Existence result for strongly nonlinear elliptic equation with a natural growth condition on the nonlinearity is proved.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N \geq 2)$ with the segment property.

Consider the nonlinear Dirichlet problem

$$A(u) + g(x, u, \nabla u) = f,$$

where $A(u) = -\text{div} a(x, u, \nabla u)$ is a Leray-Lions operator defined on $D(A) \subset W^1_0 L_M(\Omega) \rightarrow W^{-1} L_{\overline{M}}(\Omega)$ with $M$ an $N$-function and where $g$ is a nonlinearity with the “natural” growth condition

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|))$$

and which satisfies the classical sign condition $g(x, s, \xi)s \geq 0$. The right-hand side $f$ is assumed to belong to $W^{-1} E_{\overline{M}}(\Omega)$.

It is well known that Gossez [12] solved (1.1) in the case where $g$ depends only on $x$ and $u$. If $g$ depends also on $\nabla u$, existence theorems have recently been proved by Benkirane and Elmahi in [3, 4] by making some restrictions.

In [3], $g$ is supposed to satisfy a “nonnatural” growth condition of the form

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + P(|\xi|)) \quad \text{with} \quad P \ll M,$$

and in [4], $g$ is supposed to satisfy a natural growth of the form (1.2) but the result is restricted to $N$-functions $M$ satisfying a $\Delta_2$-condition.

It is our purpose in this paper to extend the result of [4] to general $N$-functions (i.e., without assuming a $\Delta_2$-condition on $M$) and hence generalize the results of [3, 4, 7].
As an example of equations to which the present result can be applied, we give

\( - \text{div} \left( \exp (m|u|) \frac{\exp (|\nabla u|) - 1}{|\nabla u|^2} \nabla u \right) + u \sin^2 u \exp (|\nabla u|) = f, \quad m \geq 0, \) 

with \( f = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \int_{\Omega} f_i \log |f_i| \, dx < \infty, \) 

(1)

\( - \text{div} \left( \frac{p(|\nabla u|)}{|\nabla u|} \nabla u \right) + ug(u)p(|\nabla u|) = f, \) 

(2)

with suitable data \( f, \) where \( p \) is a given positive and continuous function which increases from 0 to \(+\infty\) and where \( g \) is a positive function on \( \mathbb{R}. \)

For classical existence results for nonlinear elliptic equations in Orlicz-Sobolev spaces, see, for example, [2, 3, 4, 6, 8, 9, 10].

2. Preliminaries

2.1. Let \( M: \mathbb{R}^+ \to \mathbb{R}^+ \) be an \( N \)-function, that is, \( M \) is continuous and convex, with \( M(t) > 0 \) for \( t > 0, \) \( M(t)/t \to 0 \) as \( t \to 0, \) and \( M(t)/t \to \infty \) as \( t \to \infty. \)

Equivalently, \( M \) admits the following representation: \( M(t) = \int_{0}^{t} m(\tau) \, d\tau, \) where \( m: \mathbb{R}^+ \to \mathbb{R}^+ \) is nondecreasing and right continuous, with \( m(0) = 0, m(t) > 0 \) for \( t > 0, \) and \( m(t) \to \infty \) as \( t \to \infty. \)

The \( N \)-function \( \overline{M}, \) conjugate to \( M, \) is defined by \( \overline{M}(t) = \int_{0}^{t} \overline{m}(\tau) \, d\tau, \) where \( \overline{m}: \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \overline{m}(t) = \sup \{ s: m(s) \leq t \} \) (see [1, 14, 15]).

The \( N \)-function \( M \) is said to satisfy the \( \Delta_2 \)-condition if, for some \( k > 0, \)

\( M(2t) \leq kM(t) \quad \forall t \geq 0. \)  

(2.1)

When \( (2.1) \) holds only for \( t \geq t_0 > 0, \) then \( M \) is said to satisfy the \( \Delta_2 \)-condition near infinity.

