SOLUTIONS TO $H$-SYSTEMS BY TOPOLOGICAL AND ITERATIVE METHODS

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We study $H$-systems with a Dirichlet boundary data $g$. Under some conditions, we show that if the problem admits a solution for some $(H_0, g_0)$, then it can be solved for any $(H, g)$ close enough to $(H_0, g_0)$. Moreover, we construct a solution of the problem applying a Newton iteration.

1. Introduction

We consider the Dirichlet problem in a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^2$ for a vector function $X : \overline{\Omega} \to \mathbb{R}^3$ which satisfies the equation of prescribed mean curvature

$$\Delta X = 2H(u, v, X)X_u \wedge X_v \quad \text{in } \Omega,$$

$$X = g \quad \text{on } \partial \Omega,$$

where $\wedge$ denotes the exterior product in $\mathbb{R}^3$, $H : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, and the boundary data $g$ is smooth. Problem (1.1) above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied, for example, in [1, 2, 3, 4, 5].

In Section 2, we prove the following theorem.

**Theorem 1.1.** Let $X_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ be a solution of (1.1) for some $(H_0, g_0)$ with $g_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ ($2 < p < \infty$) and $H_0$ continuously differentiable with respect to $X$ over the graph of $X_0$. Set

$$k = -2 \inf_{(u, v, Y) \in \Omega \times \mathbb{R}^3, |Y| = 1} \left( \frac{\partial H_0}{\partial X} (u, v, X_0) Y \right) \left( (X_0_u \wedge X_0_v) Y \right)$$

(1.2)

and assume that

$$k + 2 \sqrt{\lambda_1} \left\| H_0(\cdot, X_0) \right\|_\infty \left\| \nabla X_0 \right\|_\infty < \lambda_1,$$

(1.3)
where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\). Then there exists a neighborhood \( B \) of \((H_0, g_0)\) in the space \( C(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}) \times W^{2,p}(\Omega, \mathbb{R}^3) \) such that (1.1) is solvable for any \((H, g) \in B\).

Remark 1.2. It is clear that

\[
0 \leq -2 \inf_{(u, v) \in \Omega} \frac{\partial H_0}{\partial X}(u, v, X_0)(X_{0u} \wedge X_{0v}) \leq k \leq 2 \left\| \frac{\partial H_0}{\partial X}(\cdot, X_0) \right\|_\infty \left\| X_{0u} \wedge X_{0v} \right\|_\infty.
\]

(1.4)

Moreover, a simple computation shows that \( k = 0 \) if and only if \((\partial H_0/\partial X)(\cdot, X_0)\) and \(X_{0u} \wedge X_{0v}\) are linearly dependent, with \((\partial H_0/\partial X)(u, v, X_0)(X_{0u} \wedge X_{0v}) \geq 0\) for every \((u, v) \in \Omega\).

In Section 3, we show that the solution provided by Theorem 1.1 can be obtained by a Newton iteration. For simplicity, we consider the case where \( H \) does not depend on \( X \) and prove the following theorem.

Theorem 1.3. Let \( X_0 \in W^{2,p}(\Omega, \mathbb{R}^3) \) be a solution of (1.1) for some \((H_0, g_0)\) with \( g_0 \in W^{2,p}(\Omega, \mathbb{R}^3) \) \((2 < p < \infty)\) and \( H_0 \) continuous, and assume that

\[
2\|H_0\|_{\infty}\|\nabla X_0\|_{\infty} < \sqrt{\lambda_1}.
\]

(1.5)

Then, if \( H \) and \( g \) are close enough to \( H_0 \) and \( g_0 \), respectively, the sequence given by

\[
\Delta X_{n+1} = 2H[(X_{nu} \wedge (X_{n+1} - X_n)_v) + (X_{n+1} - X_n)_u \wedge X_{nv}) - X_{nu} \wedge X_{nv}],
\]

\[
X_{n+1}|_{\partial \Omega} = g
\]

is well defined and converges in \( W^{2,p}(\Omega, \mathbb{R}^3) \) to a solution of (1.1).

2. Proof of Theorem 1.1

First we will prove a slight extension of a well-known result for linear elliptic second-order operators.

