ON LINEAR SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATIONS IN ONE FUNCTIONAL SPACE

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We use a special space of integrable functions for studying the Cauchy problem for linear functional-differential equations with nonintegrable singularities. We use the ideas developed by Azbelev and his students (1995). We show that by choosing the function $\psi$ generating the space, one can guarantee resolubility and certain behavior of the solution near the point of singularity.

1. Linear Volterra operators in $\Delta_\psi$ spaces

We consider the following $n$-dimensional functional-differential equation:

$$\mathcal{L}x \overset{\text{def}}{=} \dot{x} + (K + S)x + Ax(0) = f,$$

where

$$(Ky)(t) = \int_0^t K(t,s)y(s)ds,$$

$$(Sy)(t) = \begin{cases} B(t)y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1]. \end{cases}$$

The case where $K$ and $S$ are continuous on $L^p[0,1]$ operators is well studied (see, e.g., [1] and the references therein). Here we suppose that the functions $K(t,s)$ and $B(t)$ may be nonintegrable at $t = 0$. More precisely, we will formulate conditions on operators $K$ and $S$ in Sections 2 and 3. Under such conditions, those operators are not bounded on $L[0,1]$ and one has to choose other functional spaces for studying (1.1). We propose a space of integrable functions on $[0,1]$ and show that it may be useful in such a case.

We call $\Delta^p_\psi$ space the space of all measurable functions $y : [0,1] \to \mathbb{R}^n$, for which

$$\|y\|_{\Delta^p_\psi} = \sup_{0 < h < 1} \frac{1}{\psi(h)} \left( \int_0^h |y(s)|^p ds \right)^{1/p} < \infty.$$
We assume everywhere below that $\psi$ is a nondecreasing, absolutely continuous function, $\psi(0) = 0$.

**Theorem 1.1.** The space $\Delta^p_\psi$ is a Banach space.

Let $X[a,b], Y[a,b]$ be spaces of functions defined on $[a,b]$.

We will call $V : X[0,1] \rightarrow Y[0,1]$ the Volterra operator [3] if for every $\xi \in [0,1]$ and for any $x_1, x_2 \in X[0,1]$ such that $x_1(t) = x_2(t)$ on $[0,\xi]$, $(V x_1)(t) = (V x_2)(t)$ for $t \in [0,1]$.

It is possible to say that each Volterra operator $V : X[0,1] \rightarrow Y[0,1]$ generates a set of operators $V_{\xi} : X[0,\xi] \rightarrow Y[0,\xi]$, where $\xi \in (0,1]$. By $V_{\xi}$, we denote the restriction of function $y$ defined on $[0,1]$ onto segment $[0,\xi]$.

**Theorem 1.2.** Let $V : L \rightarrow L$ be a linear bounded operator. Then $V$ is a linear bounded operator in $\Delta^p_\psi$ and $\|V\|_{\Delta^p_\psi} \leq \|V\|_{L^p}$.

**Proof.** Let $y \in \Delta^p_\psi$. Then

$$\|Vy\|_{\Delta^p_\psi} = \sup_{0<h<1} \frac{1}{\psi(h)} \|(V_{\xi} y_{\xi})\|_{L^p[0,\xi]} \leq \frac{1}{\psi^p(0)} \|V_{\xi}\|_{L^p[0,\xi]} \|y_{\xi}\|_{L^p[0,\xi]} \leq \|V\|_L \|y\|_{L^p}. \quad (1.5)$$

**Theorem 1.3.** Let $V : \Delta^p_\psi \rightarrow \Delta^p_\psi$ be linear bounded operator and let

$$\sup_{t \in [0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty. \quad (1.6)$$

Then $V$ is linear and bounded in $\Delta^p_\psi$ and

$$\|V\|_{\Delta^p_\psi} \leq \|V\|_{\Delta^p_\psi} \sup_{\xi \in [0,1]} \sup_{r \in [0,\xi]} \frac{\psi_1(\xi) \psi_2(r)}{\psi_2(\xi) \psi_1(\tau)}. \quad (1.7)$$

