NONEXISTENCE RESULTS OF SOLUTIONS TO SYSTEMS OF SEMILINEAR DIFFERENTIAL INEQUALITIES ON THE HEISENBERG GROUP

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We establish nonexistence results to systems of differential inequalities on the \((2N + 1)\)-Heisenberg group. The systems considered here are of the type \((ES_m)\). These nonexistence results hold for \(N\) less than critical exponents which depend on \(p_i\) and \(\gamma_i, 1 \leq i \leq m\). Our results improve the known estimates of the critical exponent.

1. Introduction

For the reader’s convenience, we recall some background facts used here. The Heisenberg group \(\mathbb{H}^N\), whose points will be denoted by \(\eta = (x, y, \tau)\), is the Lie group \((\mathbb{R}^{2N+1}, \circ)\) with the group operation \(\circ\) defined by

\[
\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(\langle x, \tilde{y} \rangle - \langle \tilde{x}, y \rangle)),
\]

where \(\langle \cdot, \cdot \rangle\) is the usual inner product in \(\mathbb{R}^N\). The Laplacian \(\Delta_H\) over \(\mathbb{H}^N\) is obtained, from the vector fields \(X_i = \partial_{x_i} + 2y_i \partial_{\tau}\) and \(Y_i = \partial_{y_i} - 2x_i \partial_{\tau}\), by

\[
\Delta_H = \sum_{i=1}^{N} (X_i^2 + Y_i^2).
\]

Observe that the vector field \(T = \partial_{\tau}\) does not appear in (1.2). This fact makes us presume a “loss of derivative” in the variable \(\tau\). The compensation comes from the relation

\[
[X_i, Y_j] = -4T, \quad j, k \in \{1, 2, \ldots, N\}.
\]

The relation (1.3) proves that \(\mathbb{H}^N\) is a nilpotent Lie group of order 2. Incidentally, (1.3) constitutes an abstract version of the canonical relations of commutation of Heisenberg between momentums and positions. Explicit computation gives the expression

\[
\Delta_H = \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).
\]
A natural group of dilatations on $\mathbb{H}^N$ is given by
\[ \delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0, \] (1.5)
whose Jacobian determinant is $\lambda^Q$, where
\[ Q = 2N + 2 \] (1.6)
is the homogeneous dimension of $\mathbb{H}^N$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}^N$ and homogeneous with respect to the dilatations $\delta_\lambda$. More precisely, we have
\[ \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}}u)(\eta \circ \tilde{\eta}), \]
\[ \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}}u) \circ \delta_\lambda \quad \forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N. \] (1.7)

It is natural to define a distance from $\eta$ to the origin by
\[ |\eta|_{\mathbb{H}} = \left( \tau^2 + \sum_{i=1}^N (x_i^2 + y_i^2)^2 \right)^{1/4}. \] (1.8)

In [7], Pohozaev and Véron gave another proof of the result of Birindelli et al. [1] concerning the nonexistence of weak solutions of the differential inequality
\[ \Delta_{\mathbb{H}}(au) + |\eta|^{\gamma_1}_{\mathbb{H}}|v|^p \leq 0 \quad \text{in} \quad \mathbb{H}^N \] (1.9)
for $\gamma > -2, 1 < p \leq (Q + \gamma)/(Q - 2)$, and $a \in L^\infty(\mathbb{H}^N)$.

They then addressed the question of nonexistence of weak solutions of the system (ES2):
\[ -\Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|^{\gamma_i}_{\mathbb{H}} |v|^p, \quad -\Delta_{\mathbb{H}}(a_2 u_i) \geq |\eta|^{\gamma_2}_{\mathbb{H}} |u|^{p_2}, \] (1.10)
where $a_i, i \in \{1,2\}$, are measurable and bounded functions defined on $\mathbb{H}^N$, and $p_i > 1$ and $\gamma_i, i = 1,2$, are real numbers. They showed that this system admits no solution defined in $\mathbb{H}^N$ whenever $\gamma_i > -2$ and $1 < p_i \leq (Q + \gamma_i)/(Q - 2), i = 1,2$. The estimates on $p_i, i = 1,2$, are obtained using Young's inequality and are not optimal. Using the Hölder inequality, we obtain better estimates on $p_i, 1 \leq i \leq m$. The same strategy is suitable to study the systems (PS$_m$) and (HS$_m$).

