Let \( \phi \) be a semiflow of holomorphic maps of a bounded domain \( D \) in a complex Banach space. The general question arises under which conditions the existence of a periodic orbit of \( \phi \) implies that \( \phi \) itself is periodic. An answer is provided, in the first part of this paper, in the case in which \( D \) is the open unit ball of a \( J^* \)-algebra and \( \phi \) acts isometrically. More precise results are provided when the \( J^* \)-algebra is a Cartan factor of type one or a spin factor. The second part of this paper deals essentially with the discrete semiflow \( \phi \) generated by the iterates of a holomorphic map. It investigates how the existence of fixed points determines the asymptotic behaviour of the semiflow. Some of these results are extended to continuous semiflows.

1. Introduction

Let \( D \) be a bounded domain in a complex Banach space \( \mathcal{E} \) and let \( \phi : \mathbb{R}_+ \times D \to D \) be a continuous semiflow of holomorphic maps acting on \( D \).

Under which conditions does the existence of a periodic point of \( \phi \) (with a positive period) imply that the semiflow \( \phi \) itself is periodic?

An answer to this question was provided in [22] in the case in which \( \mathcal{E} \) is a complex Hilbert space and \( D \) is the open unit ball of \( \mathcal{E} \), showing that, if the orbit of the periodic point spans a dense linear subspace of \( \mathcal{E} \), then \( \phi \) is the restriction to \( \mathbb{R}_+ \) of a continuous periodic flow of holomorphic automorphisms of \( D \).

In the first part of this paper, a somewhat similar result will be established in the more general case in which \( \mathcal{E} \) is a \( J^* \)-algebra and \( D \) is the open unit ball \( B \) of \( \mathcal{E} \). The main result in this direction can be stated more easily in the case in which the periodic point is the center 0 of \( B \). It will be shown that, if the points of the orbit of 0 which are collinear to extreme points of the closure \( \overline{B} \) of \( B \) span a dense linear subspace of \( \mathcal{E} \), then the same conclusion of [22] holds,
that is, $\phi$ is the restriction to $\mathbb{R}_+$ of a continuous periodic flow of holomorphic automorphisms of $B$.

If the $f^*$-algebra $\mathcal{E}$ is a Cartan factor of type one—that is, it is the Banach space $\mathcal{L}(\mathcal{H}, \mathcal{H})$ of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with values in a complex Hilbert space $\mathcal{H}$—it was shown by Franzoni in [4] that any holomorphic automorphism of $B$ is essentially associated to a linear continuous operator preserving a Kreın space structure defined on the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H}$; a situation that has been further explored in [19, 20] in the case in which $\mathcal{H} \oplus \mathcal{H}$ carries the structure of a Pontryagin space.

Starting from a strongly continuous group $T : \mathbb{R} \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, inducing a continuous flow $\phi$ of holomorphic automorphisms of $B$, it will be shown that, if $\phi$ has a periodic point $x_0$, and if the orbit of $x_0$ is “sufficiently ample,” a rescaled version of $T$ is periodic. A theorem of Bart [1] yields a complete description of the spectral structure of the infinitesimal generator $X$ of $T$.

The particular case in which $\mathcal{H} \simeq \mathbb{C}$ and $B$ is the open unit ball of $\mathcal{H}$, which was initially explored in [22], will be revisited, showing that the periodic flow $\phi$ fixes some point of $B$ and that, if $\phi$ is eventually differentiable, the dimension of $\mathcal{H}$ is finite.

As was shown in [17, 19], in the case in which $\mathcal{H} \oplus \mathcal{H}$ carries the structure of Pontryagin space, a Riccati equation defined on $B$ is canonically associated to $X$. The periodicity of $\phi$ implies then the periodicity of the integrals of this Riccati equation.

A similar investigation to the one carried out in Sections 3 and 4 for a Cartan factor of type one is developed in Section 5 in the case in which $\mathcal{E}$ is a spin factor. In this case, the norm in $\mathcal{E}$ is equivalent to a Hilbert space norm. Assuming again, for the sake of simplicity, that the periodic point is the center $0$ of $D$, a hypothesis leading to the periodicity of $\phi$, consists in supposing that the points of the orbit of $0$ which are collinear to scalar multiples of selfadjoint unitary operators acting on $\mathcal{E}$ span a dense linear submanifold of this latter space.

The case of fixed points of the semiflow $\phi$ acting on the bounded domain $D$ is considered in the second part of this paper, where, among other things, some results which were announced in [16] for discrete semiflows generated iterating a holomorphic map $f : D \to D$ are established in the general case. (One of the basic tools in this investigation was the Earle-Hamilton theorem (see [2] or, e.g., [5, 6, 9]). This theorem, coupled with the theory of complex geodesics for the Carathéodory distance, was also used by several authors (see, e.g., [10, 11, 15, 16, 23, 24, 25, 26, 27]) to investigate the geometry of the set of fixed points of $f$. Further references to fixed points of holomorphic maps can be found in [13].) Our main purpose is to obtain some information on the asymptotic behaviour of $\phi$ in terms of “local” properties.

In this direction, extending to the continuous case a result announced in [16] for the iteration of a holomorphic map, it is shown that, if there is a sequence $\{t_n\}$ in $\mathbb{R}_+$, diverging to infinity and such that $\{\phi_{t_n}\}$ converges, for the topology of local
uniform convergence, to a function mapping $D$ into a set completely interior to $D$, then there exists a unique point $x_0 \in D$ which is fixed by the semiflow $\phi$; moreover, $\phi_s(x)$ tends to $x_0$ as $s \to +\infty$, for all $x \in D$.

If some point $x_0 \in D$ is fixed by the continuous semiflow $\phi$, the map $t \mapsto d\phi_t(x_0)$, where $d\phi_t(x_0) \in \mathcal{L}(\mathcal{E})$ is the Fréchet differential of $\phi_t(x)$ at $x = x_0$, defines a strongly continuous semigroup of bounded linear operators acting on $\mathcal{E}$.

Some situations are explored in which the behaviour of this semigroup determines the asymptotic behaviour of the semiflow $\phi$. It is shown in Sections 7 and 8 that, if the spectral radius $\rho(d\phi_t(x_0))$ of $d\phi_t(x_0)$ is $\rho(d\phi_t(x_0)) < 1$ for some $t > 0$, then, as $s \to +\infty$, $\phi_s$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

The case in which $\rho(d\phi_t(x_0)) = 1$ at some $t > 0$ is considered in Sections 9 and 10, under the additional hypothesis that $d\phi_t(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$. As is well known, the spectrum $\sigma(d\phi_t(x_0))$ of $d\phi_t(x_0)$ consists of two eigenvalues in 0 and in 1 at most.

If

$$\sigma(d\phi_t(x_0)) = \{0\}, \quad (1.1)$$

then $d\phi_s(x_0) = \{0\}$ for all $s \geq t$. As a consequence of Sections 7 and 8, if $s \to +\infty$, $\phi_s$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

If

$$\sigma(d\phi_t(x_0)) = \{1\}, \quad (1.2)$$

then $\phi$ is the restriction to $\mathbb{R}_+$ of a periodic flow of holomorphic automorphisms of $D$.

Finally, if

$$1 \in \sigma(d\phi_t(x_0)) , \quad (1.3)$$

and if there is some $t' > 0$, with $t'/t \notin \mathbb{Q}$, such that also $d\phi_{t'}(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$, then the semiflow $\phi$ is constant, that is, $\phi_t = \text{id}$ (the identity map) for all $t \geq 0$.

2. The general case of a $J^\ast$-algebra

Let $\mathcal{E}$ be a complex Banach space, let $D$ be a domain in $\mathcal{E}$, and let

$$\phi : \mathbb{R}_+ \times D \longrightarrow D \quad (2.1)$$
be a semiflow of holomorphic maps of $D$ into $D$, that is, a map such that
\begin{align}
\phi_0 &= \text{id}, \\
\phi_{t_1+t_2} &= \phi_t \phi_{t_2}, \\
\phi_t &\in \text{Hol}(D),
\end{align}
for all $t, t_1, t_2 \in \mathbb{R}_+$, where $\text{Hol}(D)$ is the semigroup of all holomorphic maps $D \to D$.

A point $x \in D$ is said to be a periodic point of $\phi$ with period $\tau > 0$ if $\phi_t(x) = x$ and $\phi_t(x) \neq x$ for all $t \in (0, \tau)$. The semiflow $\phi$ will be said to be periodic with period $\tau$ if $\phi_\tau = \text{id}$ and, whenever $0 < t < \tau$, $\phi_t$ is not the identity map.

We begin by establishing the following elementary lemma, which is a consequence of Cartan’s uniqueness theorem (see, e.g., [5]) and which might have some interest in itself.

Let $D$ be a hyperbolic domain in the Banach space $\mathcal{E}$ (or, more in general, a domain in $\mathcal{E}$ on which either the Carathéodory or the Kobayashi distances define equivalent topologies to the relative topology) and let $x_0 \in D$ be a fixed point of the semiflow $\phi$, that is, $\phi_t(x_0) = x_0$ for all $t \in \mathbb{R}_+$.

**Lemma 2.1.** If there is a vector $\xi \in \mathcal{E} \setminus \{0\}$, for which the map $t \mapsto d\phi_t(x_0)\xi$ of $\mathbb{R}_+$ into $\mathcal{L}(\mathcal{E})$ is periodic with period $\tau > 0$, and there is a set $K \subset (0, \tau)$ such that \{d$\phi_t(x_0)\xi : t \in K$\} spans a dense affine subspace $\mathcal{K}$ of $\mathcal{E}$, then $\phi_\tau = \text{id}$.

**Proof.** Let $x_0 = 0$. Since
\[ d\phi_t(0)(d\phi_t(0)\xi) = d\phi_{t+t}(0)\xi = d\phi_t(0)\xi \quad \forall t \geq 0, \]
then $d\phi_\tau(0) = \text{id}$ on $\mathcal{K}$ and therefore on $\mathcal{E}$. Cartan’s identity theorem yields the conclusion. \qed

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the complex Banach space of all continuous linear operators $\mathcal{H} \to \mathcal{K}$, endowed with the operator norm. For $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $A^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ will denote the adjoint of $A$. A $J^*$-algebra [7] is a closed linear subspace $\mathcal{A}$ of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ such that
\[ A \in \mathcal{A} \implies AA^* A \in \mathcal{A}. \tag{2.6} \]

The roles of $\mathcal{E}$ and $D$ will now be played by a $J^*$-algebra $\mathcal{A}$ and by the open unit ball $B$ of $\mathcal{A}$.

Let $S$ be the set of all extreme points of the closure $\overline{B}$ of $B$. As was noted by Harris in [7], if $\mathcal{A}$ is weakly closed in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, then $S \neq \emptyset$.

**Lemma 2.2.** Let $S \neq \emptyset$. If $0$ is a periodic point of the semiflow $\phi : \mathbb{R}_+ \times B \to B$, with period $\tau > 0$, and if there is a set $K \subset (0, \tau)$ such that, for every $t \in K$, $\phi_t(0)$ is collinear to some point of $S$, and the set \{\phi_t(0) : t \in K\} spans a dense linear subspace of $\mathcal{A}$, then the semiflow $\phi$ is periodic with period $\tau$. 
Proof. Let $\Delta$ be the open unit disc of $\mathbb{C}$. For $t \in K$,

$$\Delta \ni \zeta \mapsto \frac{\zeta}{\|\phi_t(0)\|} \phi_t(0)$$

(2.7)

is, up to parametrization, the unique complex geodesic whose support contains both 0 and $\phi_t(0)$. (For the Kobayashi or Carathéodory metrics on $B$, for the basic notions concerning complex geodesics, see, e.g., [14, 15].)

