We prove an existence result for solution to a class of nonlinear degenerate elliptic equation associated with a class of partial differential operators of the form $Lu(x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))$, with $D_j = \partial/\partial x_j$, where $a_{ij} : \Omega \to \mathbb{R}$ are functions satisfying suitable hypotheses.

1. Introduction

In this paper, we prove the existence of solution in $D(A) \subseteq H_0(\Omega)$ for the following nonlinear Dirichlet problem:

$$
-Lu(x) + g(u(x))\omega(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \quad \text{on } \Omega,
$$

$$
u(x) = 0 \quad \text{on } \partial \Omega,
$$

where $L$ is an elliptic operator in divergence form

$$
Lu(x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)), \quad \text{with } D_j = \frac{\partial}{\partial x_j}
$$

and the coefficients $a_{ij}$ are measurable, real-valued functions whose coefficient matrix $(a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$
|\xi|^2 \omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq |\xi|^2 \nu(x)
$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$ a bounded open set with piecewise smooth boundary (i.e., $\partial \Omega \in C^{0,1}$), and $\omega$ and $\nu$ two weight functions (i.e., locally integrable non-negative functions).
The basic idea is to reduce (1.1) to an operator equation

\[ Au = T, \quad u \in D(A), \]  

(1.4)

where \( D(A) = \{ u \in H_0(\Omega) : u(x)g(u(x)) \in L^1(\Omega, \omega) \} \), and apply the theorem below.

**Theorem 1.1.** Suppose that the following assumptions are satisfied.

(H1) Dual pairs. Let the dual pairs \( \{ X, X^+ \} \) and \( \{ Y, Y^+ \} \) be given, where \( X, X^+, Y, \) and \( Y^+ \) are Banach spaces with corresponding bilinear forms \( \langle \cdot, \cdot \rangle_X \) and \( \langle \cdot, \cdot \rangle_Y \) and the continuous embeddings \( Y \subseteq X \) and \( X^+ \subseteq Y^+ \).

The dual pairs are compatible, that is,

\[ \langle T, u \rangle_X = \langle T, u \rangle_Y, \quad \forall T \in X^+, u \in Y. \]  

(1.5)

Moreover, the Banach spaces \( X \) and \( Y \) are separable and \( X \) is reflexive.

(H2) Operator \( A \). Let the operator \( A : D(A) \subseteq X \rightarrow Y^+ \) be given, and let \( K \) be a bounded closed convex set in \( X \) containing the zero point as an interior point and \( K \cap Y \subseteq D(A) \).

(H3) Local coerciveness. There exists a number \( \alpha \geq 0 \) such that \( \langle Av, v \rangle_Y \geq \alpha \) for all \( v \in Y \cap \partial K \), where \( \partial K \) denotes the boundary of \( K \) in the Banach space \( X \).

(H4) Continuity. For each finite-dimensional subspace \( Y_0 \) of the Banach space \( Y \), the mapping \( u \mapsto \langle Au, v \rangle_Y \) is continuous on \( K \cap Y_0 \) for all \( v \in Y_0 \).

(H5) Generalized condition (M). Let \( \{ u_n \} \) be a sequence in \( Y \cap K \) and let \( T \in X^+ \). Then, from

\[ u_n - u \quad \text{in } X \text{ as } n \rightarrow \infty, \]  

(1.6)

\[ \langle Au_n, v \rangle_Y \rightarrow \langle T, v \rangle_X \quad \text{as } n \rightarrow \infty, \forall v \in Y, \]  

(1.7)

it follows that \( Au = T \).

(H6) Quasiboundedness. Let \( \{ u_n \} \) be a sequence in \( Y \cap K \). Then, from (1.6) and \( \langle Au_n, u_n \rangle_Y \leq C \| u \|_X \) for all \( n \), it follows that the sequence \( \{ Au_n \} \) is bounded in \( Y^+ \).

(H7) The operator \( A \) is coercive, that is, \( \langle Av, v \rangle_Y / \| v \|_X \rightarrow \infty \) as \( \| v \|_X \rightarrow \infty, v \in Y \).