We also studied the following systems:
\[ (\text{PS}_m) \frac{\partial u_i}{\partial t} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|^{\gamma_{i+1}}_{\mathbb{H}} |u_{i+1}|^{p_{i+1}}, \eta \in \mathbb{H}^N, 1 \leq i \leq m, u_{m+1} = u_1, \]
\[ (\text{HS}_m) \frac{\partial^2 u_i}{\partial t^2} - \Delta_{\mathbb{H}}(a_i u_i) \geq |\eta|^{\gamma_{i+1}}_{\mathbb{H}} |u_{i+1}|^{p_{i+1}}, \eta \in \mathbb{H}^N, 1 \leq i \leq m, u_{m+1} = u_1, \]
and showed the following results.
Theorem 1.1. Assume that the initial data \( u_i^{(0)} \in L^1(\mathbb{R}^{2N+1}) \) and \( \int u_i^{(0)}(\eta) d\eta \geq 0, \ 1 \leq i \leq m \). If

$$Q \leq \max \{ X_1, X_2, \ldots, X_m \}, \quad (1.11)$$

where the vector \( (X_1, X_2, \ldots, X_m)^T \) is the solution of (3.1), then there is no nontrivial global weak solution \( (u_1, \ldots, u_m) \) of the system \( (PS_m) \).

Theorem 1.2. Assume that initial data (for the first derivatives of \( u_i \), \( 1 \leq i \leq m \)) \( u_i^{(1)} \in L^1(\mathbb{R}^{2N+1}) \) and \( \int u_i^{(1)}(\eta) d\eta \geq 0, \ 1 \leq i \leq m \). If

$$Q \leq 1 + \max \{ X_1, X_2, \ldots, X_m \}, \quad (1.12)$$

where the vector \( (X_1, X_2, \ldots, X_m)^T \) is the solution of (3.1), then there is no nontrivial global weak solution \( (u_1, \ldots, u_m) \) of the system \( (HS_m) \).

In [2], the first author and Obeid presented results for systems of evolution type with higher-order time derivatives. Their results are the generalized versions of our previous results (Theorems 1.1 and 1.2) on \( (PS_m) \) and \( (HS_m) \).

For interesting results on elliptic equations and systems, we refer to the recent papers of Kartsatos and Kurta [3], Kurta [4, 5], and Mitidieri and Pohozaev [6].

To render the presentation very clear, we start with the case of systems of two inequalities.

2. Systems of two inequalities

In this section, we treat the case \( m = 2 \) and consider the system \( (ES_2) \).

We identify points in \( \mathbb{H}^N \) with points in \( \mathbb{R}^{2N+1} \). We also recall that the Haar measure on \( \mathbb{H}^N \) is identical to the Lebesgue measure \( d\eta = dx dy d\tau \) on \( \mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \). In the sequel, the integral \( \int_{\mathbb{R}^{2N+1}} \) will be simply denoted by \( \int \); however, the measure of integration will be specified.

Definition 2.1. Let \( a_1 \) and \( a_2 \) be two bounded measurable functions on \( \mathbb{R}^{2N+1} \). A weak solution \( (u, v) \) of the system \( (ES_2) \) on \( \mathbb{R}^{2N+1} \) is a pair of locally integrable functions \( (u, v) \) such that

$$u \in L^p_\text{loc}(\mathbb{R}^{2N+1}, |\eta|^{y_1}_{\mathbb{H}} d\eta), \quad v \in L^p_\text{loc}(\mathbb{R}^{2N+1}, |\eta|^{y_1}_{\mathbb{H}} d\eta), \quad (2.1)$$

satisfying

$$\int_{\mathbb{R}^{2N+1}} \left( a_1 u \Delta_H \varphi + |\eta|^{y_1}_{\mathbb{H}} |v|^{p_1} \varphi \right) d\eta \leq 0,$$