Since $\phi_t(0) = 0$ and

$$\phi_t(\phi_t(0)) = \phi_t(0) = \phi_t(0),$$

then $\phi_t$ is the identity on the support of the complex geodesic (2.7). Hence

$$d\phi_t(0)(\phi_t(0)) = \phi_t(0) \quad \forall t \in K,$$

(2.9)

and therefore $d\phi_t(0) = I_{\mathcal{A}}$. Thus $d\phi_t(0)$ maps the set $S$ onto itself. By Harris’ Schwarz lemma [7, Theorem 10], $\phi_t = d\phi_t(0) = \text{id.}$

Let now $x_0 \in B$ be a periodic point of $\phi$ with period $\tau > 0$.

As was shown in [7], the Moebius transformation $M_{x_0}$ is a holomorphic automorphism of $B$ which maps any $x \in B$ to the point

$$M_{x_0}(x) = (I - x_0 x_0^*)^{-1/2}(x + x_0)(I + x_0^* x)^{-1}(I - x_0^* x_0)^{1/2}$$

$$= x_0 + (I - x_0 x_0^*)^{1/2}x(I + x_0^* x)^{-1}(I - x_0^* x_0)^{1/2}.$$  

(2.10)

Furthermore,

$$M_{x_0}(0) = x_0, \quad M_{x_0}^{-1} = M_{-x_0},$$

(2.11)

and $M_{x_0}$ is the restriction to $B$ of a holomorphic function on an open neighbourhood of $\overline{B}$ in $\mathcal{A}$, mapping $\partial B$ onto itself.

Applying Lemma 2.2 to the semiflow $t \mapsto \psi_t = M_{-x_0}\phi_t M_{x_0}$, we obtain the following theorem.

**Theorem 2.3.** If $x_0 \in B$ is a periodic point of $\phi$ with period $\tau > 0$ and if there is a set $K \subset (0, \tau)$ such that

(i) for any $t \in K$, $M_{-x_0}(\phi_t(x_0))$ is collinear to some point in $S$;

(ii) the set $\{\phi_t(x_0) : t \in K\}$ spans a dense affine subspace of $\mathcal{A}$ (as was shown by Harris in [7, Corollary 8], $\overline{B}$ is the closed convex hull of $S$),

then the semiflow $\phi$ is periodic with period $\tau$. 

Remark 2.4. Under the hypotheses of Theorem 2.3, setting \( \psi_t = \phi_t \) when \( t \geq 0 \), and \( \psi_t = \phi_{-t} \) when \( t \leq 0 \), one defines a flow \( \psi : \mathbb{R} \times B \to B \) of holomorphic automorphisms of \( B \), whose restriction to \( \mathbb{R}_+ \) is \( \phi \).

The flow \( \psi \) is continuous if and only if the semiflow \( \phi \) is continuous, that is, the map \( \phi : \mathbb{R}_+ \times B \to B \) is continuous.

In the case in which \( n = \dim_{\mathbb{C}} \mathcal{A} < \infty \), a similar statement to Theorem 2.3 holds for a discrete semiflow, that is to say, for the semiflow generated by the iterates \( f^m = f \circ f \circ \cdots \circ f \) \((m = 1, 2, \ldots)\) of a holomorphic map \( f : B \to B \).

Theorem 2.5. If \( f \) has a periodic point \( x_0 \in B \), with period \( p > n \) (i.e., \( f^p(x_0) = x_0 \) if \( q = 1, \ldots, p - 1 \)), if \( M_{-x_0}(f^q(x_0)) \) is collinear to some point in the Shilov boundary of \( \overline{B} \) for \( q = 1, \ldots, p - 1 \), and if the orbit \( \{ f^q(x_0) : q = 1, \ldots, p - 1 \} \) of \( x_0 \) spans \( \mathcal{A} \), then \( f \) is periodic with period \( p \).

For example, let \( f_1 : z \mapsto e^{2\pi i/3}z \) and let \( f_2 \) be another holomorphic function \( \Delta \to \Delta \) such that \( f_2(0) = 0 \) but \( f_2 \neq 0 \). Let \( f : \Delta \times \Delta \to \Delta \times \Delta \) be the holomorphic map defined by

\[
f(z_1, z_2) = (f_1(z_1), f_2(z_2)), \quad (z_1, z_2 \in \Delta).
\]

(2.12)

If \( f_2 \) has a periodic point in \( \Delta \setminus \{0\} \), and therefore is periodic, \( f \) is periodic with period \( \geq 3 \). If \( f_2 \) is not periodic, \( f \) is not periodic. However, every point \( (z_1, 0) \) with \( z_1 \in \Delta \setminus \{0\} \) is a periodic point of \( f \) with period 3.

3. Cartan domains of type one

Let the \( J^* \)-algebra \( \mathcal{A} \) be a Cartan factor of type one, \( \mathcal{A} = \mathcal{L}(\mathcal{H}, \mathcal{H}) \). Let

\[
J = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix},
\]

(3.1)

and let \( \Gamma(f) \) be the group of all linear continuous operators \( A \) on \( \mathcal{H} \oplus \mathcal{H} \) which are invertible in \( \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) and such that

\[
A^* J A = J.
\]

(3.2)

It was shown by Franzoni in [4] that the group of all holomorphic automorphisms of the unit ball \( B \) of \( \mathcal{A} \), which is called a Cartan domain of type one, is isomorphic to a quotient of \( \Gamma(f) \), up to conjugation when \( \dim_{\mathbb{C}} \mathcal{H} = \dim_{\mathbb{C}} \mathcal{H} \).

To avoid conjugation, we will consider now the case in which \( \infty \geq \dim_{\mathbb{C}} \mathcal{H} \neq \dim_{\mathbb{C}} \mathcal{H} \leq \infty \).

Let \( T : \mathbb{R} \to \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) be a strongly continuous group such that

\[
T(t)^* J T(t) = J,
\]

(3.3)
or equivalently

\[ T(t)JT(t)^* = J, \quad (3.4) \]

for all \( t \in \mathbb{R} \).

If

\[ T(t) = \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{pmatrix} \quad (3.5) \]

is the representation of \( T(t) \) in \( \mathcal{H} \oplus \mathcal{H} \), with \( T_{11}(t) \in \mathcal{L}(\mathcal{H}), T_{12}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), T_{21}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \) and \( T_{22}(t) \in \mathcal{L}(\mathcal{H}) \), then \( (3.3) \) and \( (3.4) \) are equivalent to

\[
\begin{aligned}
T_{11}(t)^*T_{11}(t) - T_{21}(t)^*T_{21}(t) &= I_{\mathcal{H}}, \\
T_{22}(t)^*T_{22}(t) - T_{12}(t)^*T_{12}(t) &= I_{\mathcal{H}}, \\
T_{12}(t)^*T_{11}(t) - T_{22}(t)^*T_{21}(t) &= 0, \\
T_{11}(t)T_{11}(t)^* - T_{12}(t)T_{12}(t)^* &= I_{\mathcal{H}}, \\
T_{22}(t)T_{22}(t)^* - T_{21}(t)T_{21}(t)^* &= I_{\mathcal{H}}, \\
T_{21}(t)T_{11}(t)^* - T_{22}(t)T_{21}(t)^* &= 0. 
\end{aligned}
\]

Here \( T_{11}(t)^* \in \mathcal{L}(\mathcal{H}), T_{12}(t)^* \in \mathcal{L}(\mathcal{H}, \mathcal{H}), T_{21}(t)^* \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \) and \( T_{22}(t)^* \in \mathcal{L}(\mathcal{H}) \) are the adjoint operators of \( T_{11}(t), T_{12}(t), T_{21}(t), \) and \( T_{22}(t) \).

From now on, in this section, latin letters \( x \) and \( y \) indicate elements of \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \) and greek letters \( \xi \) and \( \eta \) indicate vectors in \( \mathcal{H} \) and \( \mathcal{H} \).

It was shown in [4], that, if \( x \in B, T_{21}(t)x + T_{22}(t) \in \mathcal{L}(\mathcal{H}) \) is invertible in \( \mathcal{L}(\mathcal{H}) \), and the function \( \widehat{T}(t) \), defined on \( B \) by

\[
\widehat{T}(t) : x \mapsto (T_{11}(t)x + T_{12}(t))(T_{21}(t)x + T_{22}(t))^{-1}, \quad (3.8)
\]

is, for all \( t \in \mathbb{R} \), a holomorphic automorphism of \( B \).

Setting

\[
\phi_t = \widehat{T}(t) \quad (3.9)
\]

for \( t \in \mathbb{R} \), we define a continuous flow \( \phi \) of holomorphic automorphisms of \( B \).

If \( x_0 \in B \) is a periodic point of \( \phi \) with period \( \tau > 0 \), and if the hypotheses of Theorem 2.3 are satisfied, \( \phi \) is periodic with period \( \tau \).

Since \( \widehat{T}(\tau) = \text{id} \), then

\[
T_{11}(\tau)x + T_{12}(\tau) = xT_{21}(\tau)x + xT_{22}(\tau) \quad \forall x \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \quad (3.10)
\]

whence

\[
T_{12}(\tau) = 0, \quad T_{21}(\tau) = 0, \quad (3.11)
\]
Periodicity of holomorphic maps

and therefore, by (3.6),

\[
T_{11}(\tau)^* T_{11}(\tau) = T_{11}(\tau) T_{11}(\tau)^* = I_{\mathcal{H}},
\]

\[
T_{22}(\tau)^* T_{22}(\tau) = T_{22}(\tau) T_{22}(\tau)^* = I_{\mathcal{H}},
\]

that is, \( T_{11}(\tau) \) and \( T_{22}(\tau) \) are unitary operators in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H} \). Furthermore, (3.10) becomes

\[
T_{11}(\tau) x = x T_{22}(\tau) \quad \forall x \in \mathcal{L}(\mathcal{H}, \mathcal{H}).
\]

(3.13)

Since \( T_{22}(\tau) \) is unitary, every point \( e^{i\theta \tau} (\theta \in \mathbb{R}) \) in the spectrum \( \sigma(T_{22}(\tau)) \) of \( T_{22}(\tau) \) is contained either in the point spectrum or in the continuous spectrum. In both cases, there exists a sequence \( \{\xi_v\} \) in \( \mathcal{H} \) (which may be assumed to be constant if \( e^{i\theta \tau} \) is an eigenvalue), with \( \|\xi_v\| = 1 \), such that

\[
\lim_{v \to +\infty} (T_{22}(\tau)\xi_v - e^{i\theta \tau}\xi_v) = 0.
\]

(3.14)

Since, by the Schwarz inequality,

\[
\left| \left( T_{22}(\tau)\xi_v |\xi_v \right) - e^{i\theta \tau} \right| = \left| \left( T_{22}(\tau)\xi_v - e^{i\theta \tau}\xi_v |\xi_v \right) \right| \\
\leq \left\| T_{22}(\tau)\xi_v - e^{i\theta \tau}\xi_v \right\|,
\]

then

\[
\lim_{v \to +\infty} (T_{22}(\tau)\xi_v |\xi_v \right) = e^{i\theta \tau}.
\]

(3.16)

Hence, letting, for any \( \eta \in \mathcal{H} \), \( x_v = \eta \otimes \xi_v \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \), then \( x_v(\xi_v) = \eta \) and

\[
\lim_{v \to +\infty} x_v(T_{22}(\tau)\xi_v) = \lim_{v \to +\infty} (T_{22}(\tau)\xi_v |\xi_v \right) \eta = e^{i\theta \tau} \eta.
\]

(3.17)

Thus, by (3.13),

\[
T_{11}(\tau) \eta = \lim_{v \to +\infty} T_{11}(\tau)(x_v(\xi_v)) = \lim_{v \to +\infty} x_v(T_{22}(\tau)\xi_v) = e^{i\theta \tau} \eta
\]

for all \( \eta \in \mathcal{H} \). Therefore,

\[
T_{11}(\tau) = e^{i\theta \tau} I_{\mathcal{H}},
\]

(3.19)

and (3.13) yields

\[
T_{22}(\tau) = e^{i\theta \tau} I_{\mathcal{H}}.
\]

(3.20)
In conclusion,

\[ T(\tau) = e^{i\theta \tau} I_{\mathcal{K} \oplus \mathcal{H}}. \]  

(3.21)

Thus, the rescaled group \( L : \mathbb{R} \to \mathcal{L}(\mathcal{K} \oplus \mathcal{H}) \), defined by

\[ L(t) = e^{-i \theta t} T(t), \]

(3.22)

is periodic with period \( \tau \).

