This paper deals with the problem \( \Delta u = g \) on \( G \) and \( \partial u/\partial n + uf = L \) on \( \partial G \). Here, \( G \subset \mathbb{R}^m \), \( m > 2 \), is a bounded domain with Lyapunov boundary, \( f \) is a bounded nonnegative function on the boundary of \( G \), \( L \) is a bounded linear functional on \( W^{1,2}(G) \) representable by a real measure \( \mu \) on the boundary of \( G \), and \( g \in L_2(G) \cap L_p(G) \), \( p > m/2 \). It is shown that a weak solution of this problem is bounded in \( G \) if and only if the Newtonian potential corresponding to the boundary condition \( \mu \) is bounded in \( G \).

Suppose that \( G \subset \mathbb{R}^m \), \( m > 2 \), is a bounded domain with Lyapunov boundary (i.e., of class \( C^{1+\alpha} \)). Denote by \( n(y) \) the outer unit normal of \( G \) at \( y \). If \( f, g, h \in C(\partial G) \) and \( u \in C^2(\text{cl} \, G) \) is a classical solution of

\[
\Delta u = g \quad \text{on} \ G, \\
\frac{\partial u}{\partial n} + uf = h \quad \text{on} \ \partial G,
\]

then Green’s formula yields

\[
\int_G \nabla u \cdot \nabla v d\mathcal{H}_m + \int_{\partial G} ufvd\mathcal{H}_{m-1} = \int_{\partial G} hvd\mathcal{H}_{m-1} - \int_G gvd\mathcal{H}_m
\]

for each \( v \in \mathcal{D} \), the space of all compactly supported infinitely differentiable functions in \( \mathbb{R}^m \). Here, \( \partial G \) denotes the boundary of \( G \) and \( \text{cl} \, G \) is the closure of \( G \); \( \mathcal{H}_k \) is the \( k \)-dimensional Hausdorff measure normalized so that \( \mathcal{H}_k \) is the Lebesgue measure in \( \mathbb{R}^k \). Denote by \( \mathcal{D}(G) \) the set of all functions from \( \mathcal{D} \) with the support in \( G \).

For an open set \( V \subset \mathbb{R}^m \), denote by \( W^{1,2}(V) \) the collection of all functions \( f \in L_2(V) \), the distributional gradient of which belongs to \([L_2(V)]^m\).

**Definition 1.** Let \( f \in L_\infty(\mathcal{H}) \), \( g \in L_2(G) \) and let \( L \) be a bounded linear functional on \( W^{1,2}(G) \) such that \( L(\varphi) = 0 \) for each \( \varphi \in \mathcal{D}(G) \). We say that \( u \in W^{1,2}(G) \) is a weak solution
in $W^{1,2}(G)$ of the third problem for the Poisson equation

\[ \Delta u = g \quad \text{on } G, \quad \frac{\partial u}{\partial n} + uf = L \quad \text{on } \partial G, \tag{3} \]

if

\[ \int_G \nabla u \cdot \nabla \nu \, d\mathcal{H}_m + \int_{\partial G} \nu f \, d\mu = L(\nu) - \int_G g \nu \, d\mathcal{H}_m \tag{4} \]

for each $\nu \in W^{1,2}(G)$.

Denote by $\mathcal{C}'(\partial G)$ the Banach space of all finite signed Borel measures with support in $\partial G$ with the total variation as a norm. We say that the bounded linear functional $L$ on $W^{1,2}(G)$ is representable by $\mu \in \mathcal{C}'(\partial G)$ if $L(\phi) = \int \phi \, d\mu$ for each $\phi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $W^{1,2}(G)$, the operator $L$ is uniquely determined by its representation $\mu \in \mathcal{C}'(\partial G)$.

For $x, y \in \mathbb{R}^m$, denote

\[ h_x(y) = \begin{cases} (m - 2)^{-1} A^{-1} |x - y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases} \tag{5} \]

where $A$ is the area of the unit sphere in $\mathbb{R}^m$. For the finite real Borel measure $\nu$, denote

\[ \mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) \, d\nu(y) \tag{6} \]

the Newtonian potential corresponding to $\nu$, for each $x$ for which this integral has sense.

We denote by $\mathcal{C}'_b(\partial G)$ the set of all $\mu \in \mathcal{C}'(\partial G)$ for which $\mathcal{U}\mu$ is bounded on $\mathbb{R}^m \setminus \partial G$.

