We first introduce a modified proximal point algorithm for maximal monotone operators in a Banach space. Next, we obtain a strong convergence theorem for resolvents of maximal monotone operators in a Banach space which generalizes the previous result by Kamimura and Takahashi in a Hilbert space. Using this result, we deal with the convex minimization problem and the variational inequality problem in a Banach space.

1. Introduction

Let $E$ be a real Banach space and let $T \subset E \times E^*$ be a maximal monotone operator. Then we study the problem of finding a point $v \in E$ satisfying

$$0 \in Tv.$$  \hspace{1cm} (1.1)

Such a problem is connected with the convex minimization problem. In fact, if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then Rockafellar's theorem [14, 15] ensures that the subdifferential mapping $\partial f \subset E \times E^*$ of $f$ is a maximal monotone operator. In this case, the equation $0 \in \partial f(v)$ is equivalent to $f(v) = \min_{x \in E} f(x)$.

In 1976, Rockafellar [17] proved the following weak convergence theorem.

**Theorem 1.1 (Rockafellar [17]).** Let $H$ be a Hilbert space and let $T \subset H \times H$ be a maximal monotone operator. Let $I$ be the identity mapping and let $J_r = (I + rT)^{-1}$ for all $r > 0$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n}x_n \quad (n = 1, 2, \ldots),$$  \hspace{1cm} (1.2)

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}0$.

This is called the proximal point algorithm, which was first introduced by Martinet [11]. If $T = \partial f$, where $f : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function,
then (1.2) is reduced to

\[ x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r_n} \|x_n - y\|^2 \right\} \quad (n = 1, 2, \ldots). \] (1.3)

Later, many researchers studied the convergence of the proximal point algorithm in a Hilbert space; see Brézis and Lions [4], Lions [10], Passty [12], Güler [7], Solodov and Svaiter [19] and the references mentioned there. In particular, Kamimura and Takahashi [8] proved the following strong convergence theorem.

**Theorem 1.2 (Kamimura and Takahashi [8]).** Let \( H \) be a Hilbert space and let \( T \subset H \times H \) be a maximal monotone operator. Let \( J_r = (I + rT)^{-1} \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in H \) and

\[ x_{n+1} = \alpha_n x + (1 - \alpha_n) J_r x_n \quad (n = 1, 2, \ldots), \] (1.4)

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). If \( T^{-1} 0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges strongly to \( P_{T^{-1} 0}(x) \), where \( P_{T^{-1} 0} \) is the metric projection from \( H \) onto \( T^{-1} 0 \).

Recently, using the hybrid method in mathematical programming, Kamimura and Takahashi [9] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Svaiter [19] in a Hilbert space. On the other hand, Censor and Reich [6] introduced a convex combination which is based on Bregman distance and studied some iterative schemes for finding a common asymptotic fixed point of a family of operators in finite-dimensional spaces.

In this paper, motivated by Censor and Reich [6], we introduce the following iterative sequence for a maximal monotone operator \( T \subset E \times E^* \) in a smooth and uniformly convex Banach space: \( x_1 = x \in E \) and

\[ x_{n+1} = J^{-1}(\alpha_n J x + (1 - \alpha_n) J J_r x_n) \quad (n = 1, 2, \ldots), \] (1.5)

where \( \{\alpha_n\} \subset [0, 1] \), \( \{r_n\} \subset (0, \infty) \), \( J \) is the duality mapping from \( E \) into \( E^* \), and \( J_r = (J + rT)^{-1} J \) for all \( r > 0 \). Then we extend Kamimura-Takahashi’s theorem to the Banach space (Theorem 3.3). It should be noted that we do not assume the weak sequential continuity of the duality mapping [1, 5, 13]. Finally, we apply Theorem 3.3 to the convex minimization problem and the variational inequality problem.