Note that

\[ L(t)^* JL(t) = J \quad \forall t \in \mathbb{R}. \]  

(3.23)

If

\[ L(t) = \begin{pmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{pmatrix} \]

(3.24)

is the representation of \( L(t) \) in \( \mathcal{K} \oplus \mathcal{H} \), with \( L_{11}(t) \in \mathcal{L}(\mathcal{K}), L_{12}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{K}), L_{21}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \) and \( L_{22}(t) \in \mathcal{L}(\mathcal{H}) \), then

\[ L_{\alpha,\beta}(t) = e^{-i \theta t} T_{\alpha,\beta}(t) \]

(3.25)

for \( \alpha, \beta = 1, 2 \). Therefore, setting, for \( x \in B \),

\[ \tilde{L}(t)(x) : x \mapsto (L_{11}(t)x + L_{12}(t))(L_{21}(t)x + L_{22}(t))^{-1}, \]

(3.26)

then

\[ \tilde{L}(t) = \phi_t \quad \forall t \in \mathbb{R}. \]  

(3.27)

If \( X : \mathcal{D}(X) \subset \mathcal{K} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{H} \) is the infinitesimal generator of the group \( T \), the operator \( X - i \theta I_{\mathcal{K} \oplus \mathcal{H}} \), with domain \( \mathcal{D}(X) \), generates the group \( L \).

The structure of the spectrum \( \sigma(X - i \theta I_{\mathcal{K} \oplus \mathcal{H}}) \) is described in [1] by a theorem of Bart, whereby

\( (i) \sigma(X - i \theta I_{\mathcal{K} \oplus \mathcal{H}}) \subset i(2\pi/\tau)\mathbb{Z}; \)

\( (ii) \sigma(X - i \theta I_{\mathcal{K} \oplus \mathcal{H}}) \) consists of simple poles of the resolvent function \( \zeta \mapsto (\zeta I_{\mathcal{K} \oplus \mathcal{H}} - (X - i \theta I_{\mathcal{K} \oplus \mathcal{H}}))^{-1}; \)

\( (iii) \) the eigenvectors of \( X - i \theta I_{\mathcal{K} \oplus \mathcal{H}} \) span a dense linear subspace of \( \mathcal{K} \oplus \mathcal{H} \).

According to [1], if \( X \) is the infinitesimal generator of a strongly continuous group \( T \), and if conditions (i), (ii), and (iii) hold, the group \( L \) defined by (3.22) is periodic with period \( \tau \).

Summing up, in view of Theorem 2.3, the following result has been established.
Theorem 3.1. If there is a periodic point \( x_0 \in B \) for \( \phi \), with period \( \tau > 0 \), and if there is a set \( K \subset (0, \tau) \) such that, for any \( t \in K \), \( M_{-\tau}(\phi(x_0)) \) is collinear to some point of \( S \), and the set \( \{ \phi_t(x_0) : t \in K \} \) spans a dense affine subspace of \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \), then there exist a strongly continuous group \( T : \mathbb{R} \to \mathcal{L}(\mathcal{H}, \mathcal{H}) \) and a real number \( \theta \) such that the rescaled group \( \mathbb{R} \ni t \mapsto L(t) \) is a periodic group with period \( \tau \).

If \( X : \mathcal{D}(X) \subset \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} \) is the infinitesimal generator of the group \( T \), conditions (i), (ii), and (iii) characterize the periodicity of \( L \) with period \( \tau \).

Thus, if \( X \) generates a strongly continuous group \( T \), and if conditions (i), (ii), and (iii) hold, the group \( L \) defined by (3.22) is periodic with period \( \tau \). As was proved in [19, Proposition 4.1], the group \( T \) satisfies (3.3) for all \( t \in \mathbb{R} \) if and only if the operator \( iJX \) is selfadjoint. If that is the case, setting

\[
\mathcal{H}' \oplus 0 = (\mathcal{H} \oplus 0) \cap \mathcal{D}(X), \quad 0 \oplus \mathcal{H}' = (0 \oplus \mathcal{H}) \cap \mathcal{D}(X),
\]

(3.28) [19, Lemma 5.3] implies that the linear spaces \( \mathcal{H}' \) and \( \mathcal{H}' \) are dense in \( \mathcal{H} \) and \( \mathcal{H} \).

We consider now the case in which the semigroup \( T|_{\mathbb{R}} \) is eventually differentiable (i.e., there is \( t^0 \geq 0 \) such that the function \( t \mapsto T(t)x \) is differentiable in \( (t^0, +\infty) \) for all \( x \in \mathcal{H} \oplus \mathcal{H} \) By (3.22), also \( L|_{\mathbb{R}} \) is eventually differentiable.

According to a theorem by Pazy (see, e.g., [12]), there exist \( a \in \mathbb{R} \) and \( b > 0 \) such that the set

\[
\{ \zeta \in \mathbb{C} : \Re \zeta \geq a - b \log |\Im \zeta| \}
\]

(3.29) is contained in the resolvent set of \( X - i\theta I_{\mathcal{H} \oplus \mathcal{H}} \). Thus, the intersection of \( \sigma(X - i\theta I_{\mathcal{H} \oplus \mathcal{H}}) \) with the imaginary axis is bounded. Condition (i) implies then that \( \sigma(X - i\theta I_{\mathcal{H} \oplus \mathcal{H}}) \) is finite. But then, by [1, Proposition 3.2], \( X - i\theta I_{\mathcal{H} \oplus \mathcal{H}} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \), and therefore \( X \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \), proving thereby the following proposition.

Proposition 3.2. Under the hypotheses of Theorem 3.1, if moreover the semigroup \( T|_{\mathbb{R}} \) is eventually differentiable, the group \( T \) is uniformly continuous.

Remark 3.3. The above argument holds for any strongly continuous semigroup \( T \) of linear operators, which is periodic, showing that, if \( T \) is eventually differentiable, then \( T \) is uniformly continuous.

If \( T \) is eventually norm continuous, then (see, e.g., [3]) its infinitesimal generator \( X \) is such that, for every \( r \in \mathbb{R} \), the set

\[
\{ \zeta \in \sigma(X) : \Re \zeta \geq r \}
\]

(3.30) is bounded.

At this point, [1, Proposition 3.2] implies that, if \( T \) is also periodic, then the operator \( X \) is bounded, and therefore \( T \) is uniformly continuous.

This conclusion holds, for example, if the periodic semigroup \( T \) is eventually compact.
4. The unit ball of a Hilbert space

Theorem 3.1 has been established in [22] in the case in which $B$ is the open unit ball of the Hilbert space $\mathcal{H}$ (i.e., when $\mathcal{H} = \mathbb{C}$).

In this case, $T_{11}(t) \in \mathcal{L}(\mathcal{H})$ is invertible in $\mathcal{L}(\mathcal{H})$, $T_{12}(t) \in \mathcal{H}$, $T_{21}(t) = (\bullet | T_{12}(t))$, and $T_{22}(t) \in \mathbb{C}$ are characterized by the equations

$$
\|T_{22}(t)\|^2 - \|T_{12}(t)\|^2 = 1,
$$

$$
T_{11}(t)^* T_{11}(t) = I + \frac{1}{\|T_{22}(t)\|^2} (\bullet | T_{11}(t)^* T_{12}(t)) T_{11}(t)^* T_{12}(t).
$$

(4.1)

As was shown in [22], there is a neighbourhood $U$ of $B$ such that

$$
(x | T_{11}(t)^* T_{12}(t)) + T_{22}(t) \neq 0 \quad \forall x \in U, \ t \in \mathbb{R}.
$$

(4.2)

The orbit of $x_0 \in B$ is described by

$$
\phi_t(x_0) = \tilde{T}(t)(x_0) = \frac{1}{(x_0 | T_{11}(t)^* T_{12}(t)) + T_{22}(t)} (T_{11}(t)x_0 + T_{12}(t)).
$$

(4.3)

The infinitesimal generator $X$ of $T$ is represented in $\mathcal{H} \oplus \mathbb{C}$ by the matrix

$$
X = \begin{pmatrix}
X_{11} & X_{12} \\
(\bullet | X_{12}) & iX_{22}
\end{pmatrix},
$$

(4.4)

where $X_{12} \in \mathcal{H}$, $X_{22} \in \mathbb{R}$, $iX_{11}$ is a selfadjoint operator, and the domains $\mathcal{D}(X)$ and $\mathcal{D}(X_{11})$ of $X$ and of $X_{11}$ are related by

$$
\mathcal{D}(X) = \mathcal{D}(X_{11}) \oplus \mathbb{C}.
$$

(4.5)

Since $\phi_t$ is the identity, by [17, Proposition 7.3] and by (3.27), the set

$$
\text{Fix } \phi = \{ x \in B : \phi_t(x) = x \ \forall t \in \mathbb{R} \}
$$

(4.6)

is nonempty.

The ball $B$ being homogeneous, there is no restriction in assuming $0 \in \text{Fix } \phi$. Thus, by (3.8), $T_{12}(t) = 0$ for all $t \in \mathbb{R}$, and therefore $X_{12} = 0$. Furthermore, as a consequence of (4.1),

$$
T_{22}(t) = e^{iX_{22}t},
$$

(4.7)

and the skew-selfadjoint operator $X_{11}$ generates the strongly continuous group $T_{11} : t \mapsto T_{11}(t)$ of unitary operators in $\mathcal{H}$. 
Equation (3.9), which now reads
\[ \phi_t(x) = e^{-iX_{22}t}T_{11}(t), \] (4.8)
yields the following lemma.

**Lemma 4.1.** The set \( \text{Fix} \phi \) is the intersection of \( B \) with a closed affine subspace of \( \mathcal{H} \).

Because of (3.21),
\[ X_{22} = \theta + \frac{2n\pi}{\tau} \] (4.9)
for some \( n \in \mathbb{Z} \), and therefore
\[ \phi_t(x) = e^{-\left(2n\pi/\tau\right)it}L_{11}(t)x \] (4.10)
for all \( x \in B \) and some \( n \in \mathbb{Z} \).

The strongly continuous periodic group \( L_{11} : t \mapsto L_{11}(t) \), with period \( \tau \), of unitary operators in \( \mathcal{H} \) is generated by
\[ Y_{11} := X_{11} - i\theta I_{\mathcal{H}} : \mathcal{D}(X_{11}) \subset \mathcal{H} \rightarrow \mathcal{H}. \] (4.11)

By [1], \( \sigma(Y_{11}) \subset i(2\pi/\tau)\mathbb{Z} \) consists entirely of eigenvalues, and the corresponding eigenspaces, which are mutually orthogonal, span a dense linear subspace of \( \mathcal{H} \).

For \( m \in \mathbb{Z} \), let \( P_m \) be the orthogonal spectral projector associated with \( (2\pi/\tau)m \). By [1, (3)], \( L_{11} \) is expressed by
\[ L_{11}(t)x = \sum_m e^{(2m\pi/\tau)it}P_mx \] (4.12)
for all \( x \in \mathcal{H} \) and all \( t \in \mathbb{R} \). Thus \( L_{11}(t) \) leaves invariant every space \( P_m(\mathcal{H}) \), and acts on it by the rotation
\[ x \mapsto e^{(2m\pi/\tau)it}x. \] (4.13)

Hence, the following lemma follows.

**Lemma 4.2.** If the orbit of \( x_0 \in B \) spans a dense affine subspace of \( \mathcal{H} \), then \( \dim_{\mathbb{C}} P_m(\mathcal{H}) \leq 1 \) for all \( m \in \mathbb{Z} \).