Remark that $\mathcal{C}'_b(\partial G)$ is the set of all $\mu \in \mathcal{C}'(\partial G)$ for which there is a polar set $M$ such that $\mathcal{U}\mu(x)$ is meaningful and bounded on $\mathbb{R}^m \setminus M$, because $\mathbb{R}^m \setminus \partial G$ is finely dense in $\mathbb{R}^m$ (see [1, Chapter VII, Sections 2, 6], [7, Theorems 5.10 and 5.11]) and $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set. Remark that $\mathcal{H}_{m-1}(M) = 0$ for each polar set $M$ (see [7, Theorem 3.13]). (For the definition of polar sets, see [4, Chapter 7, Section 1]; for the definition of the fine topology, see [4, Chapter 10].)

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ to $\partial G$.

**Lemma 2.** Let $\mu \in \mathcal{C}'(\partial G)$. Then the following assertions are equivalent:

1. $\mu \in \mathcal{C}'_b(\partial G)$,
2. $\mathcal{U}\mu$ is bounded in $G$,
3. $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

**Proof.** (2) $\Rightarrow$ (3). Since $\partial G$ is a subset of the fine closure of $G$ by [1, Chapter VII, Sections 2, 6] and [7, Theorems 5.10 and 5.11], $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set $M$, and $\mathcal{H}_{m-1}(M) = 0$ by [4, Theorem 7.33] and [7, Theorem 3.13], then we obtain that $\mathcal{U}\mu \in L_\infty(\mathcal{H})$. 

(3)⇒(1). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of $\mu$. For $z \in G$, denote by $\mu_z$ the harmonic measure corresponding to $G$ and $z$. If $y \in \partial G$ and $z \in G$, then

$$\int_{\partial G} h_y(x) d\mu_z(x) = h_y(z)$$

(7)

by [7, pages 264, 299]. Using Fubini’s theorem, we get

$$\int \mathcal{U} \mu^+ d\mu_z = \int_{\partial G} \int_{\partial G} h_y(x) d\mu_z(x) d\mu^+(y) = \int_{\partial G} h_y(z) d\mu^+(y) = \mathcal{U} \mu^+(z).$$

(8)

Similarly, $\int \mathcal{U} \mu^- d\mu_z = \mathcal{U} \mu^-(z)$. Since $\mathcal{U} \mu \in L_{\infty}(\mathcal{H})$, $\mu_z$ is a nonnegative measure with the total variation 1 (see [4, Lemma 8.12]) which is absolutely continuous with respect to $\mathcal{H}$ by [2, Theorem 1], then we obtain that $|\mathcal{U} \mu(z)| \leq \|\mathcal{U} \mu\|_{L_{\infty}(\mathcal{H})}$. If $z \in \mathbb{R}^m \setminus \text{cl} G$, choose a bounded domain $V$ with smooth boundary such that $\text{cl} G \cup \{z\} \subset V$. Repeating the previous reasonings for $V \setminus \text{cl} G$, we get $|\mathcal{U} \mu(z)| \leq \|\mathcal{U} \mu\|_{L_{\infty}(\mathcal{H})}$. \hfill $\square$

**Lemma 3.** Let $f \in L_{\infty}(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Then $\mathcal{U}(g \mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$. Moreover, there is a bounded linear functional $L$ on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}_b(\partial G)$ such that $\mathcal{U}(g \mathcal{H}_m)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation

$$\Delta u = -g \text{ on } G, \quad \frac{\partial u}{\partial n} + uf = L \text{ on } \partial G.$$  

(9)

**Proof.** Suppose first that $g$ is nonnegative. Since $\mathcal{U}(g \mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by [3, Theorem A.6], the energy $\int |\nabla \mathcal{U}(g \mathcal{H}_m)|^2 d\mathcal{H}_m < \infty$. According to [7, Theorem 1.20], we have

$$\int \left| \nabla \mathcal{U}(g \mathcal{H}_m) \right|^2 d\mathcal{H}_m = \int g \mathcal{U}(g \mathcal{H}_m) d\mathcal{H}_m < \infty,$$

(10)

and therefore $\mathcal{U}(g \mathcal{H}_m) \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.1.4]).

Since $\mathcal{U}(g \mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$, $f \in L_{\infty}(\mathcal{H})$ and the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], then the operator

$$L(\varphi) = \int_{G} \nabla \varphi \cdot \nabla \mathcal{U}(g \mathcal{H}_m) d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g \mathcal{H}_m) f \varphi d\mathcal{H}_{m-1} - \int_{G} g \varphi d\mathcal{H}_m$$

(11)

is a bounded linear functional on $W^{1,2}(G)$.