**2. Preliminaries**

Let \( E \) be a (real) Banach space with norm \( \|\cdot\| \) and let \( E^* \) denote the Banach space of all continuous linear functionals on \( E \). For all \( x \in E \) and \( x^* \in E^* \), we denote \( x^*(x) \) by \( \langle x, x^* \rangle \). We denote by \( \mathbb{R} \) and \( \mathbb{N} \) the set of all real numbers and the set of all positive integers, respectively. The **duality mapping** \( J \) from \( E \) into \( E^* \) is defined by

\[ J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \] (2.1)
for all \(x \in E\). If \(E\) is a Hilbert space, then \(J = I\), where \(I\) is the identity mapping. We sometimes identify a set-valued mapping \(A : E \to 2^{E^*}\) with its graph \(G(A) = \{(x,x^*) : x^* \in Ax\}\). An operator \(T \subset E \times E^*\) with domain \(D(T) = \{x \in E : Tx \neq \emptyset\}\) and range \(R(T) = \bigcup\{Tx : x \in D(T)\}\) is said to be monotone if \(\langle x - y, x^* - y^* \rangle \geq 0\) for all \((x,x^*),(y,y^*) \in T\). We denote the set \(\{x \in E : 0 \in Tx\}\) by \(T^{-1}0\). A monotone operator \(T \subset E \times E^*\) is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. If \(T \subset E \times E^*\) is maximal monotone, then the solution set \(T^{-1}0\) is closed and convex. A proper function \(f : E \to (-\infty, \infty]\) (which means that \(f\) is not identically \(\infty\)) is said to be convex if

\[
f(ax + (1-\alpha)y) \leq af(x) + (1-\alpha)f(y)
\]

for all \(x, y \in E\) and \(\alpha \in (0,1)\). The function \(f\) is also said to be lower semicontinuous if the set \(\{x \in E : f(x) \leq r\}\) is closed in \(E\) for all \(r \in \mathbb{R}\). For a proper lower semicontinuous convex function \(f : E \to (-\infty, \infty]\), the subdifferential \(\partial f\) of \(f\) is defined by

\[
\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \quad \forall y \in E\}
\]

for all \(x \in E\). It is easy to verify that \(0 \in \partial f(v)\) if and only if \(f(v) = \min_{x \in E} f(x)\). It is known that the subdifferential of the function \(f\) defined by \(f(x) = \|x\|^2/2\) for all \(x \in E\) is the duality mapping \(J\). The following theorem is also well known (see Takahashi [21] for details).

**Theorem 2.1.** Let \(E\) be a Banach space, let \(f : E \to (-\infty, \infty]\) be a proper lower semicontinuous convex function, and let \(g : E \to \mathbb{R}\) be a continuous convex function. Then

\[
\partial(f + g)(x) = \partial f(x) + \partial g(x)
\]

for all \(x \in E\).

A Banach space \(E\) is said to be strictly convex if

\[
\|x\| = \|y\| = 1, \quad x \neq y \quad \Rightarrow \quad \frac{\|x+y\|}{2} < 1.
\]

(2.5)

Also, \(E\) is said to be uniformly convex if for each \(\varepsilon \in (0,2]\), there exists \(\delta > 0\) such that

\[
\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \quad \Rightarrow \quad \frac{\|x+y\|}{2} \leq 1 - \delta.
\]

(2.6)

It is also said to be smooth if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for all \(x, y \in \{z \in E : \|z\| = 1\}\). We know the following (see Takahashi [20] for details):

1. if \(E\) is smooth, then \(J\) is single-valued;
2. if \(E\) is strictly convex, then \(J\) is one-to-one and \(\langle x - y, x^* - y^* \rangle > 0\) holds for all \((x,x^*),(y,y^*) \in J\) with \(x \neq y\);
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(3) if $E$ is reflexive, then $J$ is surjective;
(4) if $E$ is uniformly convex, then it is reflexive;
(5) if $E^*$ is uniformly convex, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

Let $E$ be a smooth Banach space. We use the following function studied in Alber [1], Kamimura and Takahashi [9], and Reich [13]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$  (2.8)

for all $x, y \in E$. It is obvious from the definition of $\phi$ that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. We also know that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$  (2.9)

for all $x, y, z \in E$. The following lemma was also proved in [9].