Since, by (3.25),
\[ \sigma(Y_{11}) = \sigma(X_{11} - i\theta I_{\mathcal{H}}) \] (4.14)
if \( \sigma(X_{11}) \) is finite, also \( \sigma(Y_{11}) \) is finite.

A similar argument to that leading to Proposition 3.2 yields now the following theorem.
Theorem 4.3. If the continuous flow $\phi$ of holomorphic automorphisms of the open unit ball $B$ of $\mathcal{H}$ defined by a strongly continuous group $T : \mathbb{R} \to \mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ has a periodic point whose orbit spans a dense affine subspace of $\mathcal{H}$, and if moreover $T$ is eventually differentiable, then $\dim_{\mathbb{C}} \mathcal{H} < \infty$.

According to [17, Theorem VII], for any $\gamma > 0$ and every choice of $x_0 \in B \cap \mathcal{D}(X_{11})$, the function

$$
\phi_* (x_0) |_{[0,\gamma]} : [0, \gamma] \to \mathcal{D}(X_{11}),
$$

defined by (4.3) for $0 \leq t \leq \gamma$, is the unique continuously differentiable map $[0, \gamma] \to \mathcal{H}$ with $x([0, \gamma]) \subset \mathcal{D}(X_{11})$, which is continuous for the graph norm

$$
x \mapsto ||x|| + ||X_{11}x||
$$

on $\mathcal{D}(X_{11})$, and satisfies the Riccati equation

$$
\frac{d}{dt} \phi_t (x_0) = X_{11} \phi_t (x_0) - ((\phi_t (x_0) | X_{12}) + iX_{22}) \phi_t (x_0) + X_{12}
$$

with the initial condition $\phi_0 (x_0) = x_0 \in B \cap \mathcal{D}(X_{11})$.

Hence, Theorem 3.1 can be rephrased.

Proposition 4.4. If the Riccati equation (4.17) has a periodic integral which spans a dense affine subspace of $\mathcal{H}$, (4.17) is periodic (i.e., all integrals of (4.17) satisfying the above regularity conditions are periodic).

We consider now the case in which one of the two spaces $\mathcal{H}$ and $\mathcal{K}$ has a finite dimension, and therefore $J$ defines in $\mathcal{H} \oplus \mathcal{K}$ the structure of a Pontryagin space. Assuming

$$
\infty > \dim_{\mathbb{C}} \mathcal{K} < \dim_{\mathbb{C}} \mathcal{H} \leq \infty,
$$

the extreme points of $\overline{B}$ are all the linear isometries $\mathcal{K} \to \mathcal{H}$; by [19, Theorem III], $X$ is represented by the matrix

$$
X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & iX_{22} \end{pmatrix},
$$

where $X_{11} : \mathcal{D}(X_{11}) \subset \mathcal{H} \to \mathcal{H}$ and $X_{22} \in \mathcal{L}(\mathcal{H})$ are skew-selfadjoint, $X_{12} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, and $\mathcal{D}(X) = \mathcal{D}(X_{11}) \oplus \mathcal{K}$.
The Riccati equation (4.17) is replaced in [19] by the operator-valued Riccati equation
\[
\frac{d}{dt}x(t) = X_{11}x(t) - x(t)X_{22} - x(t)X_{12}^*x(t) + X_{12}
\]
acting on $C^1$ maps of $[0, \gamma]$ into
\[
\mathcal{D} = \{ x \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : x\xi \in \mathcal{D}(X_{11}) \ \forall \xi \in \mathcal{K} \}
\]
which are continuous for the norm (4.16).

For any $\gamma > 0$, any choice of $u$ invertible in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ and of $v \in \mathcal{D}$ such that $x_0 = vu^{-1} \in B$, the function $t \mapsto x(t)$ expressed by (3.8), with $x = x_0$, for $t \in [0, \gamma]$ is the unique solution of (4.20) satisfying the conditions stated above, with the initial condition $x(0) = x_0$.

Theorem 3.1 yields then the following proposition.

Proposition 4.5. Let the integral $t \mapsto x(t)$ be periodic with period $\tau > 0$, and let there be a set $K \subset (0, \tau)$ such that $x(K)$ spans a dense affine subspace of $\mathcal{L}(\mathcal{H}, \mathcal{K})$. If, for any $t \in K$, $M_{-x_0}(x(t))$ is collinear to some linear isometry of $\mathcal{K}$ into $\mathcal{K}$, the Riccati equation (4.20) is periodic.

5. Spin factors

Similar results to some of those of Section 3 will now be established in the case in which the $J^*$-algebra $\mathcal{A}$ is a spin factor. In this section, $\mathcal{K}$ is, as before, a complex Hilbert space, and $C^*$ is the adjoint of $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. A Cartan factor of type four, also called a spin factor, is a closed linear subspace $\mathcal{A}$ of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ which is $^*$-invariant and such that $C \in \mathcal{A}$ implies that $C^2$ is a scalar multiple of $I_{\mathcal{K}}$.

Since, for $C_1, C_2 \in \mathcal{A}$, $C_1C_2^* + C_2^*C_1$ is a scalar multiple, $2(C_1|C_2)I_{\mathcal{K}}$, of the identity, then $C_1, C_2 \mapsto (C_1|C_2)$ is a positive-definite scalar product, with respect to which $\mathcal{A}$ is a complex Hilbert space. (For more details concerning spin factors, see, e.g., [7, 18, 21].) Denoting by $\|\cdot\|$ and by $\|\cdot\|$ the operator norm and the Hilbert space norm on $\mathcal{A}$, then
\[
\|C\|^2 = \|C\|^2 + \sqrt{\|C\|^4 - (C|C^*)^2} \ \forall C \in \mathcal{A}.
\]

The open unit ball $B$ for the norm $\|\cdot\|$, also expressed by
\[
B = \left\{ C \in \mathcal{A} : \|C\|^2 < \frac{1 + \sqrt{1 - (C|C^*)^2}}{2} < 1 \right\},
\]
is called a Cartan domain of type four. The set $S$ of all extreme points of $\overline{B}$ is the set of all multiples, by a constant factor of modulus one, of all selfadjoint unitary operators acting on the Hilbert space $\mathcal{K}$, which are contained in $\mathcal{A}$ [7, 21].
Changing again notations, we denote by $x, y$ elements of the spin factor $\mathcal{A}$, and $x \rightarrow \overline{x}$ stands for the conjugation defined by the adjunction in the Hilbert space $\mathcal{A}$. For any $M \in \mathcal{L}(\mathcal{A})$, $M^t$ will indicate the transposed of $M$. The same notation will be used to indicate the canonical transposition in $C^2$ and the transposition in $\mathcal{A} \oplus C^2$.

According to [7, 21], any holomorphic automorphism $f$ of $B$ can be described as follows.

Let

$$J = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & -I_{C^2} \end{pmatrix},$$

and let $\Lambda$ be the semigroup consisting of all $A \in \mathcal{L}(\mathcal{A} \oplus C^2)$ such that

$$A^tJA = J.$$  

Every $A \in \Lambda$ is represented by a matrix

$$A = \begin{pmatrix} M & q_1 & q_2 \\ (\bullet | r_1) & e_{11} & e_{12} \\ (\bullet | r_2) & e_{21} & e_{22} \end{pmatrix},$$

where $M \in \mathcal{L}(\mathcal{A})$ is a real operator, $q_1, q_2, r_1, r_2$ are real vectors in $\mathcal{A}$, and

$$E := \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

is a real $2 \times 2$ matrix such that $\det E > 0$, and

$$M^tM - R^tR = I_{\mathcal{A}},$$
$$M^tQ - R^tE = 0,$$
$$E^tE - Q^tQ = I_{C^2}.$$  

Here $R : \mathcal{A} \rightarrow C^2$ and $Q : C^2 \rightarrow \mathcal{A}$ are defined by

$$Rx = \begin{pmatrix} (x|r_1) \\ (x|r_2) \end{pmatrix} \in C^2 \quad \forall x \in \mathcal{A},$$
$$Qz = z_1q_1 + z_2q_2 \quad \forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in C^2.$$  

It was shown in [18] that the set $\Lambda_0 = \{ A \in \Lambda : \det E > 0 \}$ is a subsemigroup of $\Lambda$.

For $x \in \mathcal{A}$, let

$$\delta(A, x) = 2(x|r_1 - r_2) + (e_{11} - e_{22} + i(e_{12} + e_{21}))(x|x) + e_{11} + e_{22} + i(e_{21} - e_{12}).$$  

(5.11)
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One shows (see [18, 21]) that, if \( A \in \Lambda_0, \delta(A, x) \neq 0 \) for all \( x \) in an open neighbourhood \( U \) of \( B \). Hence, the map

\[
\hat{A} : U \ni x \mapsto -\frac{1}{\delta(A, x)} (2Mx + (1 + (x|\overline{x}))q_1 - i(1 - (x|\overline{x}))q_2)
\]  (5.12)

is holomorphic in \( U \). Its restriction to \( B \), which will be denoted by the same symbol \( \hat{A} \), is the most general holomorphic isometry for the Carathéodory-Kobayashi metric of \( B \) [21]. This isometry is a holomorphic automorphism of \( B \) if, and only if, \( A \) is invertible in \( \mathcal{L}(\mathcal{A} \oplus \mathbb{C}^2) \).

If \( \hat{A}(0) = 0 \), then \( q_1 - iq_2 = 0 \), and therefore \( q_1 = q_2 = 0 \) because \( q_1 \) and \( q_2 \) are real vectors; (5.9) reads now \( E \in \text{SO}(2) \), and (5.8), which now becomes \( R' E = 0 \), yields \( r_1 = r_2 = 0 \). Thus, by (5.7), \( M \) is a real linear isometry of \( \mathcal{A} \). Setting

\[
E = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}
\]  (5.13)

for some \( \alpha \in \mathbb{R} \), then

\[
\hat{A}(x) = e^{i\alpha}Mx \quad \forall x \in B. 
\]  (5.14)

As a consequence,

\[
\hat{A}(x) = x \quad \forall x \in B \iff A = \begin{pmatrix} e^{-i\alpha}I_{\mathcal{A}} & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}.
\]  (5.15)

Now, let \( T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{A} \oplus \mathbb{C}^2) \) be a strongly continuous semigroup such that \( T(t) \in \Lambda_0 \) for all \( t \geq 0 \). Setting

\[
\phi_t = T(t)
\]  (5.16)

for \( t \geq 0 \), one defines a continuous semiflow \( \phi : \mathbb{R}_+ \times B \to B \) of holomorphic isometries \( B \to B \).

If \( x_0 \in B \) is a periodic point of \( \phi \) with period \( \tau > 0 \), and if the hypotheses of Theorem 2.3 are satisfied, then

(i) \( \phi \) is the restriction to \( \mathbb{R}_+ \) of a continuous flow \( \mathbb{R} \times B \to B \), which will be denoted by the same symbol \( \phi \);
(ii) \( T \) is the restriction to \( \mathbb{R}_+ \) of a strongly continuous group \( \mathbb{R} \to \mathcal{L}(\mathcal{A} \oplus \mathbb{C}^2) \), which will be denoted by the same symbol \( T \);
(iii) (5.16) holds for all \( t \in \mathbb{R} \).

Since, \( \hat{T}(\tau)(x) = x \) for all \( x \in B \), by (5.15), there is some \( \alpha \in \mathbb{R} \) such that

\[
T(\tau) = F(\tau),
\]  (5.17)
where
\[
F(\tau) = \begin{pmatrix} e^{-i\alpha \tau} I_\mathcal{A} & 0 & 0 \\ 0 & \cos(\alpha \tau) & -\sin(\alpha \tau) \\ 0 & \sin(\alpha \tau) & \cos(\alpha \tau) \end{pmatrix}.
\] (5.18)

Thus,
\[
\sigma(T(\tau)) = \sigma(F(\tau)).
\] (5.19)

Setting
\[
L_- = \{ (\zeta, i\zeta) : \zeta \in \mathbb{C} \}, \quad L_+ = \{ (\zeta, -i\zeta) : \zeta \in \mathbb{C} \},
\] (5.20)
if \(\alpha \tau \not\in \pi \mathbb{Z}\), \(\sigma(T(\tau))\) consists of the eigenvalue \(e^{-i\alpha \tau}\), with the eigenspace \(\mathcal{A} \oplus L_- \subset \mathcal{A} \oplus \mathbb{C}^2\), and of the eigenvalue \(e^{i\alpha \tau}\), with the eigenspace \(0 \oplus L_+ \subset \mathcal{A} \oplus \mathbb{C}^2\).