Then \( X^+ \subseteq R(A) \), that is, the equation \( Au = T \) has a solution \( u \) for each \( T \in X^+ \).

**Proof.** See [7, Theorem 27.B and Corollary 27.19].

\[ \square \]

We will apply this theorem to a sufficiently large ball \( K \) in the Banach spaces \( X = H_0(\Omega), X^+ = (H_0(\Omega))^*, \) and \( Y^+ = Y^* \).

We make the following basic assumption on the weights \( \omega \) and \( v \).

The weighted Sobolev inequality (WSI). Let \( \Omega \) be an open bounded set in \( \mathbb{R}^n \). There is an index \( q = 2\sigma, \sigma > 1 \), such that for every ball \( B \) and every \( f \in \text{Lip}_0(B) \) (i.e., \( f \in \text{Lip}(B) \) whose support is contained in the interior of \( B \)),

\[ \left( \frac{1}{v(B)} \int_B |f|^q v \, dx \right)^{1/q} \leq CR_B \left( \frac{1}{\omega(B)} \int_B |\nabla f|^2 \omega \, dx \right)^{1/2}, \]  

(1.8)
with the constant $C$ independent of $f$ and $B$, $R_B$ the radius of $B$, and the symbol $\nabla$ indicating the gradient, $v(B) = \int_B v(x)dx$, and $\omega(B) = \int_B \omega(x)dx$.

Thus, we can write

$$\left( \int_B |f|^q v \omega dx \right)^{1/q} \leq C_{B,v,\omega}(|\nabla f|^2 \omega)^{1/2}, \quad (1.9)$$

where $C_{B,v,\omega}$ is called the Sobolev constant and

$$C_{B,v,\omega} = \frac{C[v(B)]^{1/q} R_B}{[\omega(B)]^{1/2}}. \quad (1.10)$$

For instance, the WSI holds if $\omega$ and $v$ are as in [6, Chapter X, Theorem 4.8], or if $\omega$ and $v$ are as in [1, Theorem 1.5].

The following theorem will be proved in Section 3.

**Theorem 1.2.** Let $L$ be the operator (1.2) and satisfy (1.3). Suppose that the following assumptions are satisfied:

(i) $(v, \omega) \in A_2$;

(ii) the function $g : \mathbb{R} \to \mathbb{R}$ is continuous with $xg(x) \geq 0$ for all $x \in \mathbb{R}$;

(iii) $f_0/v \in L^q(\Omega, v)$ and $f_j/\omega \in L^2(\Omega, \omega)$, $j = 1, 2, \ldots, n$ (where $q$ is as in WSI). Then problem (1.1) has solution $u \in D(A) \subseteq H_0^{1}(\Omega)$;

(iv) if the function $g : \mathbb{R} \to \mathbb{R}$ is monotone increasing, then the solution is unique.

**Example 1.3.** Consider the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < 1$ and $|y| < 1\}$. By Theorem 1.2, the problem

$$-Lu(x) + u(x, y) e^{x|x|} |x|^{1/2} = 1 - \frac{\partial}{\partial x} (x^2 |y|) - \frac{\partial}{\partial y} (y^2 |x|) \quad \text{on } \Omega,$$

$$u(x, y) = 0 \quad \text{on } \partial \Omega, \quad (1.11)$$

where

$$Lu(x) = \left[ \frac{\partial}{\partial x} (|x|^{1/2} \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (|x|^{-1/2} \frac{\partial u}{\partial y}) \right] \quad (1.12)$$

has a unique solution $u \in D(A) = \{u \in H_0^{1}(\Omega) : g(u(x, y))u(x, y) \in L^1(\Omega, \omega)\}$, where $g(t) = te^t$, $\omega(x, y) = |x|^{1/2}$, $v(x, y) = |x|^{-1/2}$, $f_0(x, y) = 1$, $f_1(x, y) = x^2 |y|$, and $f_2(x, y) = y^2 |x|$.