According to [7, Theorem 4.2], there is a nonnegative $\nu \in \mathcal{C}(\partial G)$ such that $\mathcal{U} \nu = \mathcal{U}(g \mathcal{H}_m)$ on $\mathbb{R}^m \setminus \text{cl} G$. A bounded domain $V$ with smooth boundary such that $\text{cl} G \subset V$. Since $\mathcal{U} \nu$ is bounded in $V \setminus \text{cl} G \subset \mathbb{R}^m \setminus \text{cl} G$, Lemma 2 yields that $\nu \in \mathcal{C}_b(\partial (V \setminus \text{cl} G))$. Therefore, $\nu \in \mathcal{C}_b(\partial G)$. According to [13, Lemma 4], there is $\tilde{\nu} \in \mathcal{C}_b(\partial G)$ such that

$$\int_{\mathbb{R}^m \setminus \text{cl} G} \nabla \varphi \cdot \nabla \mathcal{U}(g \mathcal{H}_m) d\mathcal{H}_m = \int_{\mathbb{R}^m \setminus \text{cl} G} \nabla \varphi \cdot \nabla \mathcal{U} \nu d\mathcal{H}_m = \int_{\partial G} \varphi d\tilde{\nu}$$

(12)

for each $\varphi \in \mathcal{D}$. Let $\mu = \tilde{\nu} - f \mathcal{U}(g \mathcal{H}_m) \mathcal{H}$. Since $\mathcal{U}(f \mathcal{U}(g \mathcal{H}_m) \mathcal{H}) \in \mathcal{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18] and $\mathcal{U}(f \mathcal{U}(g \mathcal{H}_m) \mathcal{H})(x) \to 0$ as $|x| \to \infty$, we have $f \mathcal{U}(g \mathcal{H}_m) \mathcal{H} \in \mathcal{C}_b^1(\partial G)$. Therefore, $\mu \in \mathcal{C}_b^1(\partial G)$. \hfill $\square$
If \( \varphi \in \mathcal{D} \), then \( \varphi = \mathcal{U}((\Delta \varphi)\mathcal{H}_m) \) by [3, Theorem A.2]. According to [7, Theorem 1.20],

\[
\int_{\mathbb{R}^m} \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) \, d\mathcal{H}_m = \int_{\mathbb{R}^m} \nabla \mathcal{U}((\Delta \varphi)\mathcal{H}_m) \cdot \nabla \mathcal{U}(g\mathcal{H}_m) \, d\mathcal{H}_m = \int_{\mathbb{R}^m} g\mathcal{U}((\Delta \varphi)\mathcal{H}_m) \, d\mathcal{H}_m = \int_{\mathbb{R}^m} g\varphi \, d\mathcal{H}_m.
\]

(13)

Since \( \mathcal{H}_m(\partial G) = 0 \),

\[
\int_G \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) \, d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi \, d\mathcal{H}_{m-1} = \int_G g\varphi \, d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi \, d\mathcal{H}_{m-1} - \int_{\mathbb{R}^m \setminus \text{cl } G} \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) \, d\mathcal{H}_m = \int_G g\varphi \, d\mathcal{H}_m + \int_{\partial G} \varphi \, d\mu.
\]

(14)

**Lemma 4.** Let \( f \in L_\infty(\mathcal{H}) \) and \( g \in L_2(G) \cap L_p(\mathbb{R}^m) \), where \( p > m/2 \), \( g = 0 \) on \( \mathbb{R}^m \setminus G \). Let \( L \) be a bounded linear functional on \( W^{1,2}(G) \) representable by \( \mu \in \mathcal{C}_b(\partial G) \). If \( u \in L_\infty(G) \cap W^{1,2}(G) \) is a weak solution in \( W^{1,2}(G) \) of problem (3), then \( \mu \in \mathcal{C}_b(\partial G) \).