If \(\alpha \tau \in \pi \mathbb{Z}\), \(T(\tau) = I_{\mathcal{A} \oplus \mathbb{C}^2}\) when \(\alpha \tau/\pi\) is even, and \(T(\tau) = -I_{\mathcal{A} \oplus \mathbb{C}^2}\) when \(\alpha \tau/\pi\) is odd.

In conclusion, the following theorem has been established.

**Theorem 5.1.** If there is a periodic point \(x_0 \in B\) for \(\phi\), with period \(\tau > 0\), and if there is a set \(K \subset (0, \tau)\) such that, for any \(t \in K\), \(M_{x_0}(\phi_t(x_0))\) is collinear to a multiple, by a constant factor of modulus one, of a selfadjoint unitary operator which acts on the Hilbert space \(\mathcal{H}\) and is contained in \(\mathcal{A}\), and the set \(\{\phi_t(x_0) : t \in K\}\) spans a dense affine subspace of \(\mathcal{A}\), then there exist a strongly continuous group \(T : \mathbb{R} \to \mathcal{L}(\mathcal{A} \oplus \mathbb{C}^2)\) and a real number \(\alpha\) for which (5.17) and (5.18) hold.

The infinitesimal generator
\[
X : \mathcal{D}(X) \subset \mathcal{A} \oplus \mathbb{C}^2 \longrightarrow \mathcal{A} \oplus \mathbb{C}^2
\] (5.21)
of the group \(T\) has a pure point spectrum, consisting of at least one and at most two distinct eigenvalues.

If \(\alpha \tau \not\in \pi \mathbb{Z}\), \(\sigma(T(\tau))\) consists of the eigenvalue \(e^{-i\alpha \tau}\), with the eigenspace \(\mathcal{A} \oplus L_-\), and of the eigenvalue \(e^{i\alpha \tau}\) with the one-dimensional eigenspace \(0 \oplus L_+\).

If \(\alpha \tau \in \pi \mathbb{Z}\), the group \(T\) is periodic with period \(\tau\) when \(\alpha \tau/\pi\) is even, and period \(2\tau\) when \(\alpha \tau/\pi\) is odd.

According to [18, Theorem 4.1], \(\mathcal{D}(X) = \mathcal{D} \oplus \mathbb{C}^2\), where \(\mathcal{D}\) is a dense linear subspace of \(\mathcal{A}\), and \(X\) is expressed by the matrix
\[
X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ \bullet | X_{12} & 0 & X_{23} \\ \bullet | X_{13} & -X_{23} & 0 \end{pmatrix},
\] (5.22)
where \(X_{23} \in \mathbb{R}\), \(X_{12}\) and \(X_{13}\) are real vectors in \(\mathcal{A}\), and \(X_{11}\) is a real, skew-selfadjoint operator on \(\mathcal{A}\) with domain \(\mathcal{D}\).
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Similar results to those established in Propositions 4.4 and 4.5 for (4.17) and (4.20) hold for the Riccati equation

\[
\frac{d}{dt} \phi_t(x_0) = (X_{11} + iX_{23}I) \phi_t(x_0) + \frac{1}{2} (X_{12} + iX_{13}) (\phi_t(x_0) | \overline{\phi_t(x_0)}) - (\phi_t(x_0) | X_{12} - iX_{13}) \phi_t(x_0) + \frac{1}{2} (X_{12} - iX_{13})
\]

(5.23)

with initial conditions \( \phi_0(x_0) = x_0 \in B \cap \mathcal{D}(X_{11}) \).

6. Fixed points of semiflows

The next sections will be devoted to investigating the fixed points of a continuous semiflow \( \phi : \mathbb{R}_+ \times D \to D \) of holomorphic maps of a bounded domain \( D \) in a complex Banach space \( \mathcal{E} \), that is to say, the points \( x \in D \) such that \( \phi_t(x) = x \) for all \( t \in \mathbb{R}_+ \).

Actually, some of the results we are going to establish hold under slightly weaker conditions. Namely, \( \phi \) will be a map of \( \mathbb{R}_+^* \times D \) into \( D \) satisfying (2.3) and (2.4) for all \( t, t_1, t_2 \in \mathbb{R}_+^* \) and such that the map \( t \mapsto \phi_t(y) \) is continuous on \( \mathbb{R}_+^* \) for all \( y \in \mathcal{E} \).

A set \( S \subset D \) is said to be completely interior to \( D \), in symbols \( S \subset D \) if \( \inf \{ \| x - y \| : x \in D, y \in \mathcal{E} \setminus D \} > 0 \).

Since

\[
\phi_{t+s} = \phi_t(\phi_s(D)) \subset \phi_t(D) \quad \forall t, s > 0,
\]

(6.1)

if

\[
\phi_t(D) \subset D,
\]

(6.2)

then

\[
\phi_r(D) \subset D \quad \forall r \geq t.
\]

(6.3)

Let \( \phi_{t_0}(D) \subset D \) for some \( t_0 > 0 \), and let \( t \geq t_0 \). By the Earle-Hamilton theorem (see [2] or, e.g., [5, Theorem V.5.2]), there is a unique point \( x_t \in D \) such that \( \phi_t(x_t) = x_t \). Hence \( x_t \) is the unique point in \( D \) such that

\[
\phi_{nt}(x_t) = x_t \quad \forall n = 1, 2, \ldots.
\]

(6.4)

Moreover, by the Earle-Hamilton theorem,

\[
\lim_{n \to +\infty} \phi_{nt}(x) = x_t \quad \forall x \in D.
\]

(6.5)
Let $p, q$ be positive integers, with $p \geq q$. There is a unique point $x_{(p/q)t} \in D$ such that

$$\phi_{(p/q)t}(x_{(p/q)t}) = x_{(p/q)t}. \quad (6.6)$$

Since

$$\phi_{n(p/q)t}(x_{(p/q)t}) = x_{(p/q)t} \quad (6.7)$$

for $n = 1, 2, \ldots$, choosing $n = mq$, $m = 1, 2, \ldots$ yields

$$\phi_{mpt}(x_{(p/q)t}) = x_{(p/q)t}. \quad (6.8)$$

Since, by (6.5),

$$\lim_{m \to +\infty} \phi_{mpt}(x_{(p/q)t}) = x_t, \quad (6.9)$$

then

$$x_{(p/q)t} = x_t \quad (6.10)$$

for all positive integers $p \geq q = 1, 2, \ldots$.

The continuity of $t \mapsto \phi_t(y)$ implies that

$$\phi_{rt}(x_t) = x_t \quad (6.11)$$

for all real numbers $r \geq 1$. Hence there is a point $x_0 \in D$ which is the unique fixed point of $\phi_t$ for every $t \geq t_0$.

Let $t_0 > 0$ and choose $s \in (0, t_0)$ and $t \geq t_0$. Then

$$\phi_s(x_0) = \phi_s(\phi_t(x_0)) = \phi_{t+s}(x_0) = x_0 \quad (6.12)$$

because $t + s > t_0$.

In conclusion, the first part of the following theorem has been established.

**Theorem 6.1.** Let $\phi : \mathbb{R}_+^* \times D \to D$ satisfy (2.3) and (2.4), and be such that $t \mapsto \phi_t(x)$ is continuous on $\mathbb{R}_+^*$ for all $x \in D$. If $D$ is bounded, and if $\phi_t(D) \subseteq D$ for some $t > 0$, there exists $x_0 \in D$ which is the unique fixed point of $\phi_s$ for every $s > 0$, and

$$\lim_{s \to +\infty} \phi_s(x) = x_0 \quad \forall x \in D. \quad (6.13)$$

**Proof.** Let $k_D$ be the Kobayashi distance in $D$. To complete the proof of the theorem note that, given $x \in D$ and $s > 0$, for every $\epsilon > 0$ there exists a positive
integer $n_0$ such that, whenever $n \geq n_0$,

$$k_D(x_0, \phi_{ns}(x)) < \epsilon.$$  \hfill (6.14)

If $n \geq n_0$ and $t > ns$,

$$k_D(x_0, \phi_t(x)) = k_D(x_0, \phi_{ns+t-ns}(x))$$
$$= k_D(\phi_{t-ns}(x_0), \phi_{t-ns}(\phi_{ns}(x)))$$
$$\leq k_D(x_0, \phi_{ns}(x)) < \epsilon. \quad \Box$$  \hfill (6.15)

**Corollary 6.2.** Under the hypotheses of Theorem 6.1, $x_0$ is the only $\omega$-stable point of $\phi$. (That means that, for every $\epsilon > 0$ and every $\tau > 0$, there is some $t \geq \tau$ for which $k_D(x_0, \phi_t(x_0)) < \epsilon$.)

**Theorem 6.3.** Let $D$ be bounded and let $\phi : \mathbb{R}_+^* \times D \to D$ satisfy the hypotheses of Theorem 6.1. If there exist a sequence $\{t_\nu\} \subset \mathbb{R}_+^*$ diverging to $+\infty$ and a map $g : D \to D$ such that $\lim_{t \to +\infty} \phi_t = g$ for the topology of local uniform convergence and if $g(D) \subseteq D$, then there exists a unique point $x_0 \in D$ such that $\phi_t(x_0) = x_0$ for all $t > 0$ and $\lim_{t \to +\infty} \phi_t(x) = x_0$ for all $x \in D$.

**Proof.** Since $g$ is holomorphic and $g(D) \subseteq D$, the Earle-Hamilton theorem implies that there is a unique point $x_0 \in D$ which is fixed by $g$.

If $\phi_t(y) = y$ for some $y \in D$ and some $t > 0$, then, if $s > t$,

$$\phi_s(y) = \phi_{s-t+t}(y) = \phi_{s-t}(\phi_t(y)) = \phi_{s-t}(y),$$  \hfill (6.16)

and therefore

$$\phi_t(\phi_s(y)) = \phi_t(\phi_{s-t}(y)) = \phi_t(y).$$  \hfill (6.17)

But then

$$g(y) = \lim_{y \to +\infty} \phi_t(y) = y,$$  \hfill (6.18)

and therefore $y = x_0$. Hence, either $\text{Fix} \phi_t = \emptyset$ for all $t > 0$, or $\text{Fix} \phi_t = \{x_0\}$ when $t \gg 0$.

Let $R > 0$ be such that

$$B(x_0, R) \subseteq D.$$  \hfill (6.19)

Since the Kobayashi distance $k_D$ and $\| \cdot \|$ are equivalent on $B(x_0, R)$, there exist real constants $c > b > 0$ such that

$$b \|x - y\| \leq k_D(x, y) \leq c \|x - y\| \quad \forall x, y \in B(x_0, R).$$  \hfill (6.20)
Let $r > 0$ be such that
\[ B_{k_D}(x_0, r) \subset B(x_0, R). \] (6.21)

For every $\epsilon > 0$, there is $\nu_0$ such that
\[ \nu \geq \nu_0 \implies \|\phi_t(\nu)(x) - g(x)\| < \epsilon \quad \forall x \in B(x_0, R) \] (6.22)
(because the sequence $\{\phi_t(\nu)\}$ converges to $g$ for the topology of local uniform convergence).