2. Definitions and basic results

Let $\omega$ be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that $\omega$ belongs to the Muckenhoupt class $A_p$, $1 < p < \infty$, or that $\omega$ is an $A_p$-weight if there is a constant $C_1 = C(p, \omega)$ such that

$$\left( \frac{1}{|B|} \int_B \omega(x)dx \right)^{1/p} \left( \frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx \right)^{p-1} \leq C_1, \quad (2.1)$$
for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [4, 5] for more information about $A_p$-weights). The weight $\omega$ satisfies the doubling condition if $\omega(2B) \leq C\omega(B)$, for all balls $B \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as $B$ which is twice as large. If $\omega \in A_p$, then $\omega$ is doubling (see [5, Corollary 15.7]).

We say that the pair of weights $(v, \omega)$ satisfies the condition $A_p$ ($1 < p < \infty$ and $(v, \omega) \in A_p$) if and only if there is a constant $C_2$ such that

$$
\left( \frac{1}{|B|} \int_B v(x)dx \right) \left( \frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx \right)^{p-1} \leq C_2,
$$

for every ball $B \subset \mathbb{R}^n$.

**Remark 2.1.** If $(v, \omega) \in A_p$ and $\omega \leq v$, then $\omega \in A_p$ and $v \in A_p$.

Given a measurable subset $\Omega$ of $\mathbb{R}^n$, we will denote by $L^p(\Omega, \omega)$, $1 \leq p < \infty$, the Banach space of all measurable functions $f$ defined on $\Omega$ for which

$$
\|f\|_{L^p(\Omega, \omega)} = \left( \int_{\Omega} |f(x)|^p \omega(x)dx \right)^{1/p} < \infty.
$$

We will denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ such that the weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$
\|u\|_{W^{k,p}(\Omega, \omega)} = \left( \int_{\Omega} |u(x)|^p \omega(x)dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}.
$$

If $\omega \in A_p$, then $W^{k,p}(\Omega, \omega)$ is the closure of $C^\infty(\bar{\Omega})$ with respect to the norm (2.4) (see [2, Proposition 3.5]). The space $W^{k,p}_0(\Omega, \omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$
\|u\|_{W^{k,p}_0(\Omega, \omega)} = \left( \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}.
$$

When $k = 1$ and $p = 2$, the spaces $W^{1,2}(\Omega, \omega)$ and $W^{1,2}_0(\Omega, \omega)$ are Hilbert spaces. We will denote by $H_0(\Omega)$ the closure of $C_0^\infty(\bar{\Omega})$ with respect to the norm

$$
\|u\|_{H_0(\Omega)} = \left( \int_{\Omega} \langle A(x)\nabla u(x), \nabla u(x) \rangle dx \right)^{1/2},
$$

where $A(x) = [a_{ij}(x)]$ (the coefficient matrix) and the symbol $\nabla$ indicates the gradient.
Remark 2.2. Using the condition (1.3), we have

\[ \|u\|_{W^{1,2}_0(\Omega,\omega)} \leq \|u\|_{H_0(\Omega)} \leq \|u\|_{W^{1,2}_0(\Omega,v)}, \]  
(2.7)

\[ W^{1,2}_0(\Omega,v) \subset H_0(\Omega) \subset W^{1,2}_0(\Omega,\omega). \]  
(2.8)

Lemma 2.3. If \( \omega \in A_2 \), then \( W^{1,2}_0(\Omega,\omega) \hookrightarrow L^2(\Omega,\omega) \) is compact and

\[ \|u\|_{L^2(\Omega,\omega)} \leq C_3 \|u\|_{W^{1,2}_0(\Omega,\omega)}. \]  
(2.9)

Proof. The proof follows the lines of [3, Theorem 4.6]. \( \square \)

We introduce the following definition of (weak) solutions for problem (1.1).