$$\int_{\mathbb{R}^{2N+1}} \left( a_2 v \Delta_H \varphi + |\eta|^{y_1}_{\mathbb{H}} |u|^{p_2} \varphi \right) d\eta \leq 0 \quad (2.2)$$

for any nonnegative test function \( \varphi \in C^2_\text{c}(\mathbb{R}^{2N+1}) \).
Theorem 2.2. Assume that

\[ Q \leq Q^* = 2 + \frac{1}{p_1 p_2 - 1} \max \{ (y_1 + 2) + p_1 (y_2 + 2) ; p_2 (y_1 + 2) + (y_2 + 2) \} \]  

Then there is no nontrivial weak solution \((u, v)\) of the system \((ES_2)\).

Proof. Let \(\varphi_R \in \mathcal{D}(\mathbb{H}^N)\) be a nonnegative function such that

\[ \varphi_R(\eta) = \Phi^\lambda \left( \frac{\tau^2 + |x|^4 + |y|^4}{R^4} \right), \]  

where \(\lambda \gg 1, R > 0\), and \(\Phi \in \mathcal{D}([0, +\infty[)\) is the “standard cutoff function”

\[ \Phi(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq 1, \\ 0, & \text{if } r \geq 2 \end{cases} \]  

Note that \(\text{supp}(\varphi_R)\) is a subset of

\[ \Omega_R = \{ \eta \equiv (x, y, \tau) \in \mathbb{H}^N ; 0 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4 \} \]  

and \(\text{supp}(\Delta_{\mathbb{H}} \varphi_R)\) is included in

\[ \mathcal{C}_R = \{ \eta \equiv (x, y, \tau) \in \mathbb{H}^N ; R^4 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4 \}. \]  

Let

\[ \rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^4}, \]  

then

\[
\Delta_{\mathbb{H}} \varphi_R(\eta) = \frac{4(N + 4) \Phi'(\rho)}{R^4} \lambda \Phi^{\lambda-1}(\rho) (|x|^2 + |y|^2) \\
+ \frac{16 \Phi''(\rho)}{R^8} \lambda \Phi^{\lambda-1}(\rho) \\
\times \left( (|x|^6 + |y|^6) + \tau^2 (|x|^2 + |y|^2) + 2\tau \langle x, y \rangle (|x|^2 - |y|^2) \right) \\
+ \frac{16 \Phi'^2(\rho)}{R^8} \lambda (\lambda - 1) \Phi^{\lambda-2}(\rho) \\
\times \left( (|x|^6 + |y|^6) + \frac{\tau^2}{4} (|x|^2 + |y|^2) + 2\tau \langle x, y \rangle (|x|^2 - |y|^2) \right).  
\]  

It follows that there is a positive constant \(C > 0\), independent of \(R\), such that