Since $g(D) \subseteq D$, there exists $a \in (0, 1)$ such that
\[ k_D(\phi_t(\nu)(x_0), x_0) \leq k_D(\phi_t(\nu)(x), g(x)) + k_D(g(x), x_0) \]
\[ \leq c\|\phi_t(\nu)(x) - g(x)\| + ak_D(x, x_0) \] (6.23)
\[ < c\epsilon + ar. \]

Let $\ell \in (a, 1)$ and $\epsilon$ be such that
\[ 0 < \epsilon < \frac{\ell - a}{c}r. \] (6.24)

Then
\[ c\epsilon + ar < (\ell - a)r + ar = \ell r, \] (6.25)
and therefore
\[ \phi_t(\nu)(B_{k_D}(x_0, r)) \subset B_{k_D}(x_0, \ell r) \quad \forall \nu \geq \nu_0. \] (6.26)

It turns out that
\[ B_{k_D}(x_0, \ell r) \subseteq B_{k_D}(x_0, r). \] (6.27)

Indeed, if $x \in B_{k_D}(x_0, \ell r)$ and $y \in B(x_0, R) \setminus B_{k_D}(x_0, r),$
\[ \|x - y\| \geq \frac{1}{c}k_D(x, y) \geq \frac{1}{c}(k_D(y, x_0) - k_D(x_0, x)) > \frac{1 - \ell}{c}r. \] (6.28)

As a consequence of (6.27),
\[ \phi_t(\nu)(B_{k_D}(x_0, r)) \subset B_{k_D}(x_0, r) \quad \forall \nu \geq \nu_0. \] (6.29)
Periodicity of holomorphic maps

If $t > t_{\nu_0}$,
\[
\phi_t(B_{k_0}(x_0, r)) = \phi_{t-t_{\nu_0}+t_{\nu_0}}(B_{k_0}(x_0, r)) = \phi_{t_{\nu_0}}(\phi_{t-t_{\nu_0}}(B_{k_0}(x_0, r))) \\
\subset \phi_{t_{\nu_0}}(B_{k_0}(x_0, r)) \subset B_{k_0}(x_0, r).
\]
(6.30)

Hence,
\[
\text{Fix} \phi_t = \{x_0\} \quad \forall \ t \geq t_{\nu_0}.
\]
(6.31)

Thus,
\[
\lim_{t \to +\infty} \phi_t(x) = x_0
\]
(6.32)
for all $x \in B_{k_0}(x_0, r)$. In particular,
\[
\lim_{\nu \to +\infty} \phi_{\nu t}(x) = x_0
\]
(6.33)
for all $x \in B_{k_0}(x_0, r)$. Hence, $g(x) = x_0$ on $B_{k_0}(x_0, r)$ and therefore also on $D$ (because the open set $D$ is connected and $g$ is holomorphic on $D$), and (6.32) holds for all $x \in D$. 

7. Convergence of iterates and its consequences

The following theorem was announced in [16] without proof.

Theorem 7.1. Let $D$ be a bounded domain in the complex Banach space $\mathcal{E}$, and let $f : D \to D$ be a holomorphic map fixing a point $x_0 \in D$. If the sequence $\{f^n\}$ of the iterates of $f$ converges for the topology of local uniform convergence on $D$, then either
\[
\sigma(df(x_0)) \subset \Delta \tag{7.1}
\]
or
\[
\sigma(df(x_0)) = \{1\} \cup (\Delta \cap \sigma(df(x_0))), \tag{7.2}
\]
and $1$ is an isolated point of $\sigma(df(x_0))$ at which the resolvent function $(\bullet I - df(x_0))^{-1}$ has a pole of order one.

Since $df^n(x_0) = (df(x_0))^n$ for $n = 0, 1, \ldots$, and $\{df^n(x_0)\}$ converges in the operator topology, Theorem 7.1 is a consequence of the following proposition, also announced in [16] without proof.

Proposition 7.2. Let $A$ and $P$ be elements of $\mathcal{L}(\mathcal{E})$. If
\[
\lim_{n \to +\infty} \|A^n - P\| = 0, \tag{7.3}
\]
there exists $k \in \mathbb{R}_+^*$, for which,

$$\|A^n\| \leq k \quad \forall n = 1, 2, \ldots,$$

and therefore the spectral radius of $A$ is

$$\rho(A) \leq 1. \quad (7.5)$$

If $\rho(A) < 1$, then $P = 0$. If $\rho(A) = 1$, then

$$\sigma(A) \cap \partial \Delta = \{1\}, \quad (7.6)$$

and 1 is an isolated point of $\sigma(A)$ which is a pole of order one of the resolvent function $(\lambda I - A)^{-1}$. Furthermore, $P$ is the projector associated to the spectral set $\{1\}$ in the spectral resolution of $A$.

Proof. For any integer $m \geq 0$,

$$A^m P = P A^m = P, \quad (7.7)$$

and therefore

$$P^2 = \lim_{m \to +\infty} A^m P = P, \quad (7.8)$$

that is, $P$ is an idempotent of $\mathcal{L}(\mathcal{E})$.

For $m = 1$, $(A - I)P = 0$, and this fact, together with (7.3), yields

$$\ker(A - I) = \text{Ran} P. \quad (7.9)$$

Thus, $P \neq 0$ if, and only if, 1 is an eigenvalue of $A$.

Since

$$\|\|A^n\| - \|P\|\| \leq \|A^n - P\|, \quad (7.10)$$

(7.3) implies (7.4), for a finite constant $k > 0$, and therefore implies (7.5) as well.

Recall that $\sigma(P) \subset \{0, 1\}$ and that $\sigma(P) = \{0\}$ if, and only if, $P = 0$, $\sigma(P) = \{1\}$ if, and only if, $P = I$. By the upper semicontinuity of the spectrum, for any open neighbourhood $U$ of $\sigma(P)$, there is an integer $n_0 \geq 0$ such that, whenever $n \geq n_0$, $\sigma(A^n) \subset U$, and therefore the image of $\sigma(A)$ by the map $\zeta \mapsto \zeta^n$ is contained in $U$. Hence,

$$P = 0 \implies \rho(A) < 1, \quad (7.11)$$

and if $1 \in \sigma(P)$, then (7.3) and the upper semicontinuity imply (7.6).
Choosing a neighbourhood $U$ of the pair $\{0, 1\}$ consisting of two mutually disjoint open discs $\Delta(0, r_1)$ and $\Delta(1, r_2)$ centered at the points 0 and 1, with radii $r_1 > 0$ and $r_2 > 0$, and using again the upper semicontinuity of the spectrum, we see that 1 is an isolated point of $\sigma(A)$ and

$$\sigma(A) = \{1\} \cup (\sigma(A) \cap \Delta). \quad (7.12)$$

What is left to prove is the final part of the proposition.

(a) It will be shown first that, for any open, relatively compact neighbourhood $U$ in $\mathbb{C}$ of $\{0, 1\}$ and for any compact set $K \subset \mathbb{C}$ such that $K \cap U = \emptyset$, there exist a constant $k_1 > 0$ and an integer $n_1 \geq 1$ such that

$$\sup \{\|(\zeta I - A^n)^{-1}\| : \zeta \in K, n \geq n_1\} \leq k_1. \quad (7.13)$$

Let now $r_1$ and $r_2$ be such that $0 < r_1 < r_1 + r_2 < 1$, so that

$$\overline{\Delta(0, r_1)} \cup \overline{\Delta(1, r_2)} \subset U. \quad (7.14)$$

There is $n_2 \geq n_1$ such that

$$\sigma(A^n) \cap \Delta \subset \Delta(0, r_1) \quad \forall n \geq n_2. \quad (7.15)$$

Given $n \geq n_2$, choose $r_3 \in (0, r_2)$ so small that the image by the map $\zeta \mapsto \zeta^n$ of $\overline{\Delta(1, r_3)}$ be contained in $\Delta(1, r_2)$. Then, for any $\zeta \in K$,

$$(\zeta I - A^n)^{-1} = \frac{1}{2\pi i} \left\{ \int_{|\tau|=r_1} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau \right\}$$

$$+ \int_{|\tau-1|=r_3} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau. \quad (7.16)$$

Let $d$ be the Euclidean distance in $\mathbb{C}$. If $\zeta \in K$, then $|\zeta| > r_1$ and, for any $|\tau| = r_1$,

$$|\zeta - \tau^n| \geq |\zeta| - |\tau|^n = |\zeta| - |\tau|^n$$

$$\geq |\zeta| - |\tau| \geq d(\zeta, \overline{\Delta(0, r_1)}) \geq d(K, U). \quad (7.17)$$

If $\tau \in \overline{\Delta(1, r_3)}$, then

$$|\zeta - \tau^n| \geq d(\zeta, \overline{\Delta(1, r_2)}) \geq d(K, U). \quad (7.18)$$
Thus, (7.16) yields
\[ \|(\zeta I - A^n)^{-1}\| \leq \frac{2}{d(K,U)} \sup \{ \|(\tau I - A)^{-1}\| : \tau \in U \} \] (7.19)
for all \( \zeta \in K \) and all \( n \geq n_1 \), proving thereby (7.13).

(b) Let
\[ k_2 = \sup \{ \|\zeta I - P\| : \zeta \in K \}. \] (7.20)

For \( \zeta \in K \),
\[ \|(\zeta I - A^n)^{-1} - (\zeta I - P)^{-1}\| = \|(\zeta I - A^n)^{-1} (\zeta I - P - (\zeta I - A^n)) (\zeta I - P)^{-1}\| \]
\[ = \|(\zeta I - A^n)^{-1} (A^n - P) (\zeta I - P)^{-1}\| \]
\[ \leq \|(\zeta I - A^n)^{-1}\| A^n P\| (\zeta I - P)^{-1}\| \]
\[ \leq k_1 k_2 \|A^n - P\|. \] (7.21)

In the following, \( K = \partial \Delta(1,r) \), and \( r \in (0,1) \) will be chosen in such a way that
\[ \overline{\Delta(1,r)} \cap \sigma(A) = \emptyset. \] (7.22)

Let
\[ (\zeta I - A^n)^{-1} = \sum_{\nu = -\infty}^{+\infty} (\zeta - 1)^\nu A^n \nu, \] (7.23)
with \( A^n \nu \in \mathcal{L}(\mathbb{C}) \), be the Laurent expansion of \( (\zeta I - A^n)^{-1} \) at 1.

Let \( P_\nu \in \mathcal{L}(\mathbb{C}) \) be the coefficient of \( (\zeta - 1)^\nu \) in the Laurent expansion of \( (\zeta I - P)^{-1} \) at 1.

Then, by (7.21), for \( \nu \geq 1 \),
\[ \|A^n \nu - P_\nu\| \leq \frac{1}{2\pi} \left\| \int_{|z| = r} (\zeta - 1)^{\nu-1} ((\zeta I - A^n)^{-1} - (\zeta I - P)^{-1}) \, d\zeta \right\| \]
\[ \leq \frac{1}{2\pi} \int_{|z| = r} |\zeta - 1|^{\nu-1} \|((\zeta I - A^n)^{-1} - (\zeta I - P)^{-1})\| \, d\zeta \]
\[ \leq r^{\nu-1} k_1 k_2 \|A^n - P\|, \] (7.24)

and therefore
\[ \lim_{n \to +\infty} \|A^n \nu - P_\nu\| = 0 \] (7.25)
for \( \nu = 1, 2, \ldots \). But, since

\[
\zeta I - P - 1 = \frac{1}{\zeta - 1} P + \frac{1}{\zeta} (I - P),
\]

(7.26)

\( P - 1 = P \) and \( P - \nu = 0 \) for \( \nu \geq 2 \). Hence,

\[
\lim_{n \to +\infty} \| A^n - P \| = 0,
\]

(7.27)

\[
\lim_{n \to +\infty} \| A^n - \nu \| = 0
\]

(7.28)

for \( \nu = 2, 3, \ldots \).

(c) Choose \( r_1 \) and \( r_2 \) in such a way that \( 0 < r_1 < r_1 + r_2 < 1 \), and \( \sigma(A) \cap \Delta \subset \Delta(0, r_1) \). For any \( n \geq 1 \), choose \( r_3 \) such that \( 0 < r_3 < r_2 \) and that the image of \( \Delta(1, r_3) \) by the map \( \zeta \mapsto \zeta^n \) be contained in \( \Delta(1, r_2) \).