Definition 2.4. A function \( u \in D(A) \subseteq H_0(\Omega) \) is (weak) solution to the problem (1.1) if

\[ \int_{\Omega} a_{ij}(x)D_iu(x)D_j\varphi(x)dx + \int_{\Omega} g(u(x))\varphi(x)\omega(x)dx \]

\[ = \int_{\Omega} f_0(x)\varphi(x)dx + \sum_{j=1}^{n} \int_{\Omega} f_j(x)D_j\varphi(x)dx, \]  
(2.10)

for all \( \varphi \in H_0(\Omega) \cap W^{k,p}(\Omega,v) \), where \( p > 4, k > n/2 \), and \( \|\varphi\|_Y = \|\varphi\|_{W^{k,p}(\Omega,v)} \), with \( D(A) = \{u \in H_0(\Omega) : g(u(x))u(x) \in L^1(\Omega,\omega)\} \).

Remark 2.5. Using that \( p > 4 \), we have that \( \nu \in A_2 \subset A_{p/2} \) and

\[ \|\cdot\|_{L^2(\Omega)} \leq \left[ \nu^{1/(1-p/2)}(\Omega) \right]^{(p-2)/2p} \|\cdot\|_{L^p(\Omega,\omega)}. \]  
(2.11)

Thus, \( W^{k,p}(\Omega,v) \subset W^{k,2}(\Omega) \subset C(\tilde{\Omega}) \) (by the Sobolev embedding theorem). Therefore \( \|\cdot\|_{C(\tilde{\Omega})} \leq C\|\cdot\|_Y \) and the embedding \( Y \subset C(\tilde{\Omega}) \) is continuous.

3. Proof of Theorem 1.2

(I) Existence. For \( u \in D(A) \) and \( \varphi \in Y \), we define

\[ B_1(u,\varphi) = \int_{\Omega} a_{ij}(x)D_iu(x)D_j\varphi(x)dx, \]
\[ B_2(u,\varphi) = \int_{\Omega} g(u(x))\varphi(x)\omega(x)dx, \]
\[ T(\varphi) = \int_{\Omega} f_0(x)\varphi(x)dx + \sum_{j=1}^{n} \int_{\Omega} f_j(x)D_j\varphi(x)dx. \]  
(3.1)

Then \( u \in D(A) \subseteq H_0(\Omega) \) is solution to problem (1.1) if

\[ B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi), \quad \forall \varphi \in Y. \]  
(3.2)
Step 1 \((T \in (H_0(\Omega))^*)\). In fact, using hypothesis (iii), Lemma 2.3, the Hölder inequality, the WSI, and (2.7), we obtain

\[
|T(\varphi)| \leq \int_{\Omega} |f_0| |\varphi| dx + \sum_{j=1}^{n} \int_{\Omega} |f_j| |D_j \varphi| dx
\]

\[
= \int_{\Omega} \left( \frac{|f_0|}{v} \right)^{1/\theta} |\varphi|^{1/\theta} dx + \sum_{j=1}^{n} \int_{\Omega} \left( \frac{|f_j|}{\omega} \right)^{1/2} |D_j \varphi|^{1/2} dx
\]

\[
\leq \left\| \frac{f_0}{v} \right\|_{L^\theta(\Omega, v)} \|\varphi\|_{L^\theta(\Omega, v)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega} \right\|_{L^2(\Omega, \omega)} \|D_j \varphi\|_{L^2(\Omega, \omega)}
\]

\[
\leq C_{B, \omega, v} \left\| \frac{f_0}{v} \right\|_{L^\theta(\Omega, v)} \|\nabla \varphi\|_{L^2(\Omega, \omega)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega} \right\|_{L^2(\Omega, \omega)} \|\nabla \varphi\|_{L^2(\Omega, \omega)}
\]

\[
\leq C \left( \left\| \frac{f_0}{v} \right\|_{L^\theta(\Omega, v)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega} \right\|_{L^2(\Omega, \omega)} \right) \|\varphi\|_{H_0(\Omega)}, \quad \forall \varphi \in H_0(\Omega).
\]