\[ |\Delta_{\mathbb{H}} \varphi_R(\eta)| \leq \frac{C}{R^2} \quad \forall \eta \in \Omega_R. \]
Let \((u, v)\) be a nontrivial weak solution of \((ES_2)\). Using \((2.2)\) with \(\varphi = \varphi_R\), one has
\[
\int |\eta|^{2_1}_{\mathbb{H}} |v|^{p_1} \varphi_R d\eta \leq - \int a_1 u \Delta_{\mathbb{H}} \varphi_R d\eta \\
\leq ||a_1||_{L^\infty} \int |u| |\Delta_{\mathbb{H}} \varphi_R| d\eta \\
\leq ||a_1||_{L^\infty} \left( \int |\eta|^{2_1}_{\mathbb{H}} |u|^{p_2} \varphi_R \right)^{1/p_1} \left( \int |\Delta_{\mathbb{H}} \varphi_R| \varphi_R |\eta|^{p_2}_{\mathbb{H}} \right)^{1/p_2^\prime},
\]
\[
\int |\eta|^{2_1}_{\mathbb{H}} |u|^{p_2} \varphi_R d\eta \leq - \int a_2 v \Delta_{\mathbb{H}} \varphi_R d\eta \\
\leq ||a_2||_{L^\infty} \left( \int |\eta|^{2_1}_{\mathbb{H}} |v|^{p_1} \varphi_R \right)^{1/p_1} \left( \int |\Delta_{\mathbb{H}} \varphi_R| \varphi_R |\eta|^{p_1}_{\mathbb{H}} \right)^{1/p_1^\prime},
\]
thanks to the Hölder inequality. Setting
\[
I(R) = \int |\eta|^{2_1}_{\mathbb{H}} |u|^{p_1} \varphi_R d\eta, \quad J(R) = \int |\eta|^{2_1}_{\mathbb{H}} |v|^{p_1} \varphi_R d\eta,
\]
we have
\[(2.13)\]
\[
J(R) \leq C_1 I(R)^{1/p_1} \mathcal{A}_{p_1, \gamma_1}(R)^{1/p_1^\prime},
\]
where
\[
\mathcal{A}_{p_1, \gamma_1}(R) = \int |\Delta_{\mathbb{H}} \varphi_R| \varphi_R |\eta|^{p_1}_{\mathbb{H}} d\eta
\]
and \(C_1\) is a positive constant independent of \(R\). Similarly, we have
\[(2.16)\]
\[
I(R) \leq C_2 J(R)^{1/p_1} \mathcal{A}_{p_1, \gamma_1}(R)^{1/p_1^\prime},
\]
where
\[
\mathcal{A}_{p_1, \gamma_1}(R) = \int |\Delta_{\mathbb{H}} \varphi_R| \varphi_R |\eta|^{p_1}_{\mathbb{H}} d\eta
\]
and \(C_2\) is a positive constant independent of \(R\).

Note that for \(\lambda\) sufficiently large, the integrals \(\mathcal{A}_{p_i, \gamma_i}(R), \ i \in \{1, 2\}\), are convergent. Indeed, in the expression \(\mathcal{A}_{p_i, \gamma_i}(R), \ i \in \{1, 2\}\), we have \(|\eta|_{\mathbb{H}} \geq R^4\), and the exponent of \(\varphi_R\) is positive for \(\lambda\) large enough.

In order to estimate the integrals \(\mathcal{A}_{p_i, \gamma_i}(R), \ i \in \{1, 2\}\), we introduce the scaled variables
\[
\tilde{\tau} = R^{-2} \tau, \quad \tilde{x} = R^{-1} x, \quad \tilde{y} = R^{-1} y.
\]
Using the fact that \(\text{supp} \varphi_R \subset \Omega_R\), we conclude that
\[(2.19)\]
\[
\mathcal{A}_{p_i, \gamma_i}(R) \leq CR^{2N+4-2p_i+\gamma_i(1-p_i)}, \quad i \in \{1, 2\}.
\]
Using \((2.16)\) and \((2.19)\) in \((2.14)\), we obtain
\[(2.20)\]
\[
J(R)^{1-1/p_1 p_2} \leq C \mathcal{A}_{p_1, \gamma_1}(R)^{1/p_1 p_2} \mathcal{A}_{p_2, \gamma_2}(R)^{1/p_2^\prime} \leq CR^q,
\]
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where

\[
\sigma_I = \frac{1}{p_2^2} (2N + 2 - 2p_2 + \gamma_2 (1 - p_2')) + \frac{1}{p_1 p_2} (2N + 2 - 2p_1 + \gamma_1 (1 - p_1')) \\
= Q \left( 1 - \frac{1}{p_1 p_2} \right) - \frac{(2p_2 + 2 + \gamma_2)p_1 + \gamma_1}{p_1 p_2}.
\]

(2.21)

Similarly, we have

\[
I(R)^{1-1/p_1 p_2} \leq C A_{p_1, \gamma_1}(R) ^{1/p_1} A_{p_2, \gamma_2}(R) ^{1/p_2} \leq CR^{\sigma_I},
\]

(2.22)

where

\[
\sigma_I = Q \left( 1 - \frac{1}{p_1 p_2} \right) - \frac{(2p_1 + 2 + \gamma_1)p_2 + \gamma_2}{p_1 p_2}.
\]