For any \( \nu \geq 1 \), Dunford’s integral and Fubini’s theorem yield

\[
A^n - \nu = \frac{1}{(2\pi i)^2} \int_{|\zeta - 1| = r_2} (\zeta - 1)^{\nu - 1} \times \left\{ \int_{|\tau| = r_1} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau \\
+ \int_{|\tau| = r_3} \frac{1}{\zeta - \tau^n} (\tau I - A)^{-1} d\tau \right\} d\zeta
\]

(7.29)

\[
= \frac{1}{(2\pi i)^2} \left\{ \int_{|\tau| = r_1} \left( \int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta \right) (\tau I - A)^{-1} d\tau \\
+ \int_{|\tau| = r_3} \left( \int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta \right) (\tau I - A)^{-1} d\tau \right\}.
\]

For \( |\tau| = r_1 \), the function

\[
\zeta \mapsto \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n}
\]

(7.30)

is holomorphic in a neighbourhood of \( \overline{\Delta(1, r_2)} \). Hence, by the Cauchy integral theorem,

\[
\int_{|\zeta - 1| = r_2} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta = 0.
\]

(7.31)

On the other hand, the Cauchy integral formula yields

\[
\frac{1}{2\pi i} \int_{|\zeta - 1| = r_3} \frac{(\zeta - 1)^{\nu - 1}}{\zeta - \tau^n} d\zeta = (\tau^n - 1)^{\nu - 1}.
\]

(7.32)
Hence, for $\nu \geq 1$,

$$A^n_{-\nu} = \frac{1}{2\pi i} \int_{|\tau - 1| = r_3} (\tau^n - 1)^{-1}(\tau I - A)^{-1} d\tau = (A^n - I)^{-1}A_{-1}, \quad (7.33)$$

and (7.27) yields

$$A^n_{-1} = P \quad \text{for } n = 1, 2, \ldots. \quad (7.34)$$

Since

$$A_{-\nu} = (A - I)^{-\nu}P \quad \forall \nu = 1, 2, \ldots, \quad (7.35)$$

(7.9) yields $A_{-\nu} = 0$ for $\nu = 2, 3, \ldots.$ □

A part of Proposition 7.2 follows also from the following lemma.

**Lemma 7.3.** If (7.4) holds, if $\partial \Delta \cap \sigma(A) \ni e^{i\theta}$ for some $\theta \in \mathbb{R}$, and if $e^{i\theta}$ is an isolated point of $\sigma(A)$ which is a pole of the resolvent function $(\bullet I - A)^{-1}$, then $e^{i\theta}$ is a pole of order one.

**Proof.** There is no restriction in assuming $e^{i\theta} = 1$. If $n > 0$ is the order of the pole, the resolvent function is represented in a neighbourhood of 1 by the Laurent series

$$\zeta^{A_{n-1}} = \sum_{n=-\infty}^{\infty} (\zeta - 1)^n A_n, \quad (7.36)$$

and the range $\text{Ran}(A_{-1})$ of $A_{-1}$ is related to $\text{ker}(I - A)^m$ by

$$\text{Ran}(A_{-1}) = \text{ker}(I - A)^m \quad \text{for } m = n, n + 1, \ldots. \quad (7.37)$$

Being

$$\text{ker}(I - A) \subset \text{ker}(I - A)^2 \subset \cdots, \quad (7.38)$$

(7.37) holds for $m = 1$ if, and only if,

$$Ax = x \quad \forall x \in \text{Ran}(A_{-1}). \quad (7.39)$$

To see that this latter condition actually holds, assume that there is some $y \in \text{Ran}(A_{-1})$ such that $(A - I)y \neq 0$, and let $\lambda$ be a continuous linear form on $\mathcal{E}$ such that

$$\langle (A - I)y, \lambda \rangle \neq 0. \quad (7.40)$$
By (7.35), \((A - I)^n y = 0\), and therefore
\[
A^N y = (A - I + I)^N y = \sum_{p=0}^{N} \binom{N}{p} (A - I)^p y = \sum_{p=0}^{n-1} \binom{N}{p} (A - I)^p y \quad (7.41)
\]
for all \(N \geq n\). Thus
\[
\langle A^N y, \lambda \rangle = \sum_{p=0}^{n-1} \binom{N}{p} \langle (A - I)^p y, \lambda \rangle, \quad (7.42)
\]
and therefore
\[
\lim_{N \to +\infty} \| \langle A^N y, \lambda \rangle \| = \infty, \quad (7.43)
\]
contradicting the fact that, in view of (7.4),
\[
\| \langle A^N y, \lambda \rangle \| \leq \| \lambda \| \| A^N \| \| y \| \leq k \| \lambda \| \| y \| \quad (7.44)
\]
for all \(N > 0\).

Thus (7.39) holds, and (7.35) yields \(A_\nu = 0\) for \(\nu = 2, 3, \ldots\) \(\square\)

If the hypotheses of Lemma 7.3 are satisfied with \(e^{i\theta} = 1\), \(\sigma(A)\) splits as the union of the two disjoint spectral sets \(\{1\}\) and \(\sigma(A) \cap \Delta\). The corresponding spectral projectors are \(P = A_{-1}\) and \(I - P\); moreover, \((A - I)P = 0\).

Setting
\[
C = A(I - P) = A - P, \quad (7.45)
\]
then \(\sigma(C) = (\sigma(A) \cap \Delta) \cup \{0\}\).

Since \(CP = PC\), then
\[
A^n = P + C^n \quad \text{for } n = 1, 2, \ldots \quad (7.46)
\]

Being \(\rho(C) < 1\), there exist \(\epsilon \in (0, 1)\) and \(n_0 \geq 1\) such that
\[
\|C^n\|^{1/n} \leq 1 - \epsilon, \quad (7.47)
\]
that is,
\[
\|C^n\| \leq (1 - \epsilon)^n \quad \forall n \geq n_0, \quad (7.48)
\]
and therefore, by (7.46), (7.3) holds.
In conclusion, the following proposition has been established.

**Proposition 7.4.** If (7.4) and (7.6) hold and if 1 is an isolated point of \( \sigma(A) \) which is also a pole of the resolvent function \((\bullet I - A)^{-1}\), then (7.3) holds, where \( P \) is the spectral projector associated to the spectral set \( \{1\} \) in the spectral resolution of \( A \).

It will be shown in **Section 8** that, if (7.1) holds, **Theorem 7.1** can be inverted.

**8. Sufficient conditions for the convergence of iterates**

Let \( D \) be a bounded domain in the complex Banach space \( \mathcal{E} \), and let \( f : D \to D \) be a holomorphic map fixing a point \( x_0 \in D \). As was noticed already, since \( D \) is bounded, \( \sigma(df(x_0)) \subset \Delta \) (see [5]).

**Theorem 8.1.** If \( \sigma(df(x_0)) \subset \Delta \), the sequence \( \{f^n\} \) of the iterates of \( f \) converges to the constant map \( x \mapsto x_0 \) for the topology of local uniform convergence on \( D \).

Obviously, there is no restriction in assuming \( D \) to be a bounded, connected, open neighbourhood of \( x_0 = 0 \).

Let \( R > 0 \) be such that

\[
D \subset B(0, R). \tag{8.1}
\]

Let

\[
f(x) = Ax + A_2(x, x) + \cdots + A_N(x, \ldots, x) + \cdots \tag{8.2}
\]

be the power series expansion of \( f \) in 0, where \( A \in \mathcal{L}(\mathcal{E}) \) and \( A_N \) is a continuous, homogeneous, polynomial of degree \( N = 2, 3, \ldots \) on \( \mathcal{E} \), with values in \( \mathcal{E} \), that is, the restriction to the diagonal of \( \mathcal{E} \times \cdots \times \mathcal{E} \) (\( n \) times) of a continuous \( N \)-linear symmetric map, which will be denoted by the same symbol \( A_N \), of \( \mathcal{E} \times \cdots \times \mathcal{E} \) into \( \mathcal{E} \). If

\[
r = \inf \{ \|y\| : y \notin D \}, \tag{8.3}
\]

the power series (8.2) converges uniformly on \( \overline{B(0, s)} \) whenever \( 0 < s < r \).

The \( n \)th iterate \( f^n \) \((n = 2, 3, \ldots)\) of \( f \) has a power series expansion in 0 which converges uniformly on \( B(0, s) \) and is expressed by

\[
f^n(x) = A^n x + C^{(n)}_2(x, x) + \cdots + C^{(n)}_N(x, \ldots, x) + \cdots, \tag{8.4}
\]

where \( C^{(n)}_N \) is a continuous homogeneous polynomial of degree \( N = 2, 3, \ldots \) on \( \mathcal{E} \) with values in \( \mathcal{E} \).
An induction argument on \( n \) will show now that, for all \( x \in \mathcal{E} \), \( N = 2, 3, \ldots \) and \( n = 2, 3, \ldots \),

\[
C^{(n)}_N(x, \ldots, x) = \sum_{q=0}^{n-1} A^q \left( A_N \left( A^{n-q-1}x, \ldots, A^{n-q-1}x \right) \right) + \sum_{m=1}^{n-1} \sum_{q=2}^{N-1} \sum_{(q,N)} C^{(m)}_q \left( A_{p_1} \left( A^{n-m-1}x, \ldots, A^{n-m-1}x \right), \ldots, A_{p_q} \left( A^{n-m-1}x, \ldots, A^{n-m-1}x \right) \right),
\]

(8.5)

where \( x \in \mathcal{E} \), \( C^{(1)}_q = A_q \), and the sum \( \sum_{(q,N)} \) is extended to all positive integers \( p_1, \ldots, p_q \) such that \( p_1 + \cdots + p_q = N \).

First of all, a simple induction on \( n \) yields

\[
C^{(n)}_2(x, x) = \sum_{q=0}^{n-1} A^q \left( A_2 \left( A^{n-q-1}x, A^{n-q-1}x \right) \right),
\]

(8.6)

which coincides with (8.5) when \( N = 2 \).

Assuming (8.5) to hold, then

\[
C^{(n+1)}_N(x, \ldots, x) = A^n \left( A_N(x, \ldots, x) \right) + \sum_{q=2}^{N} \sum_{(q,N)} C^{(n)}_q \left( A_{p_1} (x, \ldots, x), \ldots, A_{p_q} (x, \ldots, x) \right)
\]

\[
= A^n \left( A_N(x, \ldots, x) \right) + C^{(n)}_N(Ax, \ldots, Ax)
\]

\[
+ \sum_{q=2}^{N-1} \sum_{(q,N)} C^{(n)}_q \left( A_{p_1} (x, \ldots, x), \ldots, A_{p_q} (x, \ldots, x) \right)
\]

\[
= A^n \left( A_N(x, \ldots, x) \right) + \sum_{q=0}^{n-1} A^q \left( A_N \left( A^{n-q-1}Ax, \ldots, A^{n-q-1}Ax \right) \right)
\]

\[
+ \sum_{m=1}^{n-1} \sum_{q=2}^{N-1} \sum_{(q,N)} C^{(m)}_q \left( A_{p_1} \left( A^{n+1-m-1}x, \ldots, A^{n+1-m-1}x \right), \ldots, A_{p_q} \left( A^{n+1-m-1}x, \ldots, A^{n+1-m-1}x \right) \right)
\]

\[
+ \sum_{q=2}^{N-1} \sum_{(q,N)} C^{(n)}_q \left( A_{p_1} (x, \ldots, x), \ldots, A_{p_q} (x, \ldots, x) \right)
\]

\[
= \sum_{q=0}^{n+1-1} A^q \left( A_N \left( A^{n+1-q-1}Ax, \ldots, A^{n+1-q-1}Ax \right) \right)
\]

\[
+ \sum_{m=1}^{n+1-1} \sum_{q=2}^{N-1} \sum_{(q,N)} C^{(m)}_q \left( A_{p_1} \left( A^{n+1-m-1}x, \ldots, A^{n+1-m-1}x \right), \ldots, A_{p_q} \left( A^{n+1-m-1}x, \ldots, A^{n+1-m-1}x \right) \right).
\]

(8.7)
This inductive argument shows that (8.5) holds for $N = 2, 3, \ldots$ and $n = 2, 3, \ldots$.