Step 2. By condition (1.3) and the hypothesis that the matrix \(\mathcal{A}\) is symmetric, we obtain

\[
|B_1(u, \varphi)| \leq \int_{\Omega} \text{big} \left| \langle \mathcal{A} \nabla u, \nabla \varphi \rangle \right| dx
\]

\[
\leq \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla u \rangle^{1/2} \langle \mathcal{A} \nabla \varphi, \nabla \varphi \rangle^{1/2} dx
\]

\[
\leq \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)}
\]

\[
\leq \|u\|_{H_0(\Omega)} \|\varphi\|_{W_0^{1,2}(\Omega, v)}
\]

\[
\leq \|u\|_{H_0(\Omega)} \|\varphi\|_Y,
\]

for all \(u \in H_0(\Omega), \varphi \in Y\).

Hence there exists exactly one linear continuous operator

\[
A_1 : H_0(\Omega) \rightarrow Y^*,
\]

with

\[
\langle A_1 u, \varphi \rangle_Y = B_1(u, \varphi), \quad \forall u \in H_0(\Omega), \varphi \in Y.
\]

Step 3. Note that \(|g(x)| \leq xg(x) + C_4\), for all \(x \in \mathbb{R}\). Therefore, if \(u \in D(A)\), we have that \(g(u(x)) \in L^1(\Omega, \omega)\). By using hypothesis (ii), Lemma 2.3, and Remark 2.5, we obtain for \(u \in D(A)\) fixed

\[
|B_2(u, \varphi)| \leq \int_{\Omega} |g(u(x))| |\varphi(x)| \omega(x) dx
\]

\[
\leq \|\varphi\|_{C(\Omega)} \int_{\Omega} |g(u(x))| \omega(x) dx
\]

\[
\leq C \|\varphi\|_Y.
\]
Thus, there exists a unique operator
\[ A_2 : D(A) \subseteq H_0(\Omega) \to Y^*, \]  
with
\[ \langle A_2 u, \varphi \rangle_Y = B_2(u, \varphi), \quad \forall u \in D(A), \; \varphi \in Y. \]  

**Step 4.** We define the operator
\[ A : D(A) \subseteq H_0(\Omega) \to Y^*, \quad A = A_1 + A_2. \]  
We have
\[ \langle Au, \varphi \rangle_Y = \langle A_1 u, \varphi \rangle_Y + \langle A_2 u, \varphi \rangle_Y = B_1(u, \varphi) + B_2(u, \varphi). \]  
Thus, \( u \in D(A) \) is a solution to problem (1.1) if
\[ \langle Au, \varphi \rangle_Y = T(\varphi), \quad \forall \varphi \in Y. \]  

Then, the problem (1.1) corresponds to the operator equation (1.4).

**Step 5.** General coerciveness of operator \( A \). Using the condition (1.3) and hypothesis (ii), we obtain
\[ \langle A\varphi, \varphi \rangle_Y = B_1(\varphi, \varphi) + B_2(\varphi, \varphi) \]
\[ = \int_{\Omega} a_{ij}(x)D_i\varphi(x)D_j\varphi(x)dx + \int_{\Omega} g(\varphi(x))\varphi(x)\omega(x)dx \]
\[ \geq \int_{\Omega} \langle \partial_i \nabla \varphi, \nabla \varphi \rangle dx \]
\[ = \| \varphi \|_{H_0(\Omega)}^2. \]  
Thus
\[ \lim_{\| \varphi \|_{H_0(\Omega)} \to \infty} \frac{\langle A\varphi, \varphi \rangle_Y}{\| \varphi \|_{H_0(\Omega)}^2} = +\infty. \]  