(2.23)

Now, we require that \( \sigma_I \leq 0 \) or \( \sigma_J \leq 0 \), which is equivalent to

\[
Q \leq Q^*_e = \frac{1}{p_1 p_2 - 1} \max \{ p_1 (2(p_2 + 1) + \gamma_2) + \gamma_1; p_2 (2(p_1 + 1) + \gamma_1) + \gamma_2 \} \\
= 2 + \frac{1}{p_1 p_2 - 1} \max \{ (\gamma_1 + 2) + p_1 (\gamma_2 + 2); p_2 (\gamma_1 + 2) + (\gamma_2 + 2) \}.
\]

(2.24)

In this case, the integrals \( I(R) \) and \( J(R) \), increasing in \( R \), are bounded uniformly with respect to \( R \). Using the monotone convergence theorem, we deduce that \( |\eta|^{\gamma_1} |v|^{p_1} \) and \( |\eta|^{\gamma_2} |u|^{p_2} \) are in \( L^1(\mathbb{R}^{2N+1}) \). Note that instead of (2.11) we have, more precisely,

\[
\int |\eta|^{\gamma_1} |v|^{p_1} \varphi_R d\eta \leq ||a_1||_{L^p} \left( \int_{\mathbb{R}^n} |\eta|^{\gamma_2} |u|^{p_2} \varphi_R d\eta \right)^{1/p_2} \ \mathcal{A}_{p_2, \gamma_2}(R)^{1/p_2} \\
\leq C \int_{\mathbb{R}^n} |\eta|^{\gamma_2} |u|^{p_2} \varphi_R d\eta.
\]

(2.25)

Finally, using the dominated convergence theorem, we obtain that

\[
\lim_{R \to +\infty} \int_{\mathbb{R}^n} |\eta|^{\gamma_1} |v|^{p_1} \varphi_R d\eta = 0.
\]

(2.26)

Hence,

\[
\int |\eta|^{\gamma_1} |v|^{p_1} d\eta = 0,
\]

(2.27)

which implies that \( v \equiv 0 \) and \( u \equiv 0 \) via (2.12). This contradicts the fact that \((u, v)\) is a nontrivial weak solution of \((\text{ES}_2)\), which achieves the proof. \( \square \)

Remark 2.3. The critical exponent \( Q^*_e \) can be written as

\[
Q^*_e = 2 + \max \{ X_1, X_2 \},
\]

(2.28)
where the vector \((X_1, X_2)^T\) is the solution of the linear system

\[
\begin{pmatrix}
-1 & p_1 \\
p_2 & -1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 + 2 \\
\gamma_2 + 2
\end{pmatrix}.
\] (2.29)

**Comment 2.4.** In their paper, Pohozaev and Véron [7] showed that if

\[
1 < p_j \leq \frac{Q + \gamma_j}{Q - 2}, \quad j \in \{1, 2\},
\] (2.30)

then the system (ES\(_2\)) has no nontrivial weak solution. The condition (2.30) is equivalent to

\[
Q \leq 2 + \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\}.
\] (2.31)

**Theorem 2.2** gives a better estimate of the exponent. Indeed,

\[
\frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1p_2 - 1} - \frac{\gamma_2 + 2}{p_2 - 1} = -\frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1p_2 - 1} + \frac{\gamma_1 + 2}{p_1 - 1},
\] (2.32)

which implies that

\[
\max \left\{ \frac{(\gamma_1 + 2) + p_1(\gamma_2 + 2)}{p_1p_2 - 1}; \frac{p_2(\gamma_1 + 2) + (\gamma_2 + 2)}{p_1p_2 - 1} \right\} \geq \min \left\{ \frac{\gamma_1 + 2}{p_1 - 1}; \frac{\gamma_2 + 2}{p_2 - 1} \right\}.
\] (2.33)

### 3. Systems of \(m\) semilinear inequalities

In this section, we give generalizations of the last results to systems with \(m\) inequalities, \(m \in \mathbb{N}^\ast\).