**Proof.** Let $y \in \Delta^p_\psi$. Then

$$\|Vy\|_{\Delta^p_{\psi_2}} \leq \sup_{\xi \in [0,1]} \frac{\|Vy_{\xi}\|_{L^p[0,\xi]} \psi_1(\xi)}{\psi_2(\xi) \psi_1(\tau)} \leq \sup_{\xi \in [0,1]} \frac{\|Vy_{\xi}\|_{\Delta^p_{\psi_2}[0,\xi]} \psi_1(\xi)}{\psi_2(\xi)} \leq \|V\|_{M^p_{\psi_1}} \sup_{\xi \in [0,1]} \frac{\|y_{\xi}\|_{\Delta^p_{\psi_2}[0,\xi]} \psi_1(\xi)}{\psi_2(\tau)} \leq \|V\|_{M^p_{\psi_1}} \sup_{\xi \in [0,1]} \frac{\|y_{\xi}\|_{L^p[0,\xi]} \psi_1(\xi) \psi_2(r)}{\psi_2(\xi) \psi_1(\tau)} \leq \|y\|_{\Delta^p_{\psi_2}} \|V\|_{\Delta^p_{\psi_2}} \sup_{\xi \in [0,1]} \sup_{r \in [0,\xi]} \frac{\psi_1(\xi) \psi_2(r)}{\psi_2(\xi) \psi_1(\tau)}. \quad (1.8)$$

**Corollary 1.4.** If $V_1 : \Delta^p_{\psi_1} \rightarrow \Delta^p_{\psi}$ and $V_2 : \Delta^p_{\psi_2} \rightarrow \Delta^p_{\psi}$ are linear continuous Volterra operators, then $V = V_1 + V_2$ is continuous on space $\Delta^p_{\psi}$ generated by $\psi(t) = \min(\psi_1(t), \psi_2(t))$ and $\|V\|_{\Delta^p_{\psi}} \leq \|V_1\|_{\Delta^p_{\psi_1}} + \|V_2\|_{\Delta^p_{\psi_2}}$. 

\[\boxed{}\]
2. Operator $K$

In this section, we consider the integral operator (1.2). We will show that under certain conditions on matrix $K(t,s)$, a function $\psi$ may be indicated such that $K$ is bounded on $\Delta_{\psi}$ and its norm is limited by a given number.

We say that matrix $K(t,s)$ satisfies the $/H5114$ condition if for some $p$ and $p_1$ such that $1 \leq p \leq p_1 < \infty$ and for any $\epsilon \in (0,1]$,

$$\|K_\epsilon(t, \cdot)\|_{L^p[0,t]} \in L^p_\epsilon[\epsilon, 1].$$  \hspace{1cm} (2.1)$$

Here $K_\epsilon(t,s)$ is a restriction of $K(t,s)$ onto $[\epsilon, 1] \times [0, t]$, $1/p + 1/p' = 1$.

The $/H5114$ condition admits a nonintegrable singularity at point $t = 0$.

**Lemma 2.1.** Let nonnegative function $\omega : [0,1] \rightarrow \mathbb{R}$ be nonincreasing and having a nonintegrable singularity at point $t = 0$.

Then $\psi(t) = \exp[\int_1^t \omega(s) ds]$ is absolutely continuous on $[0,1]$, does not decrease, and is a solution of the equation $\int_1^t \omega(s) x(s) ds = x(t)$.

Denote

$$\psi(t) = \exp \left[ \frac{1}{C} \int_1^t \text{vraisup}_{s \in [0,\tau]} \|K(\tau,s)\| d\tau \right].$$  \hspace{1cm} (2.2)$$

**Theorem 2.2.** Let matrix $K(t,s)$ satisfy the $/H5114$ condition with $p = 1$ and let $C$ be some positive constant. Then operator $K$ is bounded in $\Delta_{\psi}$ with function $\psi$ defined by the equality (2.2) and $\|K\|_{\Delta_{\psi}} \leq C$.

