive solutions of a quasilinear elliptic equation with $p$-Laplacian, logistic-type growth rate function, and a constant yield harvesting. We use sub-super-solution methods to prove the existence of a maximal positive solution when the harvesting rate is under a certain positive constant.

1. Introduction

We consider weak solutions to the boundary value problem

$$\begin{align*}
-\Delta_p u &= f(x,u) = au^{p-1} - u^{\gamma-1} - ch(x) \quad \text{in } \Omega, \\
&\quad u > 0 \quad \text{in } \Omega, \\
&\quad u = 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(1.1)

where $\Delta_p$ denotes the $p$-Laplacian operator defined by $\Delta_p z := \text{div}(|\nabla z|^{p-2} \nabla z)$; $p > 1$, $\gamma(> p)$, $a$ and $c$ are positive constants, $\Omega$ is a bounded domain in $\mathbb{R}^N$; $N \geq 1$, with $\partial \Omega$ of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and connected (if $N = 1$, we assume $\Omega$ is a bounded open interval), and $h : \bar{\Omega} \rightarrow \mathbb{R}$ is a continuous function in $\bar{\Omega}$ satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \neq 0$, $\max_{x \in \Omega} h(x) = 1$, and $h(x) = 0$ for $x \in \partial \Omega$. By a weak solution of (1.1), we mean a function $u \in W^{1,p}_0(\Omega)$ that satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx = \int_{\Omega} [au^{p-1} - u^{\gamma-1} - ch(x)] w \, dx, \quad \forall w \in C^\infty_0(\Omega).$$

(1.2)

From the standard regularity results of (1.1), the weak solutions belong to the function class $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ (see [4, pages 115–116] and the references therein).

We first note that if $a \leq \lambda_1$, where $\lambda_1$ is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, then (1.1) has no positive solutions. This follows since if $u$ is a positive solution of (1.1), then $u$ satisfies

$$\int_{\Omega} |\nabla u|^{p} \, dx = \int_{\Omega} [au^{p-1} - u^{\gamma-1} - ch(x)] u \, dx.$$ 

(1.3)

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But \( \int_{\Omega} |\nabla u|^p \, dx \geq \lambda_1 \int_{\Omega} u^p \, dx \). Combining, we obtain \( \int_{\Omega} [a u^{p-1} - u^{r-1} - ch(x)] \, u \, dx \geq \lambda_1 \int_{\Omega} u^p \, dx \) and hence \( \int_{\Omega} (a - \lambda_1) u^p \, dx \geq \int_{\Omega} (u^{r-1} + ch(x)) \, u \, dx \geq 0 \). This clearly requires \( a > \lambda_1 \).

Next if \( a > \lambda_1 \) and \( c \) is very large, then again it can be proven that there are no positive solutions. This follows easily from the fact that if the solution \( u \) is positive, then \( \int_{\Omega} [a u^{p-1} - u^{r-1} - ch(x)] \, dx \) is nonnegative. In fact, from the divergence theorem (see [4, page 173]),

\[
\int_{\Omega} [a u^{p-1} - u^{r-1} - ch(x)] \, dx = -\int_{\partial \Omega} |\nabla u|^{p-2} \nabla u \cdot \nu \, dx \geq 0. \tag{1.4}
\]

Thus,

\[
c \int_{\Omega} h(x) \, dx \leq \int_{\Omega} [a u^{p-1} - u^{r-1}] \, dx \leq a^{(r-1)/(r-p)} |\Omega|.
\tag{1.5}
\]

Here in the last inequality, we used the fact that \( u(x) \leq a^{1/(r-p)} \) which can be proven by the maximum principle (see [4, page 173]).

This leaves us with the analysis of the case \( a > \lambda_1 \) and \( c \) small which is the focus of the paper.

**Theorem 1.1.** Suppose that \( a > \lambda_1 \). Then there exists \( c_0(a) > 0 \) such that if \( 0 < c < c_0 \), then (1.1) has a positive \( C^{1,\alpha}(\bar{\Omega}) \) solution \( u \). Further, this solution \( u \) is such that \( u(x) \geq (ch(x)/\lambda_1)^{1/(p-1)} \) for \( x \in \bar{\Omega} \).

**Theorem 1.2.** Suppose that \( a > \lambda_1 \). Then there exists \( c_1(a) \geq c_0 \) such that for \( 0 < c < c_1 \), (1.1) has a maximal positive solution, and for \( c > c_1 \), (1.1) has no positive solutions.