Let \((X_1, X_2, \ldots, X_m)\) be the solution of the linear system

\[
\begin{pmatrix}
1 & -p_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -p_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{m-1} \\
X_m
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 - y_1 - 2 \\
\gamma_2 - y_2 - 2 \\
\vdots \\
\gamma_{m-1} - y_{m-1} - 2 \\
\gamma_m - y_m - 2
\end{pmatrix},
\] (3.1)

where \(p_i > 1\) and \(\gamma_i\) are given real numbers, \(i \in \{1, 2, \ldots, m\}\).

Consider the system (ES\(_m\)):

\[
-\Delta_{\mathbb{H}} (a_i u_i) \geq |\eta|^{p_{i+1}}_{H_{\mathbb{H}}} |u_{i+1}|^{p_{i+1}}_{H_{\mathbb{H}}}, \quad \eta \in \mathbb{H}_N, \quad 1 \leq i \leq m, \quad u_{m+1} = u_1,
\] (3.2)

where \(p_{m+1} = p_1, \gamma_{m+1} = \gamma_1\).
Proof. In order to simplify the proof, we treat only the case established in the same manner.

Theorem weak solution \((u_1,\ldots,u_m)\) of the system \((ES_m)\) on \(\mathbb{R}^{2N+1}\) is a vector of locally integrable functions \((u_1,\ldots,u_m)\) such that

\[
u_i \in L^{p_i}_{\text{loc}}(\mathbb{R}^{2N+1},|\eta|^{\frac{N}{N-1}}d\eta), \quad i \in \{1,2,\ldots,m\}, \tag{3.3}
\]
satisfying

\[
\int_{\mathbb{R}^{2N+1}} \left( a_i u_1 \Delta x \phi + |\eta|^{\frac{N}{N-1}} |u_{i+1}|^{p_i-1}|\phi| \right) d\eta \leq 0, \quad i \in \{1,2,\ldots,m-1\},
\]

\[
\int_{\mathbb{R}^{2N+1}} \left( a_m u_m \Delta x \phi + |\eta|^{\frac{N}{N-1}} |u|^{p_i-1}|\phi| \right) d\eta \leq 0
\]

for any nonnegative test function \(\phi \in C^2(\mathbb{R}^{2N+1})\).

**Theorem 3.2.** If \(Q \leq 2 + \max\{X_1,X_2,\ldots,X_m\}\), then system \((ES_m)\) has no nontrivial solution.

**Proof.** In order to simplify the proof, we treat only the case \(m = 3\); the general case can be established in the same manner.

Let \((u_1,u_2,u_3)\) be a nontrivial weak solution of \((ES_m)\). The inequalities (3.4), with \(\phi = \phi_R\) defined by (2.4), imply that

\[
\int |\eta|^{\frac{N}{N-1}} |u_1|^{p_1} \phi_R d\eta \leq ||a_3||_{L^\infty} \left( \int |\eta|^{\frac{N}{N-1}} |u_3|^{p_3} \phi_R \right)^{\frac{1}{p_3}} \left( \int |\Delta x \phi_R|^{\frac{N}{N-1}} (\phi_R |\eta|^{\frac{N}{N-1}})^{1-p_3} \right)^{\frac{1}{p_3}},
\]

\[
\int |\eta|^{\frac{N}{N-1}} |u_2|^{p_2} \phi_R d\eta \leq ||a_1||_{L^\infty} \left( \int |\eta|^{\frac{N}{N-1}} |u_1|^{p_1} \phi_R \right)^{\frac{1}{p_1}} \left( \int |\Delta x \phi_R|^{\frac{N}{N-1}} (\phi_R |\eta|^{\frac{N}{N-1}})^{1-p_1} \right)^{\frac{1}{p_1}},
\]

\[
\int |\eta|^{\frac{N}{N-1}} |u_3|^{p_3} \phi_R d\eta \leq ||a_2||_{L^\infty} \left( \int |\eta|^{\frac{N}{N-1}} |u_2|^{p_2} \phi_R \right)^{\frac{1}{p_2}} \left( \int |\Delta x \phi_R|^{\frac{N}{N-1}} (\phi_R |\eta|^{\frac{N}{N-1}})^{1-p_2} \right)^{\frac{1}{p_2}}. \tag{3.5}
\]