We will extend these \( N \)-functions into even functions on all \( \mathbb{R}. \)

Let \( P \) and \( Q \) be two \( N \)-functions. \( P \ll Q \) means that \( P \) grows essentially less rapidly than \( Q, \) that is, for each \( \varepsilon > 0, \)

\( \frac{P(t)}{Q(\varepsilon t)} \to 0 \quad \text{as} \ t \to \infty. \)  

(2.2)

This is the case if and only if

\( \lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0. \)  

(2.3)

2.2. Let \( \Omega \) be an open subset of \( \mathbb{R}^N. \) The Orlicz class \( \mathcal{L}_M(\Omega) \) (resp., the Orlicz space \( L_M(\Omega) \)) is defined as the set of (equivalence classes of) real-valued measurable functions
Thus, we now turn to the Orlicz-Sobolev space. Let \( u \) be a function in \( W^1L_M(\Omega) \) (resp., \( W^1E_M(\Omega) \)) such that
\[
\int_{\Omega} M(u(x)) \, dx < +\infty \quad \text{(resp.,} \quad \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx < +\infty \, \text{for some} \, \lambda > 0\).
\] (2.4)

\( L_M(\Omega) \) is a Banach space under the norm
\[
\|u\|_M = \inf\left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \right\}
\] (2.5)

and \( \mathcal{F}_M(\Omega) \) is a convex subset of \( L_M(\Omega) \).

The closure in \( L_M(\Omega) \) of the set of bounded measurable functions with compact support in \( \overline{\Omega} \) is denoted by \( E_M(\Omega) \).

The equality \( E_M(\Omega) = L_M(\Omega) \) holds if and only if \( M \) satisfies the \( \Delta_2 \)-condition for all \( t \) or for \( t \) large according to whether \( \Omega \) has infinite measure or not.

The dual of \( E_M(\Omega) \) can be identified with \( L_{\overline{M}}(\Omega) \) by means of the pairing \( \int_{\Omega} u(x)v(x) \, dx \), and the dual norm on \( L_{\overline{M}}(\Omega) \) is equivalent to \( \| \cdot \|_{\overline{M}} \).

The space \( L_M(\Omega) \) is reflexive if and only if \( M \) and \( \overline{M} \) satisfy the \( \Delta_2 \)-condition, for all \( t \) or for \( t \) large, according to whether \( \Omega \) has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space. \( W^1L_M(\Omega) \) (resp., \( W^1E_M(\Omega) \)) is the space of all functions \( u \) such that \( u \) and its distributional derivatives up to order 1 lie in \( L_M(\Omega) \) (resp., \( E_M(\Omega) \)). It is a Banach space under the norm
\[
\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M,
\] (2.6)

thus \( W^1L_M(\Omega) \) and \( W^1E_M(\Omega) \) can be identified with subspaces of the product of \( N+1 \) copies of \( L_M(\Omega) \). Denoting this product by \( \Pi L_M \), we will use the weak topologies \( \sigma(\Pi L_M, \Pi E_M) \) and \( \sigma(\Pi L_M, \Pi L_{\overline{M}}) \).

The space \( W^1_M(\Omega) \) is defined as the (norm) closure of the Schwartz space \( \mathcal{D}(\Omega) \) in \( W^1E_M(\Omega) \) and the space \( \mathcal{D}'_M(\Omega) \) as the \( \sigma(\Pi L_M, \Pi E_{\overline{M}}) \) closure of \( \mathcal{D}(\Omega) \) in \( W^1L_M(\Omega) \).

We say that \( u_n \) converges to \( u \) for the modular convergence in \( W^1L_M(\Omega) \) if for some \( \lambda > 0 \),
\[
\int_{\Omega} M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) \, dx \to 0 \quad \forall |\alpha| \leq 1;
\] (2.7)

this implies convergence for \( \sigma(\Pi L_M, \Pi L_{\overline{M}}) \).

If \( M \) satisfies the \( \Delta_2 \)-condition on \( \mathbb{R}^+ \) (near infinity only if \( \Omega \) has finite measure), then modular convergence coincides with norm convergence.

2.4. Let \( W^{-1}_M(\Omega) \) (resp., \( W^{-1}_{\overline{M}}(\Omega) \)) denote the space of distributions on \( \Omega \) which can be written as sums of derivatives of order less than or equal to 1 of functions in \( L_{\overline{M}}(\Omega) \) (resp., \( E_{\overline{M}}(\Omega) \)). It is a Banach space under the usual quotient norm.