**Proof.** Let \( w = u - \mathcal{U}(g\mathcal{H}_m) \). According to **Lemma 3**, there is a bounded linear functional \( \tilde{L} \) on \( W^{1,2}(G) \) representable by \( \nu \in \mathcal{C}_b(\partial G) \) such that \( w \) is a weak solution in \( W^{1,2}(G) \) of the problem

\[
\Delta w = 0 \quad \text{on } G,
\]

\[
\frac{\partial w}{\partial n} + wf = L - \tilde{L} \quad \text{on } \partial G.
\]

(15)

Fix \( x \in G \). Choose a sequence \( G_j \) of open sets with \( C^\infty \) boundary such that \( \text{cl } G_j \subset G_{j+1} \subset G \), \( x \in G_1 \), and \( \cup G_j = G \). Fix \( r > 0 \) such that \( \Omega_{2r}(x) \subset G_1 \). Choose an infinitely differentiable function \( \psi \) such that \( \psi = 0 \) on \( \Omega_r(x) \) and \( \psi = 1 \) on \( \mathbb{R}^m \setminus \Omega_{2r}(x) \). According to Green’s identity,

\[
w(x) = \lim_{j \to -\infty} \left[ \int_{\partial G_j} h_x(y) \frac{\partial w(y)}{\partial n} \, d\mathcal{H}_{m-1}(y) - \int_{\partial G_j} w(y) \nu(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \right]
\]

\[
= \lim_{j \to -\infty} \left[ \int_{G_j} \nabla w(y) \cdot \nabla (h_x(y)\psi(y)) \, d\mathcal{H}_m(y)
\right.

\[
- \left. \int_{G_j} \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) \, d\mathcal{H}_m(y) \right].
\]
\[
\begin{align*}
&= \int_G \nabla w(y) \cdot \nabla (h_x(y)\psi(y)) \, d\mathcal{H}_m(y) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) \, d\mathcal{H}_m(y) \\
&= \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) \, d\mathcal{H}_m(y).
\end{align*}
\]

(16)

According to [16, Theorem 2.3.2], there is a sequence of infinitely differentiable functions \(w_n\) such that \(w_n \to w\psi\) in \(W^{1,2}(G)\). According to [6, Section 2],

\[
w(x) = \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \lim_{n \to \infty} \int_G \nabla w_n(y) \cdot \nabla h_x(y) \, d\mathcal{H}_m(y)
\]

(17)

Since the trace operator is a bounded operator from \(W^{1,2}(G)\) to \(L_2(\mathcal{H})\) by [8, Theorem 3.38], we obtain

\[
w(x) = \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y).
\]

(18)

Since \(w \in L_\infty(G)\) by Lemma 3, the trace of \(w\) is an element of \(L_\infty(\mathcal{H})\). Since

\[
\left| \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \right|
\]

\[
\leq \|w\|_{L_\infty(\mathcal{H})} \int_{\partial G} |n(y) \cdot \nabla h_x(y)| \, d\mathcal{H}_{m-1}(y)
\]

(19)

\[
\leq \|w\|_{L_\infty(\mathcal{H})} \left[ \sup_{z \in \partial G} \int_{\partial G} |n(y) \cdot \nabla h_x(y)| \, d\mathcal{H}_{m-1}(y) + \frac{1}{2} \right] < \infty
\]

by [6, Lemma 2.15 and Theorem 2.16] and the fact that \(\partial G\) is of class \(C^{1+\alpha}\), the function

\[
x \mapsto \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y)
\]

(20)

is bounded in \(G\). Since \(\mathcal{U}v\) is bounded in \(G\) and \(\mathcal{U}(f w\mathcal{H})\) is bounded in \(G\) by [6, Corollary 2.17 and Lemma 2.18], the function \(\mathcal{U}\mu\) is bounded in \(G\) by (18). Thus, \(\mu \in \mathcal{C}_b'(\partial G)\) by Lemma 2.

\[\square\]

**Notation 5.** Let \(X\) be a complex Banach space and \(T\) a bounded linear operator on \(X\). We denote by \(\text{Ker}\ T\) the kernel of \(T\), by \(\sigma(T)\) the spectrum of \(T\), by \(r(T)\) the spectral radius of \(T\), by \(X'\) the dual space of \(X\), and by \(T'\) the adjoint operator of \(T\). Denote by \(I\) the identity operator.