**Proof.** Let $x \in \Delta_{\psi}$ and $y = Kx$. From the $/H5114$ condition it follows that for almost all $t \in [0,1]$, $K(\cdot, s) \in L_\infty$. Let $\omega(t) = \text{vraisup}_{s \in [0,\tau]} \|K(\tau,s)\| d\tau$. Then

$$\left( \int_0^t \|y(s)\| ds \right) \leq \left[ \int_0^t \left( \int_0^\tau \|K(\tau,s)\| \|x(s)\| ds \right) d\tau \right]$$

$$\leq \int_0^t \left( \text{vraisup}_{s \in [0,\tau]} \|K(\tau,s)\| \right) \left( \int_0^\tau \|x(s)\| ds \right) d\tau$$

$$\leq \|x\|_{\Delta_{\psi}} \int_0^t \omega(\tau) \psi(\tau) d\tau.$$  \hspace{1cm} (2.3)$$

According to Lemma 2.1, $\psi(t) = \exp[(1/C) \int_1^t \omega(s) ds]$ is a solution of the equation $\int_1^t \omega(s) \psi(s) ds = C\psi(t)$, does not decrease, is absolutely continuous, and $\psi(0) = 0$. That implies

$$\left( \int_0^t \|y(s)\| ds \right) \leq C\|x\|_{\Delta_{\psi}} \psi(t).$$  \hspace{1cm} (2.4)$$

□
Remark 2.3. If $K(\cdot, s)$ has bounded variation on $s$, it is possible to indicate a “wider” space $\Delta_\psi$ for which conditions of Theorem 2.2 are satisfied by defining function $\psi$ as

$$
\psi(t) = \exp \left[ \frac{1}{C} \int_1^t \left( \| K(\tau, \tau) \| d\tau + \int_0^\tau d_s \text{var}_{t \in [0, r]} \| K(\tau, s) \| \right) d\tau \right].
$$

Theorem 2.4. Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$ and let $C$ be some positive constant. Then operator $K$ is bounded in space $\Delta_{\psi}^p$ generated by

$$
\psi(t) = \exp \left[ \frac{1}{pC} \int_1^t \left( \int_0^\tau \| K(\tau, s) \|^p ds \right)^{p/p'} d\tau \right]
$$

and $\| K \|_{\Delta_{\psi}^p} \leq C$.

Theorem 2.4 can be proved in a way similar to proof of Theorem 2.2.

Lemma 2.5. Let $K : \Delta_{\psi}^p \to \Delta_{\psi}^p$ $(1 < p < \infty)$ be a bounded operator and let its matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition. Then $K : \Delta_{\psi}^p \to L_p$ is a compact operator.

Proof. For every $t \in [0, 1]$, $(Ky)(t)$ is a linear bounded functional on $L_p$. Let $\{y_i\}$ be a sequence weakly converging to $y_0$ in $L_p$. If $\{y_i\} \subset \Delta_{\psi}^p$ and $\| y_i \|_{\Delta_{\psi}^p} \leq 1$, then $\| y_0 \|_{\Delta_{\psi}^p} \leq 1$. Indeed, if for some $t_1 \in [0, 1]$, $(1/\psi(t_1)) \int_0^{t_1} \| y(s) \|^{p} ds^{1/p} > 1$, then the sequence $l y_i = \int_0^{t_1} l(s)y_i(s)ds$ does not converge to $l y_0$, where

$$
l(s) = \begin{cases} 1, & \text{if } s \leq t_1, \\ 0, & \text{if } s > t_1. \end{cases}
$$

Hence, for almost all $t \in [0, 1]$, $\{(Ky_i)(t)\}$ converges and the set $Ky$ is compact in measure. Thus, for the operator $K : \Delta_{\psi}^p \to L_p$ to be compact, it is necessary and sufficient that the norms of $Ky$ are equicontinuous for $\| y \|_{\Delta_{\psi}^p} \leq M$. Let $\delta \in (0, 1)$. As $K : \Delta_{\psi}^p \to \Delta_{\psi}^p$ is a bounded operator,

$$
\left( \frac{1}{\psi(\delta)} \int_0^\delta \| (Ky)(s) \|^p ds \right)^{1/p} \leq \Delta_0.
$$

This implies that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $\delta < \delta_1$, then $(\int_0^\delta \| (Ky)(s) \|^p ds)^{1/p} \leq \varepsilon/2$.