If the open set \( \Omega \) has the segment property, then the space \( \mathcal{D}(\Omega) \) is dense in \( W^1_0L_M(\Omega) \) for the modular convergence and thus for the topology \( \sigma(\Pi L_M, \Pi L_{\overline{M}}) \) (cf. [9, 11]). Consequently, the action of a distribution \( S \) in \( W^{-1}_M(\Omega) \) on an element \( u \) of \( W^1_0L_M(\Omega) \) is well defined. It will be denoted by \( \langle S, u \rangle \).
Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$.

Let $A : D(A) \subset W^1_0 L_M(\Omega) \rightarrow W^{-1} L_M(\Omega)$ be a mapping (not everywhere defined) given by

$$A(u) = -\text{div} a(x, u, \nabla u),$$  \hspace{1cm} (3.1)

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$,

$$|a(x, s, \xi)| \leq \beta [c(x) + P^{-1} M(y|s|) + M^{-1} M(|\xi|)],$$  \hspace{1cm} (3.2)

$$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*) > 0,$$  \hspace{1cm} (3.3)

$$\alpha M(|\xi|) \leq a(x, s, \xi) \xi,$$  \hspace{1cm} (3.4)

where $c(x)$ belongs to $E_M(\Omega)$, $c \geq 0$, and $\alpha, \beta, \gamma > 0$.

Furthermore, let $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$g(x, s, \xi) \geq 0,$$  \hspace{1cm} (3.5)

$$|g(x, s, \xi)| \leq b(|s|)(c'(x) + M(|\xi|)),$$  \hspace{1cm} (3.6)

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and non decreasing function and $c'(x)$ is a given non-negative function in $L^1(\Omega)$. Finally, we assume that

$$f \in W^{-1} E_M(\Omega).$$  \hspace{1cm} (3.7)

Consider the following elliptic problem with Dirichlet boundary condition:

$$\langle A(u), v \rangle + \int_{\Omega} g(x, u, \nabla u) v \, dx = \langle f, v \rangle$$  \hspace{1cm} (3.8)

for all $v \in W^1_0 L_M(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

We will prove the following existence theorem.

**Theorem 3.1.** Assume that (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) hold true. Then there exists at least one solution $u$ of (3.8).
Remark 3.2. Note that conditions (3.4) and (3.6) can be replaced by the following ones:

\[
\alpha M \left( \frac{|\xi|}{\lambda} \right) \leq a(x, s, \xi)\xi, \quad \text{(3.9)}
\]

\[
|g(x, s, \xi)| \leq b(|s|) \left( c'(x) + M \left( \frac{|\xi|}{\lambda'} \right) \right),
\]

with \( \lambda' \geq \lambda > 0 \).

Remark 3.3. The Euler equation of the integral

\[
\int_{\Omega} \left( a(u) \int_{0}^{\|\nabla u\|} \frac{M(t)}{t} dt \right) dx - \langle f, u \rangle \quad \text{(3.10)}
\]

is

\[
- \sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} \left( a(u) \frac{M(\|\nabla u\|)}{\|\nabla u\|^2} \frac{\partial u}{\partial x_{i}} \right) + a'(u) \int_{0}^{\|\nabla u\|} \frac{M(t)}{t} dt = f, \quad \text{(3.11)}
\]

where \( a(s) \) is a smooth function satisfying \( a'(s) s \geq 0 \). Note that

\[
a'(u) \int_{0}^{\|\nabla u\|} \frac{M(t)}{t} dt \quad \text{(3.12)}
\]

satisfies the growth condition (3.6) and then Theorem 3.1 can be applied to Dirichlet problems related to (3.11).

**Proof of Theorem 3.1**

**Step 1 (a priori estimates).** Consider the sequence of approximate problems

\[
u_{n} \in W^{1}_{0} L_{M}(\Omega),
\]

\[
\langle A(u_{n}), v \rangle + \int_{\Omega} g_{n}(x, u_{n}, \nabla u_{n}) v \, dx = \langle f, v \rangle \quad \forall v \in W^{1}_{0} L_{M}(\Omega),
\]

where

\[
g_{n}(x, s, \xi) = T_{n}(g(x, s, \xi)) \quad \text{(3.14)}
\]

and where for \( k > 0 \), \( T_{k} \) is the usual truncation at height \( k \) defined by \( T_{k}(s) = \max(-k, \min(k, s)) \) for all \( s \in \mathbb{R} \).