**Theorem 6.** Let \(X\) be a complex Banach space and \(K\) a compact linear operator on \(X\). Let \(Y\) be a subspace of \(X'\) and \(T\) a closed linear operator from \(Y\) to \(X\) such that \(y(Tx) = x(Ty)\) for each \(x, y \in Y\). Suppose that \(K'(Y) \subset Y\) and \(KTy = TK' y\) for each \(y \in Y\). Let \(\alpha \in \mathcal{C} \setminus \{0\}\), \(\text{Ker}(K' - \alpha I)^2 = \text{Ker}(K' - \alpha I) \subset Y\), and \(\{\beta \in \sigma(K') : (\beta - \alpha) \cdot \alpha \leq 0\} \subset \{\alpha\}\). If \(x, y \in X\), \((K' - \alpha I)x = y\), then \(x \in Y\) if and only if \(y \in Y\).
Proof. If \( x \in Y \), then \( y \in Y \). Suppose that \( y \in Y \). Since \( K \) is a compact operator, the operator \( K' \) is a compact operator by [14, Chapter IV, Theorem 4.1]. Suppose first that \( \alpha \in \sigma(K') \). Since \( K' \) is compact, then \( \alpha \) is a pol of the resolvent by [5, Satz 50.4]. Since

\[
\ker(K' - \alpha I)^2 = \ker(K' - \alpha I),
\]

the ascent of \((K' - \alpha I)\) is equal to 1. Since \( \alpha \) is a pol of the resolvent and the ascent of \((K' - \alpha I)\) is equal to 1, [5, Satz 50.2] yields that the space \( X' \) is the direct sum of \( \ker(K' - \alpha I) \) and \((K' - \alpha I)(X')\) and the descent of \((K' - \alpha I)\) is equal to 1. Since the descent of \((K' - \alpha I)\) is equal to 1, we have

\[
(K' - \alpha I)^2(X') = (K' - \alpha I)(X').
\]

Since the space \( X' \) is the direct sum of \( \ker(K' - \alpha I) \) and \((K' - \alpha I)(X')\), the operator \((K' - \alpha I)\) is invertible on \((K' - \alpha I)(X')\). If \( \alpha \not\in \sigma(K') \), then the space \( X' \) is the direct sum of \( \ker(K' - \alpha I) \) and \((K' - \alpha I)(X')\), and the operator \((K' - \alpha I)\) is invertible on \((K' - \alpha I)(X')\). Therefore, there are \( x_1 \in \ker(K' - \alpha I) \subset Y \) and \( x_2 \in (K' - \alpha I)(X') \) such that \( x_1 + x_2 = x \). We have \((K' - \alpha I)x_2 = y\).

Denote by \( Z \) the closure of \( Y \). Since \( K'(Y) \subset Y \), we obtain \( K'(Z) \subset Z \). Denote by \( K'_Z \) the restriction of \( K' \) to \( Z \). Then \( K'_Z \) is a compact operator in \( Z \). Since \( \ker(K' - \alpha I)^2 \subset Y \), we have

\[
\ker(K'_Z - \alpha I)^2 = \ker(K' - \alpha I)^2 = \ker(K' - \alpha I) = \ker(K'_Z - \alpha I).
\]

If \( \alpha \not\in \sigma(K'_Z) \), then the space \( Z \) is the direct sum of \( \ker(K'_Z - \alpha I) \) and \((K'_Z - \alpha I)(Z)\), and the operator \((K'_Z - \alpha I)\) is invertible on \( Z \). Suppose that \( \alpha \in \sigma(K'_Z) \). Since \( K'_Z \) is compact, then \( \alpha \) is a pol of the resolvent by [5, Satz 50.4]. Since

\[
\ker(K'_Z - \alpha I)^2 = \ker(K'_Z - \alpha I),
\]

the ascent of \((K'_Z - \alpha I)\) is equal to 1. Since \( \alpha \) is a pol of the resolvent and the ascent of \((K'_Z - \alpha I)\) is equal to 1, [5, Satz 50.2] yields that the space \( Z \) is the direct sum of \( \ker(K'_Z - \alpha I) \) and \((K'_Z - \alpha I)(Z)\) and the descent of \((K'_Z - \alpha I)\) is equal to 1. Since the descent of \((K'_Z - \alpha I)\) is equal to 1, we have

\[
(K'_Z - \alpha I)^2(Z) = (K' - \alpha I)(Z).
\]