**Step 6.** Generalized condition (M). Let \( T \in (H_0(\Omega))^* \) and let \( \{ u_n \} \) be a sequence in \( Y \) with
\[ u_n \rightharpoonup u \quad \text{in} \; H_0(\Omega), \]
\[ \langle Au_n, \varphi \rangle_Y \to T(\varphi) \quad \text{as} \; n \to \infty, \; \forall \varphi \in Y, \]
\[ \lim_{n \to \infty} \langle Au_n, u_n \rangle \leq T(u). \]  
We want to show that this implies that \( Au = T \).
Using that the operator \( A_1 \) is linear and continuous, we obtain
\[ \langle A_1 u_n, \varphi \rangle_Y \to \langle A_1 u, \varphi \rangle_Y, \quad \forall \varphi \in Y. \]
Because of \((3.16)\), it is sufficient to prove that \(u \in D(A)\) and
\[
\langle A_2 u_n, \varphi \rangle_Y \rightarrow \langle A_2 u, \varphi \rangle_Y, \quad \forall \varphi \in Y.
\]
Therefore, it is sufficient to show that
\[
\int_{\Omega} [g(u_n(x)) - g(u(x))] \varphi(x) \omega(x) dx \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
Using the same argument in Step 3, we obtain
\[
\left| \int_{\Omega} (g(u_n(x)) - g(u(x))) \varphi(x) \omega(x) dx \right|
\leq \int_{\Omega} |g(u_n(x)) - g(u(x))| \varphi(x) \omega(x) dx
\leq \|\varphi\|_{C(\bar{\Omega})} \int_{\Omega} |g(u_n(x)) - g(u(x))| \omega(x) dx
\leq C\|\varphi\|_Y \int_{\Omega} |g(u_n(x)) - g(u(x))| \omega(x) dx.
\]
Therefore, it is sufficient to show that
\[
g(u_n(x)) \rightarrow g(u(x)) \quad \text{in} \quad L^1(\Omega, \omega).
\]
Note that it is sufficient to prove \((3.22)\) for a subsequence of \(\{u_n\}\).

If \((\nu, \omega) \in A_2\) and \(\omega \leq \nu\), then \(\omega \in A_2\) (see Remark 2.1). By Lemma 2.3,
\[
W^{1,2}_0(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega)
\]
is compact and \(\|u\|_{L^2(\Omega, \omega)} \leq C_2 \|u\|_{W^{1,2}_0(\Omega, \omega)}\). Using \((2.7)\), we also have that
\[
H_0(\Omega) \hookrightarrow L^2(\Omega, \omega)
\]
is compact. This implies \(u_n \rightharpoonup u\) in \(L^2(\Omega, \omega)\). Using again that \(\omega \in A_2\), we have \(u_n \rightharpoonup u\) in \(L^1(\Omega)\). Thus, there exists a subsequence, again denoted by \(\{u_n\}\), such that \(u_n(x) \rightarrow u(x)\) for almost all \(x \in \Omega\). The continuity of \(g\) implies that \(g(u_n(x)) \rightarrow g(u(x))\) for almost all \(x \in \Omega\). Moreover, since \(u_n \rightharpoonup u\) in \(H_0(\Omega)\), it follows that
\[
\sup \|u_n\|_{H_0(\Omega)} \leq C, \quad \text{independent of} \quad n.
\]
Hence, using \((1.2)\), we obtain
\[
\langle A_1 u_n, u_n \rangle_Y \leq \Lambda \|u_n\|^2_{H_0(\Omega)} \leq \Lambda C^2, \quad \text{with} \quad C \text{independent of} \quad n.
\]
Therefore, using \((3.16)\), we obtain
\[
\lim_{n \rightarrow \infty} \langle A_2 u_n, u_n \rangle_Y = \lim_{n \rightarrow \infty} \int_{\Omega} g(u_n(x)) u_n(x) \omega(x) dx \leq C,
\]
with \(C\) independent of \(n\).
The continuity of $g$ implies that $g(u_n(x))u_n(x)\omega(x) \rightarrow g(u(x))u(x)\omega(x)$ for almost all $x \in \Omega$. Therefore, by Fatou lemma, we have

$$
\int_{\Omega} g(u(x))u(x)\omega(x)dx < \infty, \quad (3.28)
$$

that is, $u \in D(A)$.