Then, from the $\mathcal{N}$ condition, there exists $\delta_2$ such that if $\varepsilon \leq \delta_2$ for some $\varepsilon \subset [\delta, 1]$, then $(\int_\varepsilon^1 \| (Ky)(s) \|^p ds)^{1/p} \leq \varepsilon/2$.

Finally, for $\varepsilon_1 \subset [\delta, 1]$ such that $\varepsilon_1 \leq \min \{ \delta_1, \delta_2 \}$,

$$
\left( \int_{\varepsilon_1} \| (Ky)(s) \|^p ds \right)^{1/p} \leq \left( \int_0^\delta \| (Ky)(s) \|^p ds \right)^{1/p} + \left( \int_\delta^1 \| (Ky)(s) \|^p ds \right)^{1/p} \leq \varepsilon.
$$

Lemma 2.6. Let $\{y_i\} \rightarrow y_0$ in $L_p$ $(1 < p < \infty)$ and let the sequence $\{(1/u)y_i\}$ be bounded in $\Delta_{\psi}^p$ for some continuous increasing function $u$, $u(0) = 0$. Then $\{y_i\} \rightarrow y_0$ in $\Delta_{\psi}^p$. 

□
Proof. We have
\[
\left( \int_0^t \| y_i(s) \|^p ds \right)^{1/p} \leq u(t) \left( \int_0^t \| y_i(s) \|^p ds \right)^{1/p} \leq Mu(t) \psi(t).
\]
(2.10)

Thus, \( y_i \in \Delta^p_\psi \). Beginning with some \( N \) for any \( t \in [0,1] \) and for any given \( \varepsilon > 0 \),
\[
\left( \int_0^t \| y_i(s) - y_0(s) \|^p ds \right)^{1/p} \leq \varepsilon.
\]
(2.11)

Hence,
\[
\left( \int_0^t \| y_0(s) \|^p ds \right)^{1/p} \leq \left( \int_0^t \| y_i(s) - y_0(s) \|^p ds \right)^{1/p} + \left( \int_0^t \| y_i(s) \|^p ds \right)^{1/p} \leq \varepsilon + Mu(t) \psi(t) \leq Mu(t) \psi(t),
\]
(2.12)

beginning with some \( N_\delta \) for any \( \delta > 0 \), \( \| y_0 - y_i \|_{\Delta^p_\psi} < \delta \). Indeed, Lemma 2.5 guarantees the existence of \( \tau \in (0,1] \) such that for all \( t \in [0,\tau] \),
\[
\left( \int_0^t \| y_i(s) - y_0(s) \|^p ds \right)^{1/p} \leq \delta \psi(t).
\]
(2.13)

Let \( t \in [\tau,1] \). Then for \( \varepsilon = \delta \psi(\tau) \), (2.11) yields (2.13) for all \( t \in [0,1] \).

Let \( u : [0,1] \to \mathbb{R} \) be a continuous increasing function, \( u(0) = 0 \). Denote
\[
\psi(t) = \exp \left[ \int_1^t \frac{1}{u(\tau)} \left( \int_0^\tau \| K(\tau,s) \|^p ds \right)^{p/p'} d\tau \right].
\]
(2.14)

Lemmas 2.5 and 2.6 imply the following theorem.