Since the space \( Z \) is the direct sum of \( \ker(K'_Z - \alpha I) \) and \((K'_Z - \alpha I)(Z)\), the operator \((K'_Z - \alpha I)\) is invertible on \((K'_Z - \alpha I)(Z)\). Since \( y \in Y \subset Z \), there are \( y_1 \in \ker(K'_Z - \alpha I) \) and \( y_2 \in (K'_Z - \alpha I)(Z) \) such that \( y = y_1 + y_2 \). Since \( X' \) is the direct sum of \( \ker(K' - \alpha I) = \ker(K'_Z - \alpha I) \) and \((K' - \alpha I)(X') \subset (K'_Z - \alpha I)(Z) \) and \( y \in (K' - \alpha I)(X') \), we obtain that \( y_1 = 0 \) and \( y_2 = y \). Thus, \( y \in (K'_Z - \alpha I)(Z) \). Since \((K'_Z - \alpha I)\) is invertible on \((K'_Z - \alpha I)(Z)\), there is \( z \in (K'_Z - \alpha I)(Z) \) such that \((K'_Z - \alpha I)(z) = y \). Since \((K' - \alpha I)\) is invertible on \((K' - \alpha I)(X')\), we deduce that \( x_2 = z \in (K'_Z - \alpha I)(Z) \subset Z \).
Now, let $w \in \text{Ker}(K' - \alpha I)$. Fix a sequence $\{z_k\} \subset Y$ such that $z_k \to z = x_2$. Then

$$w(Ty) = y(Tw) = [(K' - \alpha I)x_2](Tw) = \lim_{k \to \infty} [(K' - \alpha I)z_k](Tw) = \lim_{k \to \infty} z_k((K - \alpha I)Tw) = \lim_{k \to \infty} z_k(T(K' - \alpha I)w) = \lim_{k \to \infty} z_k(0) = 0.$$ (26)

Since $w(Ty) = 0$ for each $w \in \text{Ker}(K' - \alpha I)$, [15, Chapter 10, Theorem 3] yields $Ty \in (K - \alpha I)(X)$.

Denote by $\tilde{K}'$ the restriction of $K'$ to $(K' - \alpha I)(X)$. If we denote by $P$ the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator $K'$, then $P(X') = (K' - \alpha I)(X')$ by [5, Satz 50.2] and $\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Therefore,

$$\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\} \subset \{\beta; (\beta - \alpha) \cdot \alpha > 0\} \subset \bigcup_{t > 0} \{\beta; |\beta - \alpha - ta| < |t\alpha|\}. \quad (27)$$

Since $\{\beta; |\beta - \alpha - t_1\alpha| < |t_1\alpha|\} \subset \{\beta; |\beta - \alpha - t_2\alpha| < |t_2\alpha|\}$ for $0 < t_1 < t_2$ and $\sigma(\tilde{K}')$ is a compact set (see [14, Chapter VI, Theorem 1.3, and Lemma 1.5]), there is $t > 0$ such that $\sigma(\tilde{K}') \subset \{\beta; |\beta - \alpha - ta| < |t\alpha|\}$. Therefore, $r(\tilde{K}' - \alpha I - taI) < |t\alpha|$. Since we have $r(t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - taI)) < 1$, the series

$$V = \sum_{k=0}^{\infty} (-1)^k [t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - taI)]^k \quad (28)$$

converges. Easy calculation yields that $V$ is the inverse operator of the operator $I + t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - taI) = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)$. Since $t^{-1}\alpha^{-1}y = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)x_2$, we have $x_2 = t^{-1}\alpha^{-1}Vy$. Denote $z_k = t^{-1}\alpha^{-1} [-t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - taI)]^k y$. Then

$$x_2 = \sum_{k=0}^{\infty} z_k. \quad (29)$$

Since $K'(Y) \subset Y$, $z_k \in Y$ for each $k$. Since $KT = TK'$ on $Y$, we have $Tz_k = t^{-1}\alpha^{-1} [-t^{-1}\alpha^{-1}(K - \alpha I - taI)]^k Ty$.

Since $(K - \alpha I)$, $(K - \alpha I)^2$, $(K' - \alpha I)$, and $(K' - \alpha I)^2$ are Fredholm operators with index 0 (see [14, Chapter V, Theorem 3.1]), [14, Chapter VII, Theorem 3.2] yields

$$\dim \text{Ker}(K - \alpha I)^2 = \dim \text{Ker}(K' - \alpha I)^2 = \dim \text{Ker}(K' - \alpha I) = \dim \text{Ker}(K - \alpha I), \quad (30)$$

and thus $\text{Ker}(K - \alpha I)^2 = \text{Ker}(K - \alpha I)$. If $\alpha \notin \sigma(K)$, then the space $X$ is the direct sum of $\text{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$, and the operator $(K - \alpha I)$ is invertible on $X$. Suppose that $\alpha \in \sigma(K)$. Since $K$ is compact, then $\alpha$ is a pole of the resolvent by [5, Satz 50.4]. Since