**Lemma 2.2 (Kamimura-Takahashi [9])**. Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $T \subset E \times E^*$ be a monotone operator. Then $T$ is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$ (see Barbu [2] and Takahashi [21]). If $T \subset E \times E^*$ is a maximal monotone operator, then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in D(T)$ satisfying

$$J(x) \in J(x_r) + rTx_r.$$  (2.10)

We define the *resolvent* of $T$ by $J_r x = x_r$. In other words, $J_r = (J + rT)^{-1}J$ for all $r > 0$. The resolvent $J_r$ is a single-valued mapping from $E$ into $D(T)$. If $E$ is a Hilbert space, then $J_r$ is *nonexpansive*, that is, $\|J_r x - J_r y\| \leq \|x - y\|$ for all $x, y \in E$ (see Takahashi [20]). It is easy to show that $T^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ denotes the set of all fixed points of $J_r$. We can also define, for each $r > 0$, the *Yosida approximation* of $T$ by $A_r = (J - J_r)/r$. We know that $(J_r x, A_r x) \in T$ for all $r > 0$ and $x \in E$. Let $C$ be a nonempty closed convex subset of the space $E$. By Alber [1] or Kamimura and Takahashi [9], for each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $P_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$  (2.11)

The mapping $P_C$ is called the *generalized projection* from $E$ onto $C$. If $E$ is a Hilbert space, then $P_C$ is coincident with the metric projection from $E$ onto $C$. We also know the following lemma.

**Lemma 2.3 ([1], see also [9])**. Let $E$ be a smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $x \in E$ and $x_0 \in C$. Then the following are equivalent:

1. $\phi(x_0, x) = \min_{y \in C} \phi(y, x);
2. $\langle y - x_0, Jx - Jx_0 \rangle \leq 0$ for all $y \in C$. 

3. Strong convergence theorem

The resolvents of maximal monotone operators have the following property, which was proved in the case of the resolvents of normality operators in Kamimura and Takahashi [9].

**Lemma 3.1.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, let $T \subset E \times E^*$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$, and let $J_r = (J + rT)^{-1}J$ for each $r > 0$. Then

$$\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x) \quad (3.1)$$

for all $r > 0$, $u \in T^{-1}0$, and $x \in E$.

**Proof.** Let $r > 0$, $u \in T^{-1}0$, and $x \in E$ be given. By the monotonicity of $T$, we have

$$\phi(u, x) = \phi(u, J_r x) + \phi(J_r x, x) + 2 \langle u - J_r x, J J_r x - Jx \rangle$$

$$= \phi(u, J_r x) + \phi(J_r x, x) + 2r \langle u - J_r x, -A x \rangle \quad (3.2)$$

$$\geq \phi(u, J_r x) + \phi(J_r x, x). \quad \square$$

Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $J$ be the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single-valued, one-to-one, and surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad (3.3)$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $g$ defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from $E^*$ into $\mathbb{R}$. We can prove the following lemma.

**Lemma 3.2.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $V$ be as in (3.3). Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \quad (3.4)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

**Proof.** Let $x \in E$ be given. Define $g(x^*) = V(x, x^*)$ and $f(x^*) = \|x^*\|^2$ for all $x^* \in E^*$. Since $J^{-1}$ is the duality mapping from $E^*$ into $E$, we have

$$\partial g(x^*) = \partial (-2 \langle x, \cdot \rangle + f)(x^*) = -2x + 2J^{-1}(x^*) \quad (3.5)$$

for all $x^* \in E^*$. Hence, we have

$$g(x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq g(x^* + y^*), \quad (3.6)$$
that is,
\[ V(x, x^*) + 2 \langle f^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \]  \hspace{1cm} (3.7)
for all \( x^*, y^* \in E^* \).

Now we can prove the following strong convergence theorem, which is a generalization of Kamimura-Takahashi’s theorem (Theorem 1.2).

**Theorem 3.3.** Let \( E \) be a smooth and uniformly convex Banach space and let \( T \subseteq E \times E^* \) be a maximal monotone operator. Let \( J_r = (I + rT)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and

\[ x_{n+1} = J^{-1}(\alpha_nJx + (1 - \alpha_n)JJ_nx_n) \quad (n = 1, 2, \ldots) \tag{3.8} \]

where \( \alpha_n \in [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). If \( T^{-1}0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges strongly to \( PT^{-1}0(x) \), where \( PT^{-1}0 \) is the generalized projection from \( E \) onto \( T^{-1}0 \).