**Theorem 2.7.** Let matrix \( K(t,s) \) satisfy the \( \mathcal{N} \) condition with \( 1 < p < \infty \). And let \( \psi \) be defined by (2.14). Then \( K : \Delta^p_\psi \to \Delta^p_\psi \) is a compact operator and its spectral radius is equal to zero.

### 3. Operator \( S \)

Denote
\[
(S_g y)(t) = \begin{cases} y [g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1], \end{cases}
\]
(3.1)

\[
(S y)(t) = B(t) (S_g) (t).
\]

In [2], it is shown that \( S_g \) is bounded in \( L_p \) if \( r = \left( \sup (\text{mes} g^{-1}(E)/\text{mes} E) \right)^{1/p} < \infty \) and
\[
\| S_g \|_{L_p} = r,
\]
where sup is taken on all measurable sets from \([0,1]\).
Let $Ω_m$ be a set of points from $[0,1]$ for which $g(t) ≥ mt$, $β(t)$ is a nonincreasing majorant of function $∥B(t)∥$, and

$$φ(t) = \lim_{\text{mes} \to 0} \frac{\text{mes} g^{-1}(e)}{\text{mes} e}, \quad (3.2)$$

where $e$ is a closed interval containing $t$.

We say that operator $S_g$ satisfies the $M$ condition if $\text{vraisup}_{t \in [ε,1]} φ(t) < ∞$ for any $ε \in (0,1]$ and there exists $m \in [0,1)$ such that

$$ε \in (0,1] \text{vraisup}_{t \in [ε,1]} ∥B(t)∥ < ∞, \quad (3.3)$$

and there exists $m \in [0,1)$ such that

$$μ_m = \text{vraisup}_{t \in g(Ω_m)} (β(t)^p φ(t)) < ∞. \quad (3.4)$$

Lemma 3.1. There exists nonincreasing function $u : (0,1] → \mathbb{R}$ such that $β(t)^p φ(t) ≤ u(t)$ and the function

$$ψ(t) = \begin{cases} 
  t^u(t) & \text{if } t \in (0,1], \\
  0 & \text{if } t = 0,
\end{cases} \quad (3.5)$$

is absolutely continuous on $[0,1]$.

Proof. Let $\{t_i\}$ be a decreasing sequence, $t_i = 1, t_i → 0$. Denote

$$n_i = \text{vraisup}_{t \in (t_{i-1},t_i)} (β(t)^p φ(t)), \quad u(t) = \frac{n_i + n_{i+1}}{t_{i+1} - t_i} (t - t_i) + n_i, \quad (3.6)$$

where $t \in (t_{i+1},t_i)$. Then $β(t)^p φ(t) ≤ u(t)$, $u$ increases and is absolutely continuous on $[0,1]$.

Let

$$ν_m = m^{u(1)} \left[ u(1) - \frac{1}{\ln m} \right]. \quad (3.7)$$

Theorem 3.2. Let operator $S_g$ satisfy the $M$ condition and let function $u$ satisfy conditions of Lemma 3.1. Then $S_g$ is bonded in $Δ_p^ν$ with $ψ(t) = t^u(t)$ and

$$||S_g||_{Δ_p^ν} ≤ (ν_m + μ_m)^{1/p}. \quad (3.8)$$

Proof. Let $y ∈ Δ_p^ν$, $∥y∥_{Δ_p^ν} = 1$, and $δ ∈ (0,1)$. Denote measures $λ$ and $μ$ on $[δ,1]$ by $λ(e) = \int_δ^1 β(s)^p ds$ and $μ(e) = \int_δ^1 β(s)^p ds$. Then by the Radon-Nikodym [2] theorem, we have