Now we want to show that $g(u_n(x)) \rightarrow g(u(x))$ in $L^1(\Omega, \omega)$. Let $a > 0$ be fixed. For each $x \in \Omega$, we have either

$$
|u_n(x)| \leq a \quad \text{or} \quad |g(u_n(x))| \leq a^{-1}g(u_n(x))u_n(x) \quad (3.29)
$$

(if $x \neq 0$, we can write $g(x) = x^{-1}[g(x)x]$). We get $|g(x)| \leq c(a)$ if $|x| \leq a$ (because $g$ is continuous).

Let $X$ be a measurable subset of $\Omega$. Then

$$
\int_X |g(u_n(x))|\omega(x)dx = \int_{X \cap \{x:|u_n(x)| \leq a\}} |g(u_n(x))|\omega(x)dx
+ \int_{X \cap \{x:|u_n(x)| > a\}} |g(u_n(x))|\omega(x)dx
\leq c(a)\omega(X) + a^{-1}\int_X g(u_n(x))u_n(x)\omega(x)dx
\leq c(a)\omega(X) + a^{-1}C \quad (by \ (3.27)).
$$

Hence, for all $\epsilon > 0$, we have

$$
\int_X |g(u_n(x))|\omega(x)dx \leq \frac{\epsilon}{2} \quad (3.31)
$$

if $a$ is sufficiently large and $\omega(X)$ is sufficiently small. Therefore, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that

$$
\int_X |g(u_n(x)) - g(u(x))|\omega(x)dx
\leq \int_X |g(u_n(x))|\omega(x)dx + \int_X |g(u(x))|\omega(x)dx \leq \epsilon, \quad (3.32)
$$

with $\omega(X) < \delta$. Thus, the Vitali convergence theorem tells us that (3.22) holds.

**Step 7. Quasiboundedness of the operator $A$.** Let $\{u_n\}$ be a sequence in $Y$ with $u_n \rightharpoonup u$ in $H_0(\Omega)$ and suppose that

$$
\langle Au_n, u_n \rangle_Y \leq C||u_n||_{H_0(\Omega)}, \quad \forall n. \quad (3.33)
$$

We want to show that the sequence $\{Au_n\}$ is bounded in $Y^\ast$. In fact, the boundedness of $\{u_n\}$ in $H_0(\Omega)$ implies that

$$
\lim_{n \rightarrow \infty} \langle Au_n, u_n \rangle_Y \leq C. \quad (3.34)
$$
Suppose by contradiction that the sequence \( \{Au_n\} \) is unbounded in \( Y^* \). Then there exists a subsequence, again denoted by \( \{u_n\} \), such that
\[
\|Au_n\|_{Y^*} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty. \tag{3.35}
\]
By the same arguments as in Step 6, we obtain that
\[
\langle Au_n, \varphi \rangle_Y \longrightarrow \langle Au, \varphi \rangle_Y \quad \text{as} \quad n \longrightarrow \infty, \quad \forall \varphi \in Y. \tag{3.36}
\]
The uniform boundedness principle tells us that the sequence \( \{Au_n\} \) is bounded (which is a contradiction with (3.35)).

Therefore, by Theorem 1.1, the equation \( Au = T \), with \( T \in (H_0(\Omega))^* \), has a solution \( u \in D(A) \subseteq H_0(\Omega) \), and it is the solution for problem (1.1).

\[ \text{(II) Uniqueness.} \]
If the function \( g : \mathbb{R} \rightarrow \mathbb{R} \) is monotone increasing, we have that \( (g(a) - g(b))(a - b) \geq 0 \), for all \( a, b \in \mathbb{R} \). Then
\[
\langle Au - Av, u - v \rangle_Y = \int_{\Omega} \langle \mathcal{A} \nabla (u - v), \nabla (u - v) \rangle dx \\
+ \int_{\Omega} (g(u(x)) - g(v(x))) (u(x) - v(x)) \omega(x) dx \\
\geq \int_{\Omega} \langle \mathcal{A} \nabla (u - v), \nabla (u - v) \rangle dx = \|u - v\|_{H_0(\Omega)}^2, \tag{3.37}
\]
for all \( u, v \in D(A) \).

Therefore, if \( u, v \in D(A) \) and \( Au = Av = T \), we obtain that \( u = v \).

References