Let

\[
I_i(R) = \int |\eta|^{\frac{N}{N-1}} |u_i|^{p_i} \phi_R d\eta, \quad 1 \leq i \leq 3,
\]

\[
\mathcal{A}_i(R) = \int |\Delta x \phi_R|^{\frac{N}{N-1}} (\phi_R |\eta|^{\frac{N}{N-1}})^{1-p_i}, \quad 1 \leq i \leq 3, \tag{3.6}
\]

then there is a positive constant \(C\) such that

\[
I_1 \leq CI_3^{\frac{1}{p_3}} \mathcal{A}_3^{\frac{1}{p_3}}, \quad I_2 \leq CI_1^{\frac{1}{p_1}} \mathcal{A}_1^{\frac{1}{p_1}}, \quad I_3 \leq CI_2^{\frac{1}{p_2}} \mathcal{A}_2^{\frac{1}{p_2}}. \tag{3.7}
\]
Hence, the estimates
\[
I_{1}^{1-1/p_{1}p_{2}p_{3}} \leq C \mathcal{A}_{1}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{2}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{3}^{1/p_{1}p_{2}p_{3}},
\]
\[
I_{2}^{1-1/p_{1}p_{2}p_{3}} \leq C \mathcal{A}_{1}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{2}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{3}^{1/p_{1}p_{2}p_{3}},
\]
\[
I_{3}^{1-1/p_{1}p_{2}p_{3}} \leq C \mathcal{A}_{1}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{2}^{1/p_{1}p_{2}p_{3}} \mathcal{A}_{3}^{1/p_{1}p_{2}p_{3}},
\]
(3.8)
hold true.

In order to estimate the expressions \(I_{i}, 1 \leq i \leq 3\), we use the scaled variables (2.18) and obtain
\[
I_{1}^{1-1/p_{1}p_{2}p_{3}} \leq CR^{\sigma_{i}}, \quad 1 \leq i \leq 3,
\]
(3.9)
where
\[
\sigma_{1} = \left(1 - \frac{1}{p_{1}p_{2}p_{3}}\right)\left(Q - 2 - \frac{(y_{1} + 2) + p_{1}(y_{2} + 2) + p_{1}p_{2}(y_{3} + 2)}{p_{1}p_{2}p_{3} - 1}\right),
\]
\[
\sigma_{2} = \left(1 - \frac{1}{p_{1}p_{2}p_{3}}\right)\left(Q - 2 - \frac{p_{2}p_{3}(y_{1} + 2) + (y_{2} + 2) + p_{2}(y_{3} + 2)}{p_{1}p_{2}p_{3} - 1}\right),
\]
\[
\sigma_{3} = \left(1 - \frac{1}{p_{1}p_{2}p_{3}}\right)\left(Q - 2 - \frac{p_{3}(y_{1} + 2) + p_{1}p_{3}(y_{2} + 2) + (y_{3} + 2)}{p_{1}p_{2}p_{3} - 1}\right).
\]
(3.10)

Now, we require that, at least, one of \(\sigma_{i}, 1 \leq i \leq 3\), is less than zero, which is equivalent to \(Q \leq 2 + \max\{X_{1}, X_{2}, X_{3}\}\), where the vector \((X_{1}, X_{2}, X_{3})^{T}\) is the solution of
\[
\begin{pmatrix}
1 & -p_{1} & 0 \\
0 & 1 & -p_{2} \\
-p_{3} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_{1} \\
X_{2} \\
X_{3}
\end{pmatrix}
= \begin{pmatrix}
-y_{1} - 2 \\
-y_{2} - 2 \\
-y_{3} - 2
\end{pmatrix}.
\]
(3.11)

Following the arguments used in the proof of Theorem 2.2, we conclude that \((u_{1}, u_{2}, u_{3}) \equiv (0,0,0)\). This ends the proof by contradiction. \(\square\)

References


Nonexistence results to semilinear inequalities


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