$$\int_δ^1 \frac{t}{g^{-1}(t)} |S_g y(t)|^p ds \leq \int_{g^{-1}([δ,t])} ||y[σ(s)]||^p dλ(s) \leq \int_{g^{-1}([δ,t])} ||y(s)||^p \frac{dμ}{dλ}(s) dλ(s). \quad (3.9)$$
Then as \( g(t) \leq t \),
\[
\frac{d\mu}{d\lambda}(s) = \lim_{\text{mes} \to 0} \frac{\int_{g^{-1}(e)} \beta(s)^p ds}{\int e \beta(s)^p ds} \leq \lim_{\text{mes} \to 0} \frac{\varpi_{\text{sup}} g^{-1}(e) \beta(s)^p ds}{\varpi_{\text{sup}} \beta(s)^p} \varphi(s) = \varphi(s)
\]
(3.10)
or
\[
\left\| \int_{\delta}^t \left| (S_g y)(t) \right|^p ds \right\| \leq \int_{g^{-1}([0,t] \cap [\delta,1])} \beta(s)^p \right\| y(s) \right\|^p \varphi(s) ds + \int_{g^{-1}([\delta,1])} \beta(s)^p \right\| y(s) \right\|^p \varphi(s) ds
\]
(3.11)
\[
\leq \int_0^t \beta(s)^p \right\| y(s) \right\|^p \varphi(s) ds + \int_0^t \right\| y(s) \right\|^p \mu_m ds
\]
\[
\leq \int_0^t \right\| y(s) \right\|^p u(s) ds + \mu \psi(t)^p.
\]

We denote function \( u_k : (0,1] \to \mathbb{R} \) by \( u_k(t) = u(t_i) \), where \( t_i = (2^k - i)/2^k \), \( i = 0, 1, 2, \ldots, 2^k - 1 \). From \( u_k \to u \), it follows that
\[
\int_0^t \right\| y(s) \right\|^p u(s) ds = \lim_{k \to 0} \int_0^t \right\| y(s) \right\|^p u_k(s) ds.
\]
(3.12)
We write function \( u_k \) in the form
\[
u_k(t) = \begin{cases} u(t_0), & \text{if } t \in (t_1, t_0], \\ u(t_0) + [u(t_1) - u(t_0)], & \text{if } t \in (t_2, t_1], \\ \vdots \\ u(t_{k-2}) + [u(t_{k-1}) - u(t_{k-2})], & \text{if } t \in (t_k, t_{k-1}]. \end{cases}
\]
(3.13)
The condition \( t < t_i \) implies that
\[
\int_0^t \right\| y(s) \right\|^p u(s) ds \leq \psi(t)^p (mt) \leq m^{p(t)} \right\| u(t) \right\| \psi(t)
\]
and
\[
\int_0^t \right\| y(s) \right\|^p u(s) ds \leq \psi(t) \sum_{i=1}^{2^k} m^{p(t)} \left[ u(t_i) - u(t_{i-1}) \right] \psi(t) + u(1) m^{p(t)} \psi(t)
\]
(3.14)
\[
\leq \psi(t) \left( m^t ds + m^{u(t)} u(1) \right)
\]
\[
\leq \psi(t) m^{u(t)} \left[ u(1) - \frac{1}{\ln m} \right],
\]
simultaneously for all \( k \). Finally,
\[
\left\| \int_0^t \left| (S_g y)(s) \right|^p ds \right\| = \lim_{\delta \to 0} \left\| \int_0^t \left| (S_g y)(s) \right|^p ds \right\|
\]
(3.15)
\[
\leq \psi(t) m^{u(t)} \left[ u(1) - \frac{1}{\ln m} \right] + \psi(t) \mu_m + \psi(t) (\nu_m + \mu_m)
\]
which proves the theorem. \( \square \)
Remark 3.3. From (3.7) and (3.8), it follows that if \( \lim_{m \to \infty} < 1 \), then there exists function \( \psi \) such that the norm of operator \( S_{\psi} : \Delta_{\psi}^p \to \Delta_{\psi}^p \) is less than 1.

In some particular cases, it is possible to give less strict conditions on function \( \psi \) generating the space \( \Delta_{\psi}^p \). Direct calculations prove the following theorem.