$$\text{Ker}(K - \alpha I)^2 = \text{Ker}(K - \alpha I), \quad (31)$$

the ascent of $(K - \alpha I)$ is equal to 1. Since $\alpha$ is a pole of the resolvent and the ascent of $(K - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space $X$ is the direct sum of $\text{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$ and the descent of $(K - \alpha I)$ is equal to 1. Since the descent of $(K - \alpha I)$ is equal to 1, we have $(K - \alpha I)^2(x) = (K - \alpha I)(X)$. Since the space $X$ is the direct sum
of \( \ker(K - \alpha I) \) and \((K - \alpha I)(X) = (K - \alpha I)^2(X)\), the operator \((K - \alpha I)\) is invertible on \((K - \alpha I)(X)\). Denote by \( \hat{K} \) the restriction of \( K \) to \((K - \alpha I)(X)\). If we denote by \( \beta \) the spectral projection corresponding to the spectral set \( \{ \alpha \} \) and the operator \( K \), then \( Q(X) = (K - \alpha I)(X) \) by [5, Satz 50.2] and \( \sigma(\hat{K}) = \sigma(K) \setminus \{ \alpha \} \) by [14, Chapter VI, Theorem 4.1]. Since \( \sigma(K) = \sigma(K') \) by [14, Chapter VI, Theorem 4.6], we obtain \( \sigma(\hat{K}) \subset \{ \beta; |\beta - \alpha - t\alpha| < |t\alpha| \} \). Therefore, \( r(\hat{K} - \alpha I - t\alpha I) < |t\alpha| \). Since \( Ty \in (K - \alpha X) \) and \( r(t^{-1}\alpha^{-1}(\hat{K} - \alpha I - t\alpha I)) < 1 \), the series

\[
\sum_{k=0}^{\infty} Tz_k = \sum_{k=0}^{\infty} t^{-1}\alpha^{-1}[ -t^{-1}\alpha^{-1}(\hat{K} - \alpha I - t\alpha I)]^k Ty
\]

converges. Since \( T \) is closed, \( x_2 = \sum z_k \), and \( \sum Tz_k \) converges, then the vector \( x_2 \) lies in \( Y \), the domain of \( T \).

**Theorem 7.** Let \( f \in L_{\infty}(\mathcal{H}) \), \( f \geq 0 \), and \( g \in L_2(G) \cap L_p(\mathbb{R}^m) \), where \( p > m/2 \), \( g = 0 \) on \( \mathbb{R}^m \setminus G \). Let \( L \) be a bounded linear functional on \( W^{1,2}(G) \) representable by \( \mu \in \mathcal{C}'(G) \). If \( u \) is a weak solution in \( W^{1,2}(G) \) of problem (3), then \( u \in L_{\infty}(G) \) if and only if \( \mu \in \mathcal{C}'(\partial G) \).

**Proof.** If \( u \in L_{\infty}(G) \), then \( \mu \in \mathcal{C}'(\partial G) \) by Lemma 4.

Suppose now that \( \mu \in \mathcal{C}'(\partial G) \). Let \( w = u - \mathcal{U}(g\mathcal{H}_m) \). According to Lemma 3, there is a bounded linear functional \( \tilde{L} \) on \( W^{1,2}(G) \) representable by \( \tilde{\mu} \in \mathcal{C}'(\partial G) \) such that \( w \) is a weak solution in \( W^{1,2}(G) \) of the problem

\[
\Delta w = 0 \quad \text{on } G, \\
\frac{\partial w}{\partial n} + wf = \tilde{L} \quad \text{on } \partial G.
\]

Define for \( \varphi \in L_{\infty}(\mathcal{H}) \) and \( x \in \partial G \),

\[
T\varphi(x) = \frac{1}{2} \varphi(x) + \int_{\partial G} \varphi(y) \frac{\partial}{\partial n(y)} h_x(y) d\mathcal{H}(y) + \mathcal{U}(f \varphi \mathcal{H}).
\]

Since \( \mathcal{U}(f \mathcal{H}) \in \mathcal{C}(\mathbb{R}^m) \) by [6, Corollary 2.17 and Lemma 2.18], the operator \( T \) is a bounded linear operator on \( L_{\infty}(\mathcal{H}) \) by [11, Proposition 8] and [6, Lemma 2.15]. The operator \( T - (1/2)I \) is compact by [12, Theorem 20] and [6, Theorem 4.1 and Corollary 1.11]. According to [10, Theorem 1], there is \( v \in \mathcal{C}'(\partial G) \subset (L_{\infty}(\mathcal{H}))' \) such that \( T'v = \tilde{\mu} \) and

\[
\int_G \nabla \mathcal{U} v \cdot \nabla v d\mathcal{H}_m + \int_{\partial G} \mathcal{U} vf d\mathcal{H} = \int v d\tilde{\mu},
\]

for each \( v \in \mathcal{D} \).