Lemma 2.1. Let \( L : W^{2,p}(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \) be the linear elliptic operator given by \( LX = \Delta X + AX_u + BX_v + CX \) with \( A, B, C \in L^\infty(\Omega, \mathbb{R}^{3 \times 3}) \) \((2 < p < \infty)\), and assume that \( r := (|A|^2 + |B|^2/\lambda_1)^{1/2} < 1 \) and that \( CY \cdot Y \leq \kappa |Y|^2 \) for every \( Y \in \mathbb{R}^3 \) with \( \kappa < \lambda_1(1 - r) \). Then \( L|_{W^{1,p}_0(\Omega, \mathbb{R}^3)} : W^{2,p} \cap W^{1,p}_0(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \) is an isomorphism.
Proof. Let $Z_n \in W^{2,p} \cap W^{1,p}_0(\Omega, \mathbb{R}^3)$ be a sequence such that $\|LZ_n\|_p \to 0$. Then $\|LZ_n\|_2 \to 0$, and from the inequalities
\[
- \int LZ_nZ_n \geq \|\nabla Z_n\|_2^2 - \left(\|A\|^2 + \|B\|^2\right)^{1/2} \|\nabla Z_n\|_2^2 - \int CZ_nZ_n \\
\geq \left(1 - r - \frac{\kappa}{\lambda_1}\right)\|\nabla Z_n\|_2^2,
\]
we deduce that $\|\nabla Z_n\|_2 \to 0$. Thus, $\|Z_n\|_2 \to 0$ and hence $\|\Delta Z_n\|_2 \to 0$. From the invertibility of $\Delta$, there exists a subsequence (still denoted $Z_n$) such that $\|Z_n\|_{1,p} \to 0$ and we conclude that $\|\Delta Z_n\|_p \to 0$.

In order to prove that $L$ is onto, it suffices to consider for any $\phi \in L^p(\Omega)$, the homotopy
\[
\Delta X = \sigma(\phi - AX_u - BX_v - CX)
\]
and apply a Leray-Schauder argument. □

Now we are able to prove Theorem 1.1. Consider a pair $(H,g)$ with $\|g - g_0\|_{2,p} < \delta$ and $\|(H - H_0)\|_K \|_\infty < \epsilon$ for some compact $K$ containing a neighborhood of the graph of $X_0$. Setting $Y = X - X_0$, equation (1.1) is equivalent to the problem
\[
LY = F(u,v,Y,Y_u,Y_v) \quad \text{in } \Omega, \\
Y = g - g_0 \quad \text{on } \partial \Omega,
\]
where $L$ is the linear operator given by
\[
LY = \Delta Y - 2H_0(u,v,X_0) \left[ X_{0_u} \wedge Y_v + Y_u \wedge X_{0_v} \right] - 2 \left( \frac{\partial H_0}{\partial X} (u,v,X_0) Y \right) X_{0_u} \wedge X_{0_v}
\]
and
\[
F(u,v,Y,Y_u,Y_v) \quad := \quad 2 \left( H(u,v,X_0 + Y) Y_u \wedge Y_v \right. \\
+ \left[H(u,v,X_0 + Y) - H_0(u,v,X_0) \right] \left( X_{0_u} \wedge Y_v + Y_u \wedge X_{0_v} \right) \\
+ \left[H(u,v,X_0 + Y) - H_0(u,v,X_0) \right] \left( \frac{\partial H_0}{\partial X} (u,v,X_0) Y \right) X_{0_u} \wedge X_{0_v} \right).
\]

We define an operator $T : C^1(\overline{\Omega}, \mathbb{R}^3) \to C^1(\overline{\Omega}, \mathbb{R}^3)$ given by $T(\overline{Y}) = Y$ where $Y$ is the unique solution of the linear problem
\[
LY = F(u,v,\overline{Y},\overline{Y_u},\overline{Y_v}) \quad \text{in } \Omega, \\
Y = g - g_0 \quad \text{on } \partial \Omega.
\]
As $L$ satisfies the hypothesis of Lemma 2.1, it is immediate to prove that $T$ is well defined and continuous. Furthermore, the range of a bounded set is bounded with $\|\cdot\|_{2,p}$, and by Sobolev imbedding, we conclude that $T$ is compact. More precisely, for $\|\bar{Y}\|_{1,\infty} \leq R$, we obtain

$$
\|T(\bar{Y})\|_{1,\infty} \leq \|g - g_0\|_{1,\infty} + c\|T(\bar{Y}) - (g - g_0)\|_{2,p}
$$

$$
\leq \|g - g_0\|_{1,\infty} + c_1\left(\|L(T(\bar{Y}))\|_p + \|L(g - g_0)\|_p\right)
$$

(2.7)

for some constants $k_0$ and $c_1$.