**Theorem 3.4.** Let \( B(t) \leq C_1/t^\alpha \) and \( g(t) = C_2t^\beta \) with \( \beta > 1 \). Then \( \| S_{\psi} \|_{\Delta_{\psi}^p} \leq C_1/C_2 \), where \( \psi(t) = t^\gamma \), \( \gamma \geq (\alpha p + \beta - 1)/p(\beta - 1) \). If \( \gamma > (\alpha p + \beta - 1)/p(\beta - 1) \), then the spectral radius of \( S_{\psi} \) is equal to zero.

4. The Cauchy problem

We consider the Cauchy problem for (1.1):

\[(\mathcal{L}x)(t) = f(t), \quad x(0) = \alpha. \tag{4.1}\]

The theorems of this section are immediate corollaries of Theorems 2.2, 2.4, 2.7, 3.2, and 3.4.

**Theorem 4.1.** Let matrix \( K(t,s) \) satisfy the \( \mathcal{N} \) condition and let operator \( S_{\psi} \) satisfy the \( \mathcal{M} \) condition. Let also \( \sup_{t \in [0,1]} \| u(t) \| = \infty \), \( (\mu_m)^{1/p} \leq q < 1 \), and let the function \( \psi_1 \) be given by (2.14). Then if \( C < 1 - q \), the Cauchy problem (4.1) has a unique solution in \( \Delta_{\psi}^p \) with \( \psi(t) = \min \{ \psi_1(t), t^\gamma \} \) for \( f \) and \( \alpha \) such that \( (f - \alpha A) \in \Delta_{\psi}^p \).

Let \( \omega \) be a solution of the equation

\[m^\omega \left( \omega - \frac{1}{\ln m} \right) \leq C_1^p - q, \quad \gamma = \sup_{t \in [0,1]} \{ u(t), \omega \}, \tag{4.2}\]

where \( 0 \leq q \leq C_1^p < 1 \), and \( u \) satisfies conditions of Lemma 3.1.

**Theorem 4.2.** Let matrix \( K(t,s) \) and operator \( S_{\psi} \) satisfy the \( \mathcal{N} \) and \( \mathcal{M} \) conditions, respectively. Let \( \sup_{t \in [0,1]} \| u(t) \| < \infty \) and \( (\mu_m)^{1/p} \leq q < 1 \). Then if \( q < C_1 \), \( (C_1 + C_2) < 1 \), then the Cauchy problem (4.1) has a unique solution in \( \Delta_{\psi}^p \) with \( \psi(t) = \min \{ \psi_1(t), t^\gamma \} \) for \( f \) and \( \alpha \) such that \( (f - \alpha A) \in \Delta_{\psi}^p \).

**Theorem 4.3.** Let matrix \( K(t,s) \) satisfy the \( \mathcal{N} \) condition, \( B(t) \leq C_1/t^\alpha \), \( g(t) = C_2t^\beta \) (\( \beta > 1 \)), and \( \gamma > (\alpha p + \beta - 1)/p(\beta - 1) \). Let also \( C < 1 \) and \( \psi(t) = \min \{ \psi_1(t), t^\gamma \} \). Then the Cauchy problem (4.1) has a unique solution for \( f \) and \( \alpha \) such that \( (f - \alpha A) \in \Delta_{\psi}^p \).

**Example 4.4.** The Cauchy problem

\[
\dot{x}(t) + p(t) \frac{x[h(t)]}{t_k} + q(t)\dot{x}(t^2) = f(t), \quad t \in [0,1],
\]

\[x(\xi) = 0, \quad \text{if } h(\xi) \leq 0, \tag{4.3}\]

where \( h(t) \leq t \), \( k > 1 \), and \( p \) and \( q \) are bounded functions, has a solution if \( \int_{0}^{1} |f(s)|ds \leq M \exp(-t^{1-k}) \). If \( (t - h(t)) \geq \tau > 0 \), then it has a solution if \( \int_{0}^{1} |f(s)|ds \leq Mt^\gamma \) for \( \gamma > 1 \).
References


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