**Proof.** Put \( y_n = J_{r_n}x_n \) for all \( n \in \mathbb{N} \). We denote the mapping \( PT^{-1}0 \) by \( P \). We first prove that \( \{x_n\} \) is bounded. From Lemma 3.1, we have

\[ \phi(Px, x_{n+1}) = \phi(Px, J^{-1}(\alpha_nJx + (1 - \alpha_n)JJ_nx_n)) \]
\[ = V(Px, \alpha_nJx + (1 - \alpha_n)JJ_nx_n) \]
\[ \leq \alpha_n V(Px, Jx) + (1 - \alpha_n) V(Px, Jx_n) \tag{3.9} \]
\[ = \alpha_n \phi(Px, x) + (1 - \alpha_n) \phi(Px, J_nx_n) \]
\[ \leq \alpha_n \phi(Px, x) + (1 - \alpha_n) \phi(Px, x_n) \]

for all \( n \in \mathbb{N} \). Hence, by induction, we have \( \phi(Px, x_n) \leq \phi(Px, x) \) for all \( n \in \mathbb{N} \). Since \( (||u|| - ||v||)^2 \leq \phi(u, v) \) for all \( u, v \in E \), the sequence \( \{x_n\} \) is bounded. Since \( \phi(Px, y_n) = \phi(Px, J_nx_n) \leq \phi(Px, x_n) \) for all \( n \in \mathbb{N}, \{y_n\} \) is also bounded. We next prove

\[ \lim_{n \to \infty} \langle x_n - Px, Jx - JPx \rangle \leq 0. \tag{3.10} \]

Put \( z_n = x_{n+1} \) for all \( n \in \mathbb{N} \). Since \( \{z_n\} \) is bounded, we have a subsequence \( \{z_{n_i}\} \) of \( \{z_n\} \) such that

\[ \lim_{i \to \infty} \langle z_{n_i} - Px, Jx - JPx \rangle = \limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle \tag{3.11} \]

and \( \{z_{n_i}\} \) converges weakly to some \( v \in E \). From the definition of \( \{x_n\} \), we have

\[ Jz_n - Jy_n = \alpha_n(Jx - Jy_n) \tag{3.12} \]

for all \( n \in \mathbb{N} \). Since \( \{y_n\} \) is bounded and \( \alpha_n \to 0 \) as \( n \to \infty \), we have

\[ \lim_{n \to \infty} ||Jz_n - Jy_n|| = \lim_{n \to \infty} \alpha_n||Jx - Jy_n|| = 0. \tag{3.13} \]
Since \( E \) is uniformly convex, \( E^* \) is uniformly smooth, and hence the duality mapping \( J^{-1} \) from \( E^* \) into \( E \) is uniformly norm-to-norm continuous on each bounded subset of \( E^* \). Therefore, we obtain from (3.13) that

\[
\lim_{n \to \infty} \| z_n - y_n \| = \lim_{n \to \infty} \| J(z_n) - J(y_n) \| = 0. \tag{3.14}
\]

This implies that \( y_{n_i} \rightharpoonup v \) as \( i \to \infty \), where \( \rightharpoonup \) implies the weak convergence. On the other hand, from \( r_n = \infty \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \| A_{r_n} x_n \| = \lim_{n \to \infty} \frac{1}{r_n} \| J x_n - J y_n \| = 0. \tag{3.15}
\]

If \((z, z^*) \in T\), then it holds from the monotonicity of \( T \) that

\[
\langle z - y_n, z^* - A_{r_n} x_n \rangle \geq 0 \tag{3.16}
\]

for all \( i \in \mathbb{N} \). Letting \( i \to \infty \), we get \( \langle z - v, z^* \rangle \geq 0 \). Then, the maximality of \( T \) implies \( v \in T^{-1} \). Applying Lemma 2.3, we obtain

\[
\limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle = \lim_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle = \langle v - Px, Jx - JPx \rangle \leq 0. \tag{3.17}
\]