On the other hand, a simple computation shows that

$$
\|F(\cdot, \bar{Y}, Y_u, Y_v)\|_p \leq k_1 R^2 + k_2 \epsilon R + k_3 \epsilon
$$

(2.8)

for some constants $k_1$, $k_2$, and $k_3$. Hence, if $\delta$ and $\epsilon$ are small, it is possible to choose $R$ such that $T(B_R) \subset B_R$ and the result follows by Schauder’s Theorem.

3. A Newton iteration for problem (1.1)

In this section, we apply a Newton iteration to (1.1). For simplicity, we will assume that $H$ does not depend on $X$.

Let $X_0$ be a solution of (1.1) for some $H_0$ and $g_0$ with

$$
2\|H_0\|_\infty \|\nabla X_0\|_\infty < \sqrt{\lambda_1}.
$$

(3.1)

In order to define a sequence that converges to a solution of (1.1) for $(H, g)$ close to $(H_0, g_0)$, we consider the function $F : g + (W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)) \to L^p(\Omega, \mathbb{R}^3)$ given by

$$
F(X) = \Delta X - 2HX_u \wedge X_v.
$$

(3.2)

Thus, the problem is equivalent to find a zero of $F$. The well-known Newton method consists in defining a recursive sequence

$$
X_{n+1} = X_n - (DF(X_n))^{-1}(F(X_n))
$$

(3.3)

or equivalently

$$
DF(X_n)(X_{n+1} - X_n) = -F(X_n).
$$

(3.4)
A simple computation shows that in this case,

$$DF(X)(Y) = \Delta Y - 2H (X_u \wedge Y_v + Y_u \wedge X_v).$$  \hfill (3.5)

According to this, we start at $X_0$ and define the sequence $\{X_n\}$ from the following problem:

$$\Delta X_{n+1} - 2H (X_n \wedge (X_{n+1} - X_n)_v + (X_{n+1} - X_n)_u \wedge X_n) = 2HX_n \wedge X_n$$  \hfill (3.6)

with Dirichlet condition

$$X_{n+1}|_{\partial \Omega} = g.$$  \hfill (3.7)

We will prove that if $H$ and $g$ are close enough to $H_0$ and $g_0$, respectively, this sequence is well defined (i.e., $DF(X_n)$ is invertible for every $n$) and converges.

Fix a positive $R$ such that

$$R < \frac{\sqrt{\lambda_1}}{2\|H_0(\cdot, X_0)\|_{\infty}} - \|\nabla X_0\|_{\infty}$$  \hfill (3.8)

and set

$$\mathcal{C} = \left\{ X \in W^{2,p}(\Omega, \mathbb{R}^3) : X|_{\partial \Omega} = g, \|X - X_0\|_{2,p} \leq \frac{R}{4} \right\}.$$  \hfill (3.9)

We will assume that

$$\|H - H_0\|_{\infty} < \epsilon, \quad \|g - g_0\|_{2,p} < \delta \leq R$$  \hfill (3.10)

with

$$\epsilon < \frac{\sqrt{\lambda_1}}{2(\|\nabla X_0\|_{\infty} + R)} - \|H(\cdot, X_0)\|_{\infty}.$$  \hfill (3.11)

For each $X \in \mathcal{C}$, we define the linear operator $L_X$ given by

$$L_X Y = \Delta Y - 2H (X_u \wedge Y_v + Y_u \wedge X_v).$$  \hfill (3.12)

By Lemma 2.1, $L_X|_{W_0^{1,p}(\Omega)}$ is invertible for any $X \in \mathcal{C}$. Furthermore, we claim that $\|L_X^{-1}\|$ is bounded over $\mathcal{C}$. Indeed, for $Z \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)$ and $X, Y \in \mathcal{C}$, we have

$$\|L_Y Z\|_p \geq \|L_X Z\|_p - \|(L_X - L_Y) Z\|_p \geq \left( \frac{1}{\|L_X^{-1}\|} - 2\|H\|_{\infty} \|\nabla (X - Y)\|_{\infty} \right) \|Z\|_{2,p}.$$  \hfill (3.13)
Solutions to $H$-systems by topological and iterative methods