**Remark 1.3.** Theorem 1.2 holds even when \( h(x) > 0 \) in \( \bar{\Omega} \).

We establish Theorem 1.1 by the method of sub-supersolutions. By a supersolution (subsolution) \( \phi \) of (1.1), we mean a function \( \phi \in W_0^{1,p}(\Omega) \) such that \( \phi = 0 \) on \( \partial \Omega \) and

\[
\int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla w \, dx \geq (\leq) \int_{\Omega} [a \phi^{p-1} - \phi^{r-1} - ch(x)] \, w \, dx, \quad \forall w \in W,
\tag{1.6}
\]

where \( W = \{ v \in C_0^\infty(\Omega) \mid v \geq 0 \text{ in } \Omega \} \). Now if there exist subsolutions and supersolutions \( \psi \) and \( \phi \), respectively, such that \( 0 \leq \psi \leq \phi \) in \( \Omega \), then (1.1) has a positive solution \( u \in W_0^{1,p}(\Omega) \) such that \( \psi \leq u \leq \phi \). This follows from a result in [3].

Equation (1.1) arises in the studies of population biology of one species with \( u \) representing the concentration of the species and \( ch(x) \) representing the rate of harvesting. The case when \( p = 2 \) (the Laplacian operator) and \( r = 3 \) has been studied in [6]. The purpose of this paper is to extend some of this study to the \( p \)-Laplacian case. In [3], the authors studied (1.1) in the case when \( c = 0 \) (nonharvesting case). However, the \( c > 0 \) case is a semipositone problem \( (f(x,0) < 0) \) and studying positive solutions in this case is significantly harder. Very few results exist on semipositone problems involving the \( p \)-Laplacian operator (see [1, 2]), and these deal with only radial positive solutions with the domain \( \Omega \) a ball or an annulus. In Section 2, when \( a > \lambda_1 \) and \( c \) is sufficiently small, we will construct nonnegative subsolutions and supersolutions \( \psi \) and \( \phi \), respectively, such that \( \psi \leq \phi \), and
establish Theorem 1.1. We also establish Theorem 1.2 in Section 2 and discuss the case when \( h(x) > 0 \) in \( \Omega \).

2. Proofs of theorems

Proof of Theorem 1.1. We first construct the subsolution \( \psi \). We recall the antimaximum principle (see [4, pages 155–156]) in the following form. Let \( \lambda_1 \) be the principal eigenvalue of \( -\Delta_p \) with Dirichlet boundary conditions. Then there exists a \( \delta(\Omega) > 0 \) such that the solution \( z_\lambda \) of

\[
-\Delta_p z - \lambda z^{p-1} = -1 \quad \text{in } \Omega, \\
z = 0 \quad \text{on } \partial \Omega,
\]

for \( \lambda \in (\lambda_1, \lambda_1 + \delta) \) is positive for \( x \in \Omega \) and is such that \((\partial z_\lambda / \partial \nu)(x) < 0 \), \( x \in \partial \Omega \).

We construct the subsolution \( \psi \) of (1.1) using \( z_\lambda \) such that \( \lambda_1 \psi(x)^{p-1} \geq c h(x) \). Fix \( \lambda_* \in (\lambda_1, \min\{a, \lambda_1 + \delta\}) \). Let \( \alpha = \|z_\lambda\|_\infty \), \( K_0 = \inf \{ K \mid \lambda_1 K^{p-1} z_{\lambda_*}^{p-1} \geq h(x) \} \), and \( K_1 = \max\{1, K_0\} \). Define \( \psi = K c^{1/(p-1)} z_{\lambda_*} \), where \( K \geq K_1 \) is to be chosen. Let \( w \in W \). Then

\[
- \int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w \, dx + \int_\Omega [a(\psi)^{p-1} - (\psi)^{y-1} - c h(x)] w \, dx \\
= \int_\Omega [-c K^{p-1}(\lambda_* z_{\lambda_*}^{p-1} - 1) + ac(Kz_{\lambda_*})^{p-1} - (Kc^{1/(p-1)} z_{\lambda_*})^{y-1} - c h(x)] w \, dx \\
\geq \int_\Omega [-c K^{p-1}(\lambda_* z_{\lambda_*}^{p-1} - 1) + ac(Kz_{\lambda_*})^{p-1} - (Kc^{1/(p-1)} z_{\lambda_*})^{y-1} - c] w \, dx \\
= \int_\Omega [(a - \lambda_*)(Kz_{\lambda_*})^{p-1} - (Kz_{\lambda_*})^{y-1} c^{(y-p)/(p-1)} + (K^{p-1} - 1)] w \, dx.
\]

(2.2)

Define \( H(y) = (a - \lambda_*)(K \alpha)^{p-1} - y^{y-1} c^{(y-p)/(p-1)} + (K^{p-1} - 1) \). Then \( \psi(x) \) is a subsolution if \( H(y) \geq 0 \) for all \( y \in [0, K \alpha] \). But \( H(0) = K^{p-1} - 1 \geq 0 \) since \( K \geq 1 \) and \( H'(y) = y^{y-2} [(a - \lambda_*)(p - 1) - c^{(y-p)/(p-1)} (y - 1) y^{p-1}] \). Hence \( H(y) \geq 0 \) for all \( y \in [0, K \alpha] \) if \( H(K \alpha) = (a - \lambda_*)(K \alpha)^{p-1} - (K \alpha)^{y-1} c^{(y-p)/(p-1)} + (K^{p-1} - 1) \geq 0 \), that is, if

\[
c \leq \left( \frac{(a - \lambda_*)(K \alpha)^{p-1} + (K^{p-1} - 1)}{(K \alpha)^{y-1}} \right)^{(p-1)/(y-p)}.
\]