Note that \( g_{n}(x, s, \xi)s \geq 0 \), \(|g_{n}(x, s, \xi)| \leq |g(x, s, \xi)|\), and \(|g_{n}(x, s, \xi)| \leq n\). Since \( g_{n} \) is bounded for any fixed \( n > 0 \), there exists at least one solution \( u_{n} \) of (3.13) (see [13, Propositions 1 and 5]).

Using in (3.13) the test function \( u_{n} \), we get

\[
\int_{\Omega} a(x, u_{n}, \nabla u_{n}) \nabla u_{n} \, dx \leq \langle f, u_{n} \rangle. \quad \text{(3.15)}
\]
Consequently, one has that \((u_n)\) is bounded in \(W_0^1 L_M(\Omega)\). By \([13, \text{Proposition 5}]\) (see \([13, \text{Remark 8}]\)), \((a(x, u_n, \nabla u_n))_n\) is bounded in \((L_M^\infty(\Omega))^N\),

\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq C, \tag{3.16}
\]

where \(C\) is a real constant which does not depend on \(n\).

Passing to a subsequence, if necessary, we can assume that

\[
\begin{align*}
\{a(x, u_n, \nabla u_n) - h\} & \quad \text{weakly in } L_M(\Omega), \\
\int_{\Omega} gn(x, u_n, \nabla u_n) \, dx & \quad \leq C, \tag{3.16}
\end{align*}
\]

for some \(h \in (L_M^\infty(\Omega))^N\).

**Step 2** (almost everywhere convergence of the gradients). Fix \(k > 0\) and let \(\varphi(t) = te^{\sigma t^2}\), \(\sigma > 0\). It is well known that when \(\sigma \geq (b(k)/2\alpha)^2\), one has

\[
\varphi'(t) - \frac{b(k)}{\alpha} |\varphi(t)| \geq \frac{1}{2} \quad \forall t \in \mathbb{R}. \tag{3.18}
\]

Take a sequence \((v_j) \subset \mathcal{D}(\Omega)\) which converges to \(u\) for the modular convergence in \(W_0^1 L_M(\Omega)\) (cf. \([11]\)) and set \(\theta_n^j = T_k(u_n) - T_k(v_j), \quad \theta^j = T_k(u) - T_k(v_j), \quad \text{and } z_n^j = \varphi(\theta_n^j).\)

Using in (3.13) the test function \(z_n^j\), we get

\[
\langle A(u_n), z_n^j \rangle + \int_{\Omega} g_n(x, u_n, \nabla u_n) z_n^j \, dx = \langle f, z_n^j \rangle. \tag{3.19}
\]

Denote by \(\varepsilon_i(n, j)\) \((i = 0, 1, 2, \ldots)\) various sequences of real numbers which tend to 0 when \(n\) and \(j \to \infty\), that is,

\[
\lim_{j \to \infty} \lim_{n \to \infty} \varepsilon_i(n, j) = 0. \tag{3.20}
\]

In view of (3.17), we have \(z_n^j \rightharpoonup \varphi(\theta^j)\) weakly in \(W_0^1 L_M(\Omega)\) for \(\sigma(\Pi L_M, \Pi E_M)\) as \(n \to \infty\) and then \(\langle f, z_n^j \rangle \rightharpoonup \langle f, \varphi(\theta^j) \rangle\) as \(n \to \infty\). Using, now, the modular convergence of \((v_j)\), we get \(\langle f, \varphi(\theta^j) \rangle \to 0\) as \(j \to \infty\) so that

\[
\langle f, z_n^j \rangle = \varepsilon_0(n, j). \tag{3.21}
\]

Since \(g_n(x, u_n, \nabla u_n)z_n^j \geq 0\) on the subset \(\{x \in \Omega : |u_n| > k\}\), we have

\[
\langle A(u_n), z_n^j \rangle + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^j \, dx \leq \varepsilon_0(n, j). \tag{3.22}
\]
The first term on the left-hand side of (3.22) reads as

$$\langle A(u_n), z_{n}^{j} \rangle = \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \right] \varphi' (\theta_n^{j}) \, dx$$