Taking, for example, $Y$ such that $\|\nabla(Y - X)\|_\infty \leq 1/(4\|H\|_\infty\|L^{-1}_X\|) := R_X$, we obtain

$$\|L_Y^{-1}\| \leq 2\|L_X^{-1}\|. \quad (3.14)$$

By compactness, there exist $X^1, \ldots, X^n \in \mathcal{C}$ such that

$$\mathcal{C} \subset \bigcup_{i=1}^n \{ Y : \|\nabla(Y - X^i)\|_\infty \leq R_{X^i} \} \quad (3.15)$$

and hence,

$$\|L_X^{-1}\| \leq 2 \max_{1 \leq i \leq n} \|L_{X^i}^{-1}\|. \quad (3.16)$$

Let $Z_n = X_{n+1} - X_n$. For $n = 0$, we have

$$\|Z_0\|_{2,p} \leq \|g - g_0\|_{2,p} + \|Z_0 - (g - g_0)\|_{2,p} \leq 2\delta + \|X_0\|_{2,p} + c\|\nabla X_0\|_p \leq c\delta + c\|\nabla X_0\|_p \quad (3.17)$$

As

$$\|L_{X_0}Z_0\|_p = \|2(H - H_0)X_0 \wedge X_0\|_{2,p} \leq \epsilon \|\nabla X_0\|_p, \quad (3.18)$$

we conclude that

$$\|Z_0\|_{2,p} \leq 2\delta(1 + (\|H_0\|_\infty + \epsilon))\|\nabla X_0\|_\infty + c\|L_{X_0}Z_0\|_p \quad (3.19)$$

Then we may establish a more precise version of Theorem 1.3.

**Theorem 3.1.** With the previous notations, assume that

$$c(\delta, \epsilon) \leq \frac{R}{1 + R_0c(\|H_0\|_\infty + \epsilon)}, \quad (3.20)$$

where $c_0$ is the constant of the imbedding $W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow C^1(\overline{\Omega}, \mathbb{R}^3)$. Then the sequence given by (1.6) is well defined and converges in $W^{2,p}(\Omega, \mathbb{R}^3)$ to a solution of (1.1).

**Proof.** By (3.20), we have that $\|Z_0\|_{2,p} \leq c(\delta, \epsilon) \leq R$, proving that $X_1 \in \mathcal{C}$. For $n > 0$, we assume as inductive hypothesis that $X_k \in \mathcal{C}$ for $k \leq n$, and then

$$\|Z_n\|_{2,p} \leq c\|L_{X_n}Z_n\|_p = 2c\|HZ_{n-1} \wedge Z_{n-1}\|_p \leq c\|H\|\|\nabla Z_{n-1}\|_\infty\|\nabla Z_{n-1}\|_p \leq c_0c\|H\|\|Z_{n-1}\|_{2,p}^2. \quad (3.21)$$
Inductively,
\[ \|Z_n\|_{2,p} \leq (c_0 c \|H\|_{\infty})^{2^n-1} \|Z_0\|_{2,p} = A^{2^n-1} \|Z_0\|_{2,p}, \]  
(3.22)
where \( A = c_0 c \|H\|_{\infty} \|Z_0\|_{2,p} \). By hypothesis, it is immediate that \( A < 1 \), and hence
\[ \|X_{n+1} - X_0\|_{2,p} \leq \sum_{j=0}^{n} \|Z_j\|_{2,p} \leq \|Z_0\|_{2,p} \frac{1}{1 - A} \leq R. \]  
(3.23)
Thus, \( X_n \in \mathcal{C} \) for every \( n \), and
\[ \|X_{n+k} - X_n\|_{2,p} \leq \frac{A^{2^n-1}}{1 - A} \]  
(3.24)
for every \( k \geq 0 \). Then \( X_n \) is a Cauchy sequence, and the result follows. \( \square \)

**Remark 3.2.** It is clear from definition that \( c(\delta, \varepsilon) \to 0 \) for \( (\delta, \varepsilon) \to (0, 0) \).

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**References**


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