Remark that \( \mathcal{C}'(\partial G) \) is a closed subspace of \( (L_{\infty}(\mathcal{H}))' \). According to [11, Proposition 8], we have \( T'(\mathcal{C}'(\partial G)) \subset \mathcal{C}'(\partial G) \). Denote by \( \tau \) the restriction of \( T' \) to \( \mathcal{C}'(\partial G) \). According to [10, Lemma 11] and [14, Chapter VI, Theorem 1.2], we have \( \sigma(\tau) \subset \{ \beta; \beta \geq 0 \} \). Since \( \sigma(T') = \sigma(\tau) \) (see [15, Chapter VIII, Section 6, Theorem 2]), each \( \beta \in \sigma(T) \) is an eigenvalue (see [14, Chapter VI, Theorem 1.2]), and \( T \) is the restriction of \( \tau' \) to \( L_{\infty}(\mathcal{H}) \), we obtain that \( \sigma(T') = \sigma(T) \subset \{ \beta; \beta \geq 0 \} \) by [15, Chapter VIII, Section 6, Theorem 2].
According to [9, Theorem 1.11], we have $\ker T' \subset \mathcal{C}_b'(\partial G)$. According to [9, Lemma 1.10] and [10, Lemmas 12 and 13], $\ker T'' = \ker (T')^2$. Denote, for $\rho \in \mathcal{C}_b'(\partial G)$, by $\nabla \rho$ the restriction of $\nabla \rho$ to $\partial G$. Then $V$ is a closed operator from $\mathcal{C}_b'(\partial G)$ to $L^\infty(\mathcal{H})$ by [13, Lemma 5]. If $\rho \in \mathcal{C}_b'(\partial G)$, then $VT' \rho = TV \rho$ by [13, Lemma 4]. If $\rho_1, \rho_2 \in \mathcal{C}_b'(\partial G)$, then $\rho_1$ and $\rho_2$ have finite energy by [13, Proposition 23], [7, Theorem 1.20], and

$$\int \nabla \rho_1 \cdot \nabla \rho_2 \, d\mathcal{H}_m = \int \nabla \rho_2 \cdot d\mathcal{H}_m.$$  \hspace{1cm} (36)

Since $T' \nu = \tilde{\mu} \in \mathcal{C}_b'(\partial G)$, Theorem 6 yields that $\nu \in \mathcal{C}_b'(\partial G)$. Since $\nu$ has finite energy $\int \nabla \nu \cdot d\mathcal{H}_m$ by [7, Theorem 1.20], we obtain that $\mathcal{U} \nu \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.14]). Since $\mathcal{D}$ is dense in $W^{1,2}(G)$ by [16, Theorem 2.3.2], relation (35) yields that the function $\mathcal{U} \nu$ is a weak solution in $W^{1,2}(G)$ of

$$\Delta \nu = 0 \quad \text{on } G,$$

$$\frac{\partial \nu}{\partial n} + \nu f = 0 \quad \text{on } \partial G,$$

and $f \geq 0$, we obtain

$$0 = \int_G \nabla \nu \cdot \nabla \nu \, d\mathcal{H}_m + \int_{\partial G} \nu f \, d\mathcal{H}_m \geq \int_G |\nabla \nu|^2 \, d\mathcal{H}_m \geq 0.$$  \hspace{1cm} (38)

Therefore, $\nabla \nu = 0$ on $G$ and there is a constant $c$ such that $\nu(x) = c$ for $\mathcal{H}_m$-a.a. $x \in G$ by [16, Corollary 2.1.9]. Since $\nu \in \mathcal{C}_b'(\partial G)$, the function $\mathcal{U} \nu$ is bounded in $G$. Since $u(x) = \mathcal{U}(g \mathcal{H}_m)(x) + \mathcal{U} \nu(x) - c$ for $\mathcal{H}_m$-a.a. $x \in G$ and $\mathcal{U}(g \mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by Lemma 3, we obtain $u \in L^\infty(G)$.

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References


Boundedness of solutions


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