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi' (\theta_n^{j}) \, dx$$

$$= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \right] \varphi' (\theta_n^{j}) \, dx$$

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi' (\theta_n^{j}) \, dx$$

and then

$$\langle A(u_n), z_{n}^{j} \rangle = \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_{s}^{j}) \right]$$

$$\times \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{s}^{j} \right] \varphi' (\theta_n^{j}) \, dx$$

$$+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_{s}^{j}) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{s}^{j} \right] \varphi' (\theta_n^{j}) \, dx$$

$$- \int_{\Omega \cap \Omega_{j}^{k}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \varphi' (\theta_n^{j}) \, dx$$

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi' (\theta_n^{j}) \, dx,$$

where $\chi_{s}^{j}$ denotes the characteristic function of the subset

$$\Omega_{j}^{s} = \{ x \in \Omega : |\nabla T_k(v_j) | \leq s \}.$$  

We will pass to the limit in $n$ and in $j$ for $s$ fixed in the last three terms of the right-hand side of (3.24).

Starting with the fourth term, observe that, since

$$| \nabla T_k(v_j) \chi_{|u_n| > k} \varphi' (\theta_n^{j}) | \leq \varphi' (2k) | \nabla T_k(v_j) | \leq \varphi' (2k) \| \nabla v_j \|_{\infty} = a_j \in \mathbb{R},$$

we have

$$\nabla T_k(v_j) \chi_{|u_n| > k} \varphi' (\theta_n^{j}) \rightarrow \nabla T_k(v_j) \chi_{|u| > k} \varphi' (\theta^{j}) \text{ strongly in } (E_{M}(\Omega))^{N} \text{ as } n \rightarrow \infty,$$

and hence

$$\int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi' (\theta_n^{j}) \, dx \rightarrow \int_{\{|u| > k\}} h \nabla T_k(v_j) \varphi' (\theta^{j}) \, dx \quad \text{as } n \rightarrow \infty.$$

Observe that

$$| \nabla T_k(v_j) \chi_{|u| > k} \varphi' (\theta^{j}) | \leq \varphi' (2k) | \nabla T_k(v_j) | \leq \varphi' (2k) | \nabla v_j |;$$
then, by using the modular convergence of $|\nabla v_j|$ in $L_M(\Omega)$ and Vitali’s theorem, we get
\[ \nabla T_k(v_j) \chi_{\{|u| \geq k\}} \varphi'(\theta^I) \to 0 \] (3.30)
for the modular convergence in $(L_M(\Omega))^N$, and thus
\[ \int_{\{|u| \geq k\}} h \nabla T_k(v_j) \varphi'(\theta^I) dx \to 0 \quad \text{as} \quad j \to \infty. \] (3.31)

We have then proved that
\[ \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi'(\theta^I_n) dx = \epsilon_1(n, j). \] (3.32)

The second term on the right-hand side of (3.24) tends to (by letting $n \to \infty$)
\[ \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j \nabla T_k(u) - \nabla T_k(v_j) \chi_j \varphi'(\theta^I) dx \] (3.33)
since $a(x, T_k(u_n), \nabla T_k(v_j) \chi_j \varphi'(\theta^I_n) \to a(x, T_k(u), \nabla T_k(v_j) \chi_j \varphi'(\theta^I)$ strongly in $(E_{\Pi}(\Omega))^N$ as $n \to \infty$ by [3, Lemma 2.3], while $\nabla T_k(u_n) \to \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ by (3.17).

Since $\nabla T_k(v_j) \chi_j \to \nabla T_k(u) \chi_j$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, where $\chi_j$ denotes the characteristic function of $\Omega$, $\chi_j \in \{x \in \Omega : |\nabla T_k(u)| \leq s\}$, it is easy to see that
\[ \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j \nabla T_k(u) - \nabla T_k(v_j) \chi_j \varphi'(\theta^I) dx \to 0 \quad \text{as} \quad j \to \infty, \] (3.34)
and thus
\[ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j \varphi'(\theta^I_n) dx = \epsilon_2(n, j). \] (3.35)

Concerning the third term on the right-hand side of (3.24), we have
\[ -\int_{\Omega \setminus \Omega} a(x, T_k(u_n), \nabla T_k(u_n) \nabla T_k(v_j) \varphi'(\theta^I_n) dx \to -\int_{\Omega \setminus \Omega} h_k \nabla T_k(v_j) \varphi'(\theta^I) dx \] (3.36)
as $n \to \infty$ by using the fact that $\nabla T_k(v_j)$ belongs to $(E_M(\Omega))^N$.