Finally, we prove that \( x_n \to Px \) as \( n \to \infty \). Let \( \varepsilon > 0 \) be given. From (3.10), we have \( m \in \mathbb{N} \) such that

\[
\langle x_n - Px, Jx - JPx \rangle \leq \varepsilon \tag{3.18}
\]

for all \( n \geq m \). If \( n \geq m \), then it holds from (3.18) and Lemmas 3.1 and 3.2 that

\[
\phi(Px, x_{n+1}) = V(Px, \alpha_n Jx + (1 - \alpha_n) J y_n) \\
\leq V(Px, \alpha_n Jx + (1 - \alpha_n) J y_n - \alpha_n (Jx - JPx)) \\
\quad - 2 \langle J^{-1} (\alpha_n Jx + (1 - \alpha_n) J y_n) - Px, - \alpha_n (Jx - JPx) \rangle \\
= V(Px, (1 - \alpha_n) J y_n + \alpha_n Jx) + 2 \langle x_{n+1} - Px, \alpha_n (Jx - JPx) \rangle \\
\leq (1 - \alpha_n) V(Px, J y_n) + \alpha_n V(Px, Jx) + 2 \alpha_n (x_{n+1} - Px, Jx - JPx) \tag{3.19}
\]

\[
\leq (1 - \alpha_n) \phi(Px, y_n) + \alpha_n \phi(Px, Px) + 2 \alpha_n \varepsilon \\
= (1 - \alpha_n) \phi(Px, J r_n x_n) + 2 \alpha_n \varepsilon \\
\leq (1 - \alpha_n) \phi(Px, x_n) + 2 \alpha_n \varepsilon \\
= 2 \varepsilon \{ 1 - (1 - \alpha_n) \} + (1 - \alpha_n) \phi(Px, x_n).
\]
Therefore, we have
\[
\phi(Px, x_{n+1}) \\
\leq 2\varepsilon\left[1 - (1 - \alpha_n)\right] + (1 - \alpha_n)\left[2\varepsilon\left[1 - (1 - \alpha_{n-1})\right] + (1 - \alpha_{n-1})\phi(Px, x_{n-1})\right] \\
= 2\varepsilon\left[1 - (1 - \alpha_n)\right] + (1 - \alpha_n)\left(1 - \alpha_{n-1}\right)\phi(Px, x_{n-1}) \\
\leq \cdots \leq 2\varepsilon\left[1 - \prod_{i=m}^{n} (1 - \alpha_i)\right] + \prod_{i=m}^{n} (1 - \alpha_i)\phi(Px, x_m)
\]
for all \(n \geq m\). Since \(\sum_{i=1}^{\infty} \alpha_i = \infty\), we have \(\prod_{i=m}^{\infty} (1 - \alpha_i) = 0\) (see Takahashi [21]). Hence, we have
\[
l_{\text{lim sup}} \phi(Px, x_n) = \limsup_{l \to \infty} \phi(Px, x_{m+l+1}) \\
\leq \limsup_{l \to \infty} 2\varepsilon\left[1 - \prod_{i=m}^{m+l} (1 - \alpha_i)\right] + \prod_{i=m}^{m+l} (1 - \alpha_i)\phi(Px, x_m) = 2\varepsilon. \tag{3.21}
\]
This implies \(l_{\text{lim sup}} \phi(Px, x_n) \leq 0\) and hence we get
\[
\lim_{n \to \infty} \phi(Px, x_n) = 0. \tag{3.22}
\]
Applying Lemma 2.2, we obtain
\[
\lim_{n \to \infty} \|Px - x_n\| = 0. \tag{3.23}
\]
Therefore, \(\{x_n\}\) converges strongly to \(P_{\partial f^{-1}(0)}(x)\). \(\Box\)

4. Applications

In this section, we first study the problem of finding a minimizer of a proper lower semi-continuous convex function in a Banach space.