(2.3)

We define

\[
c_1 = \sup_{K \geq K_1} \left( \frac{(a - \lambda_*)(K \alpha)^{p-1} + (K^{p-1} - 1)}{(K \alpha)^{y-1}} \right)^{(p-1)/(y-p)}.
\]

(2.4)

Then for \( 0 < c < c_1 \), there exists \( K \geq K_1 \) such that

\[
c < \left( \frac{(a - \lambda_*)(K \alpha)^{p-1} + (K^{p-1} - 1)}{(K \alpha)^{y-1}} \right)^{(p-1)/(y-p)}
\]

(2.5)

and hence \( \psi(x) = K c^{1/(p-1)} z_{\lambda_*} \) is a subsolution.
We next construct the supersolution \( \phi(x) \) such that \( \phi(x) \geq \psi(x) \). Let \( G(y) = ay^{p-1} - y^{\gamma-1} \). Since \( G'(y) = p y^{p-2} [a(p - 1) - (\gamma - 1)y^{\gamma - p}] \), \( G(y) \leq L = G(y_0) \), where \( y_0 = [a(p - 1)/(\gamma - 1)]^{1/(\gamma - p)} \). Let \( \phi \) be the positive solution of

\[
-\Delta_p \phi = L \quad \text{in } \Omega, \\
\phi = 0 \quad \text{on } \partial \Omega.
\]

Then for \( w \in W \),

\[
\int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla w \, dx = \int_{\Omega} L w \, dx \\
\geq \int_{\Omega} [a\phi^{p-1} - \phi^{\gamma-1}] w \, dx \\
\geq \int_{\Omega} [a\phi^{p-1} - \phi^{\gamma-1} - c(x)] w \, dx.
\]

Thus \( \phi \) is a supersolution of (1.1). Also since \(-\Delta_p \psi \leq a\psi^{p-1} - \psi^{\gamma-1} - c(x) \leq L = -\Delta_p \phi \), by the weak comparison principle (see [4, 5]), we obtain \( \phi \geq \psi \geq 0 \). Hence there exists a solution \( u \in W^{1,p}_0(\Omega) \) such that \( \phi \geq u \geq \psi \). From the regularity results (see [4, pages 115–116] and the references therein), \( u \in C^{1,a}(\bar{\Omega}) \).

**Remark 2.1.** If \( \tilde{u} \) is any \( C^{1,a}(\bar{\Omega}) \) solution of (1.1), then by the weak comparison principle, \( \|\tilde{u}\|_\infty \leq \|\phi\|_\infty \), where \( \phi \) is as in (2.6).

**Proof of Theorem 1.2.** From Theorem 1.1, we know that for \( c \) small, there exists a positive solution. Whenever (1.1) has a positive solution \( u \), (1.1) also has a maximal positive solution. This easily follows since \( \phi \) in (2.6) is always a supersolution such that \( \phi \geq u \). Next if for \( c = \tilde{c} \), we have a positive solution \( u_{\tilde{c}} \), then for all \( c < \tilde{c} \), \( u_c \) is a positive subsolution. Hence again using \( \phi \) in (2.6) as the supersolution, we obtain a maximal positive solution for \( c \). From (1.3), it is easy to see that for large \( c \), there does not exist any positive solution. Hence there exists a \( c_1(a) > 0 \) such that there exists a maximal positive solution for \( c \in (0, c_1) \) and no positive solution for \( c > c_1 \).

**Remark 2.2.** The use of the antimaximum principle in the creation of the subsolution helps us to easily modify the proof of Theorem 1.1 to obtain a positive maximal solution for all \( c < c_2(a) = \sup_{K \geq 1} (((a - \lambda_1)(K \alpha)^{p-1} + (K^{p-1} - 1))/(K \alpha)^{\gamma-1})^{(p-1)/(\gamma-p)} \) even in the case when \( h(x) > 0 \) in \( \bar{\Omega} \). Here \( c_2(a) \geq c_0(a) \). (Of course when \( h(x) > 0 \) in \( \bar{\Omega} \), our solution does not satisfy \( u(x) \geq (ch(x)/\lambda_1)^{1/(p-1)} \) for \( x \in \bar{\Omega} \).) Hence Theorem 1.2 also holds in the case when \( h(x) > 0 \) in \( \bar{\Omega} \).

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