In view of the modular convergence of $(\nabla v_j)$ in $(L_M(\Omega))^N$, we have
\[ -\int_{\Omega \setminus \Omega} h_k \nabla T_k(v_j) \varphi'(\theta^I) dx \to -\int_{\Omega \setminus \Omega} h_k \nabla T_k(u) dx \quad \text{as} \quad j \to \infty \] (3.37)
and thus
\[ -\int_{\Omega \setminus \Omega} a(x, T_k(u_n), \nabla T_k(u_n) \nabla T_k(v_j) \varphi'(\theta^I_n) dx = \epsilon_3(n, j) - \int_{\Omega \setminus \Omega} h_k \nabla T_k(u) dx. \] (3.38)
Combining now (3.32), (3.35), and (3.38), we obtain
\[
\langle A(u_n), z_n^j \rangle = \int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^i) \right] \\
\times \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^i \right] \varphi'(\theta_n^i) \, dx - \int_{\Omega \setminus \Omega_i} h_k \nabla T_k(u) \, dx + \varepsilon_4(n, j).
\]
(3.39)

We now turn to the second term on the left-hand side of (3.22). We have
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^i \, dx \right|
= \left| \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) z_n^i \, dx \right|
\leq \int_\Omega b(k) c'(x) |\varphi(\theta_n^i)| \, dx + b(k) \int_\Omega M(1 |\nabla T_k(u_n)|) |\varphi(\theta_n^i)| \, dx
\leq \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi(\theta_n^i)| \, dx + \varepsilon_5(n, j).
\]
(3.40)

The first term of the right-hand side of this inequality reads as
\[
\frac{b(k)}{\alpha} \int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^i) \right] \\
\times \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^i \right] \varphi(\theta_n^i) \, dx
+ \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^i) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^i \right] \varphi(\theta_n^i) \, dx
- \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^i \varphi(\theta_n^i) \, dx
\]
(3.41)

and, as above, it is easy to see that
\[
\frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^i) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^i \right] \varphi(\theta_n^i) \, dx = \varepsilon_6(n, j)
\]
(3.42)

and that
\[
- \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^i \varphi(\theta_n^i) \, dx = \varepsilon_7(n, j)
\]
(3.43)

so that
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^i \, dx \right|
\leq \frac{b(k)}{\alpha} \int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^i) \right] \\
\times \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^i \right] \varphi(\theta_n^i) \, dx + \varepsilon_6(n, j).
\]
(3.44)
Combining this inequality with (3.50) and (3.48), we obtain
\[
\int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx + \int_{\Omega \setminus \Omega_h} h_k \nabla T_k(u) dx.
\]

Consequently,
\[
\int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx \leq 2\varepsilon_g(n,j) + 2 \int_{\Omega \setminus \Omega_h} h_k \nabla T_k(u) dx.
\]

On the other hand,
\[
\int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \chi^\delta \right] dx
\]
\[
= \int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx
\]
\[
+ \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \left[ \nabla T_k(v_j) \chi_j^\delta - \nabla T_k(u) \chi^\delta \right] dx
\]
\[
- \int_\Omega a(x, T_k(u_n), \nabla T_k(u) \chi^\delta) \left[ \nabla T_k(u_n) - \nabla T_k(u) \chi^\delta \right] dx
\]
\[
+ \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx.
\]

We will pass to the limit in \( n \) and in \( j \) in the last three terms on the right-hand side of the above equality. Similar tools as in (3.24) and (3.41) give
\[
\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \left[ \nabla T_k(v_j) \chi_j^\delta - \nabla T_k(u) \chi^\delta \right] dx = \varepsilon_{10}(n,j),
\]
\[
\int_\Omega a(x, T_k(u_n), \nabla T_k(u) \chi^\delta) \left[ \nabla T_k(u_n) - \nabla T_k(u) \chi^\delta \right] dx = \varepsilon_{11}(n,j),
\]
\[
\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx = \varepsilon_{12}(n,j)
\]
which imply that
\[
\int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \chi^\delta \right] dx
\]
\[
= \int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^\delta) \right] \left[ \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^\delta \right] dx
\]
\[
+ \varepsilon_{13}(n,j).
\]
For $r \leq s$, one has