**Theorem 4.1.** Let \(E\) be a smooth and uniformly convex Banach space and let \(f : E \to (-\infty, \infty]\) be a proper lower semicontinuous convex function such that \((\partial f)^{-1}(0) \neq \emptyset\). Let \(\{x_n\}\) be a sequence defined as follows: \(x_1 = x \in E\) and
\[
y_n = \arg\min_{y \in E} \left\{f(y) + \frac{1}{2r_n}||y||^2 - \frac{1}{r_n}\langle y, Jx_n \rangle \right\} \quad (n = 1, 2, \ldots), \tag{4.1}
\]
\[
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n) \quad (n = 1, 2, \ldots),
\]
where \(\{\alpha_n\} \subset [0, 1]\) and \(\{r_n\} \subset (0, \infty)\) satisfy \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \) and \(\lim_{n \to \infty} r_n = \infty\). Then the sequence \(\{x_n\}\) converges strongly to \(P_{(\partial f)^{-1}(0)}(x)\).

**Proof.** By Rockafellar’s theorem [14, 15], the subdifferential mapping \(\partial f \subset E \times E^*\) is maximal monotone (see also Borwein [3], Simons [18], or Takahashi [21]). Fix \(r > 0, z \in E,\) and let \(J_r\) be the resolvent of \(\partial f\). Then we have
\[
J_z \in J(J_rz) + r\partial f(J_rz) \tag{4.2}
\]
and hence,
\[ 0 \in \partial f(Jrz) + \frac{1}{r} J(Jrz) - \frac{1}{r} Jz = \partial \left( f + \frac{1}{2r} \| \cdot \|^2 - \frac{1}{r} Jz \right)(Jrz). \] (4.3)

Thus, we have
\[ Jrz = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r} \| y \|^2 - \frac{1}{r} \langle y, Jz \rangle \right\}. \] (4.4)

Therefore, \( y_n = Jrx_n \) for all \( n \in \mathbb{N} \). Using Theorem 3.3, \( \{ x_n \} \) converges strongly to \( P_{\partial f^{-1}(0)}(x) \).

We next study the problem of finding a solution of a variational inequality. Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) and let \( A : C \to E^* \) be a single-valued, monotone, and hemicontinuous operator such that \( \text{VI}(C,A) \neq \emptyset \). Let \( \{ x_n \} \) be a sequence defined as follows: \( x_1 = x \in E \) and
\[
\begin{align*}
  y_n &= \text{VI} \left( C, A + \frac{1}{r_n} (J - Jx_n) \right) \quad (n = 1, 2, \ldots), \\
  x_{n+1} &= J^{-1} \left( \alpha_n Jx + (1 - \alpha_n) Jy_n \right) \quad (n = 1, 2, \ldots),
\end{align*}
\] (4.7)

where \( \{ \alpha_n \} \subset [0, 1] \) and \( \{ r_n \} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). Then, the sequence \( \{ x_n \} \) converges strongly to \( P_{\text{VI}(C,A)}(x) \).

Proof. By Rockafellar’s theorem [16], the mapping \( T \in E \times E^* \) defined by
\[
Tx = \begin{cases} 
  A(x) + N_C(x), & \text{if } x \in C, \\
  \emptyset, & \text{otherwise},
\end{cases}
\] (4.8)

is maximal monotone and \( T^{-1} 0 = \text{VI}(C,A) \). Fix \( r > 0 \), \( z \in E \), and let \( J_r \) be the resolvent of \( T \). Then we have
\[ Jz \in J(Jrz) + rT(Jrz) \] (4.9)
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and hence,

\[-A(Jr,z) + \frac{1}{r}(Jz - J(Jr,z)) \in N_C(Jr,z). \tag{4.10}\]

Thus, we have

\[\langle y - Jrz, A(Jr,z) + \frac{1}{r}(J(Jr,z) - Jz) \rangle \geq 0 \tag{4.11}\]

for all \(y \in C\), that is,

\[Jr z = VI(C, A + \frac{1}{r}(J - Jz)). \tag{4.12}\]

Therefore, \(y_n = Jr_n x_n\) for all \(n \in \mathbb{N}\). Using Theorem 3.3, \(\{x_n\}\) converges strongly to \(P_{VI(C,A)}(x)\). \(\square\)

References


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