\[
0 \leq \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] [\nabla T_k(u_n) - \nabla T_k(u)] dx
\]

\[
\leq \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] [\nabla T_k(u_n) - \nabla T_k(u)] dx
\]

\[
= \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \chi^s \right] [\nabla T_k(u_n) - \nabla T_k(u) \chi^s] dx
\]

\[
\leq \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \chi^s \right] [\nabla T_k(u_n) - \nabla T_k(u) \chi^s] dx
\]

\[
= \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \right] [\nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j] dx
+ \varepsilon_{13}(n, j)
\]

\[
\leq \varepsilon_{14}(n, j) + 2 \int_{\Omega \setminus \overline{\Omega}} h_k \nabla T_k(u) dx.
\]

(3.52)

This implies that, by passing at first to the limit sup over $n$ and next over $j$,

\[
0 \leq \limsup_{n \to \infty} \int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] [\nabla T_k(u_n) - \nabla T_k(u)] dx
\]

\[
\leq 2 \int_{\Omega \setminus \overline{\Omega}} h_k \nabla T_k(u) dx.
\]

(3.53)

Using the fact that $h_k \nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$, we get

\[
\int_{\Omega} \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] [\nabla T_k(u_n) - \nabla T_k(u)] dx \to 0
\]

(3.54)

as $n \to \infty$.

As in [3], we deduce that there exists a subsequence still denoted by $u_n$ such that

\[
\nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in } \Omega,
\]

(3.55)

which implies that

\[
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_\infty(\Omega))^N \quad \text{for } \sigma(PL_{\infty}, \Pi E_M).
\]

(3.56)

**Step 3 (modular convergence of the truncations).** Going back to (3.46), we can write

\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi^s_j dx
\]

\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j] dx
\]

\[
+ 2\varepsilon_{14}(n, j) + 2 \int_{\Omega \setminus \overline{\Omega}} h_k \nabla T_k(u) dx.
\]

(3.57)
which implies, by using (3.50),

$$
\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^2 dx + \epsilon_{15}(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx.
$$

(3.58)

Passing to the limit sup over $n$ in both sides of this inequality yields

$$
\limsup_{n \to \infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(v_j) \chi_j^2 dx + \lim_{n \to \infty} \epsilon_{15}(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx,
$$

(3.59)

in which we can pass to the limit in $j$ to obtain

$$
\limsup_{n \to \infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx
$$

(3.60)

which gives, by letting $s \to \infty$, 

$$
\limsup_{n \to \infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
$$

(3.61)

On the other hand, we have, by using Fatou’s lemma,

$$
\int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \to \infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx,
$$

(3.62)

which implies that

$$
\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \to \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \quad \text{as} \quad n \to \infty,
$$

(3.63)

and by using [4, Lemma 2.4], we conclude that

$$
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{in} \quad L^1(\Omega).
$$

(3.64)

This implies, by using (3.4), that

$$
T_k(u_n) \to T_k(u) \quad \text{in} \quad W^{1}_0 L_M(\Omega)
$$

(3.65)

for the modular convergence.
Step 4 (equi-integrability of the nonlinearities and passage to the limit). We will prove that \( g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \) strongly in \( L^1(\Omega) \) by using Vitali’s theorem.

Since \( g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \) a.e. in \( \Omega \), thanks to (3.55), it suffices to prove that \( g_n(x, u_n, \nabla u_n) \) are uniformly equi-integrable in \( \Omega \). Let \( E \subset \Omega \) be a measurable subset of \( \Omega \). We have, for any \( m > 0 \),

\[
\int_E \left| g_n(x, u_n, \nabla u_n) \right| \ dx = \int_{E \cap \{|u_n| \leq m\}} \left| g_n(x, u_n, \nabla u_n) \right| \ dx + \int_{E \cap \{|u_n| > m\}} \left| g_n(x, u_n, \nabla u_n) \right| \ dx \leq b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \ dx + b(m) \int_E c'(x) \ dx + \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \ dx. \tag{3.66}
\]

Standard arguments allow to deduce, using the strong convergence (3.64), that there exists \( \mu > 0 \) such that

\[
|E| < \mu \Rightarrow \int_E \left| g_n(x, u_n, \nabla u_n) \right| \ dx \leq \epsilon, \quad \forall n, \tag{3.67}
\]

which shows that \( g_n(x, u_n, \nabla u_n) \) are uniformly equi-integrable in \( \Omega \) as required.

In order to pass to the limit, we have, by going back to approximate equations (3.13),

\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla w \ dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) w \ dx = \langle f, w \rangle \tag{3.68}
\]

for all \( w \in \mathcal{D}(\Omega) \), in which, we can easily pass to the limit as \( n \rightarrow \infty \) to get

\[
\int_{\Omega} a(x, u, \nabla u) \nabla w \ dx + \int_{\Omega} g(x, u, \nabla u) w \ dx = \langle f, w \rangle. \tag{3.69}
\]

Let now \( v \in W^1_0 L^1(\Omega) \cap L^\infty(\Omega) \). There exists \( (w_j) \subset \mathcal{D}(\Omega) \) such that \( \|w_j\|_{\infty, \Omega} \leq (N + 1)\|v\|_{\infty, \Omega} \) for all \( j \in \mathbb{N} \) and

\[
w_j \rightharpoonup v \tag{3.70}
\]

for the modular convergence in \( W^1_0 L^1(\Omega) \). Taking \( w = w_j \) in (3.69) and letting \( j \rightarrow \infty \) yields

\[
\int_{\Omega} a(x, u, \nabla u) \nabla v \ dx + \int_{\Omega} g(x, u, \nabla u) v \ dx = \langle f, v \rangle. \tag{3.71}
\]

By choosing \( v = T_k(u) \) in the last equality, we get

\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k(u) \ dx + \int_{\Omega} g(x, u, \nabla u) T_k(u) \ dx = \langle f, T_k(u) \rangle. \tag{3.72}
\]

From (3.16), we deduce by Fatou’s lemma that \( g(x, u, \nabla u) u \in L^1(\Omega) \) and since \( |g(x, u, \nabla u) u| \leq g(x, u, \nabla u) u \) and \( T_k(u) \rightarrow u \) in \( W^1_0 L^1(\Omega) \) for the modular convergence and
Existence of solutions for elliptic equations a.e. in $\Omega$ as $k \to \infty$, it is easy to pass to the limit in both sides of (3.72) (by using Lebesgue theorem) to obtain
\[
\int_{\Omega} a(x,u,\nabla u) \nabla u \, dx + \int_{\Omega} g(x,u,\nabla u) u \, dx = \langle f, u \rangle. \tag{3.73}
\]
This completes the proof of Theorem 3.1. \hfill \Box

Remark 3.4. If we replace, as in [5], (3.2) by the general growth condition
\[
|a(x,s,\xi)| \leq \overline{b}(|s|)(c(x) + M^{-1} M(|\xi|)), \tag{3.74}
\]
where $\gamma > 0$, $c \in E_M(\Omega)$, and $\overline{b} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function, we prove the existence of solutions for the following problem:
\[
u \in W_0^1 L_M(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \quad g(x,u,\nabla u) u \in L^1(\Omega), \quad \langle A(u), T_k(u - v) \rangle + \int_{\Omega} g(x,u,\nabla u) T_k(u - v) \, dx \leq \langle f, T_k(u - v) \rangle \tag{3.75}
\]
\forall v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega).

Indeed, we consider the following approximate problems:
\[
u_n \in W_0^1 L_M(\Omega), \quad -\text{div} a(x,T_n(u_n),\nabla u_n) + g_n(x,u_n,\nabla u_n) = f \quad \text{in} \ \Omega, \tag{3.76}
\]
and we conclude by adapting the same steps.

As an application of this result, we can treat the following model equations:
\[
- \text{div} \left( (1 + |u|)^m \exp \left( \frac{(|\nabla u|) - 1}{|\nabla u|^2} \nabla u \right) \right) + u \cos^2 u \exp (|\nabla u|) = f, \quad m \geq 0. \tag{3.77}
\]
Remark that the solutions of (3.77) belong to $L^\infty(\Omega)$ so that (3.77) holds in the distributional sense.

References


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