**Lemma 8.2.** If $\|A\| < 1$, for $N = 2, 3, \ldots$, there is a positive constant $c_N$ such that

$$
\|C_N^{(n)}\| \leq c_N \|A\|^{n-N+1} \quad \forall n \geq N - 1.
$$

(8.8)

Here, $\|C_N^{(n)}\|$ is the norm of the continuous polynomial $x \mapsto C_N^{(n)}(x, \ldots, x)$

$$
\|C_N^{(n)}\| = \sup \{\|C_N^{(n)}(x, \ldots, x)\| : \|x\| \leq 1\},
$$

(8.9)

and is related to the norm

$$
\|\|C_N^{(n)}\|\| = \sup \{\|C_N^{(n)}(x, \ldots, y)\| : \|x\| \leq 1, \ldots, \|y\| \leq 1\}
$$

(8.10)

of the continuous, symmetric $N$-linear map $(x, \ldots, y) \mapsto C_N^{(n)}(x, \ldots, y)$ by the inequalities (see, e.g., [5])

$$
\|C_N^{(n)}\| \leq \|\|C_N^{(n)}\|\| \leq \frac{N^N}{N!} \|C_N^{(n)}\|.
$$

(8.11)

**Proof of Lemma 8.2.** By (8.5),

$$
C_2^{(n)}(x, x) = \sum_{q=0}^{n-1} A^q (A_2 (A^{n-q-1} x, A^{n-q-1} x)),
$$

(8.12)

and therefore

$$
\|C_2^{(n)}(x, x)\| \leq \|A_2\| \sum_{q=0}^{n-1} \|A\|^{2n-2q-2+q} \|x\|^2
$$

$$
= \|A_2\| \|A\|^{n-1} \sum_{q=0}^{n-1} \|A\|^{n-q+1} \|x\|^2
$$

$$
= \|A_2\| \|A\|^{n-1} \frac{1 - \|A\|^n}{1 - \|A\|} \|x\|^2
$$

$$
\leq \|A_2\| \|A\|^{n-1} \frac{\|x\|^2}{1 - \|A\|}.
$$

(8.13)
Assuming the lemma to hold for \( q = 2, 3, \ldots, N - 1 \), and choosing \( n \geq N - 1 \), then
\[
\left\| C_N^{(n)}(x, \ldots, x) \right\| \leq \left\{ \left\| A_N \right\| \left\| A \right\|^n \left\| A \right\|^{N(n-1)} \right. \\
+ \sum_{m=1}^{n-1} \sum_{q=2}^{N-1} \frac{q^q}{q!} \left( \left\| A \right\|^{m-q+1} \left\| A_{p_1} \right\| \cdots \left\| A_{p_q} \right\| \left\| A \right\|^{N(m-1)} \right) \right\} \left\| x \right\|^{N(n-1)}
\]

In view of (8.1), the Cauchy inequalities yield
\[
\left\| C_N^{(n)}(x, \ldots, x) \right\| \leq c_N \left\| A \right\|^{n-N+1} \left\| x \right\|, \quad (8.16)
\]
with
\[
c_N = \left\| A_N \right\| + \sum_{q=2}^{N-1} \left( \frac{q^q}{q!} c_q \sum_{(q,N)} \left\| A_{p_1} \right\| \cdots \left\| A_{p_q} \right\| \right) \frac{1}{1 - \left\| A \right\|^{N-1}}. \quad (8.17)
\]

In view of (8.1), the Cauchy inequalities yield
\[
\left\| C_N^{(n)} \right\| \leq \frac{R}{r^n} \quad \forall N \geq 1, \; n \geq 1. \quad (8.18)
\]
Hence, if $s \in (0, 1)$ is sufficiently small, in such a way that $B(0, s) \subset D$, and if $x \in B(0, s/2)$, $n \geq 1$, and $N_0 \geq 2$,

$$
\|f^n(x)\| \leq \|A^n x\| + \|C_2^{(n)}(x, x)\| + \cdots + \|C_{N_0}^{(n)}(x, \ldots, x)\| + R \sum_{N=N_0+1}^{+\infty} \left( \frac{\|x\|}{s} \right)^N \\
\leq \|A^n x\| + \|C_2^{(n)}(x, x)\| + \cdots + \|C_{N_0}^{(n)}(x, \ldots, x)\| \\
+ R \left( \frac{\|x\|}{s} \right)^{N_0+1} \frac{1}{1 - \|x\|/s} \\
\leq \|A\|^n \|x\| + c_2 \|A\|^{n-1} \|x\|^2 + \cdots + c_{N_0} \|A\|^{n-N_0+1} \|x\|^{N_0} + \frac{R}{2^{N_0}}.
$$

(8.19)

Let $c = \max\{1, c_2, \ldots, c_{N_0}\}$. Then

$$
\|f^n(x)\| \leq \|A\|^{n-N_0+1} (\|A\|^{N_0-1} + \|A\|^{N_0-2} + \cdots + 1)s + \frac{R}{2^{N_0}}.
$$

(8.20)

For $\epsilon > 0$, choosing $N_0 \gg 0$ and $n_0 \gg 0$ in such a way that

$$
\frac{Rr^{N_0+1}}{1-r} < \frac{\epsilon}{2}, \quad c \left( \frac{\|A\|^{n-N_0+1}}{1 - \|A\|^{-r}} \right) < \frac{\epsilon}{2} \quad \forall n \geq n_0,
$$

(8.21)

then

$$
\|f^n(x)\| < \epsilon \quad \forall x \in B\left(0, \frac{s}{2}\right), \quad \forall n \geq n_0.
$$

(8.22)

That proves the following lemma.

**Lemma 8.3.** If $\|A\| < 1$, for any $\epsilon > 0$ and any $s \in (0, 1)$ such that $B(0, s) \subset D$, there is $n_0 \geq 1$ such that (8.22) holds.

**Proposition 8.4.** If $\sigma(A) \subset \Delta$, for any $\epsilon > 0$ and any $s \in (0, 1)$ such that $B(0, s) \subset D$, there is $n_0 \geq 1$ such that (8.22) holds.

**Proof.** There is $n_1 \geq 1$ such that $\|A^{n_1}\| < 1$. By **Lemma 8.3**, there is $n_2 \geq 1$ such that

$$
\|f^{n_1n}(x)\| < \epsilon \quad \forall x \in B\left(0, \frac{s}{2}\right), \quad n \geq n_2.
$$

(8.23)
Let $\omega$ be the Poincaré distance in $\Delta$. Since holomorphic maps contract the Kobayashi distance, for $m \geq 1$, $n \geq n_2$, and $x \in B(0,s/2)$, then

$$
\omega(0, \frac{\|f^{n_1n+m}(x)\|}{R}) = k_{B(0,R)}(0, f^{n_1n+m}(x)) \leq k_D(0, f^{n_1n+m}(x)) \\
\leq k_D(0, f^{n_1n}(x)) \leq k_{B(0,s)}(0, f^{n_1n}(x)) \\
= \omega(0, \frac{\|f^{n_1n}(x)\|}{s}) < \omega(0, \frac{\epsilon}{s}).
$$

(8.24)

Thus, the sequence $\{f^n\}$ converges to 0 uniformly on $B(0,s/2)$, and therefore converges to zero everywhere on $D$ by Vitali’s theorem [8, Theorem 3.18.1]. The convergence being uniform on $B(0,s/2)$, the sequence $\{f^n\}$ tends to zero for the topology of local uniform convergence on $D$ [5, page 104].

The proof of Theorem 8.1 is complete. □

As in Section 6, let $\phi$ be a map of $\mathbb{R}_+^* \times D$ into $D$ satisfying (2.3) and (2.4) for all $t, t_1, t_2 \in \mathbb{R}_+$, and such that the map $t \mapsto \phi_t(x)$ is continuous on $\mathbb{R}_+$ for all $x \in \mathcal{C}$.

Let $\phi_t(x_0) = x_0$ for all $t > 0$ and for some point $x_0$ in the bounded domain $D \subset \mathcal{C}$.

If $\sigma(d\phi_{t_0}(x_0)) \subset \Delta$ for some $t_0 > 0$, Theorem 8.1 applied to the function $f = \phi_{t_0}$, implies that, as $n \to +\infty$, the sequence $\{\phi_{nt_0} : n = 1,2,\ldots\}$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence.

Let $r > 0$ be such that

$$
B_{k_D}(x_0, r) \subset D.
$$

(8.25)

Since the distances $\|\|$ and $k_D$ are equivalent on $B_{k_D}(x_0, r)$, for any $\epsilon > 0$, there is $n_0 \geq 1$ such that

$$
\phi_{nt_0}(B_{k_D}(x_0, r)) \subset B_{k_D}(x_0, \epsilon),
$$

(8.26)

whenever $n \geq n_0$. For all $t > nt_0$,

$$
\phi_t(B_{k_D}(x_0, r)) = \phi_{t-nt_0+nt_0}(B_{k_D}(x_0, r)) = \phi_{t-nt_0}(\phi_{nt_0}(B_{k_D}(x_0, r))) \\
\subset \phi_{t-nt_0}(B_{k_D}(x_0, \epsilon)) \subset B_{k_D}(x_0, \epsilon)
$$

(8.27)

because holomorphic maps contract the Kobayashi distance.

Thus the following theorem holds.
Theorem 8.5. If \( \phi : \mathbb{R}_+^* \times D \to D \) fixes a point \( x_0 \in D \) of the bounded domain \( D \), and if \( \sigma(d(\phi_t)(x_0)) \subseteq \Delta \) for some \( t_0 > 0 \), then, as \( t \to +\infty \), \( \phi_t \) converges to the constant map \( x \to x_0 \) for the topology of local uniform convergence.

9. Fixed points and idempotents

As at the beginning of Section 8, let \( D \) be a bounded domain in \( \mathbb{C} \) and let \( f : D \to D \) be a holomorphic map fixing a point \( x_0 \in D \).

If \( f \) is an idempotent of the semigroup \( \text{Hol}(D) \), a direct inspection of the power series expansion of \( f \) at \( x_0 \) shows that \( df(x_0) \) is an idempotent of \( \mathcal{L}(\mathbb{C}) \).

In this section, we show that, if the geometry of \( D \) satisfies suitable conditions, the fact that \( df(x_0) \) is an idempotent of \( \mathcal{L}(\mathbb{C}) \) implies that the iterates of \( f \) converge for the topology of local uniform convergence to an idempotent of \( \text{Hol}(D) \).

As before, let \( D \) be a bounded, open, connected neighbourhood of 0, and let \( f(0) = 0 \). Let \( f \) be expressed in \( B(0,r) \) by the power series (8.2) (and \( r \) is given by (8.3)).

Let \( A = df(0) \) be an idempotent of \( \mathcal{L}(\mathbb{C}) \).

Since \( A^2 = A \), (8.12) reads, for \( n \geq 2 \),

\[
C_2^{(n)}(x,x) = AA_2(x,x) + A_2(Ax,Ax) + (n - 2)AA_2(Ax,Ax) \tag{9.1}
\]

for all \( x \in \mathbb{C} \). If \( AA_2(Ax,Ax) \neq 0 \), there are \( y \in \mathbb{C} \) and \( \lambda \in \mathbb{C}' \) (the topological dual of \( \mathbb{C} \)) such that

\[
\langle AA_2(Ay,Ay),\lambda \rangle \neq 0. \tag{9.2}
\]

The Cauchy inequalities (8.18) yield, for \( N = 2 \) and \( n = 1, 2, \ldots \),

\[
|\langle AA_2(y,y) + A_2(Ay,Ay) + (n - 2)AA_2(Ay,Ay),\lambda \rangle| \leq \frac{R}{r^2} ||y||^2 |\lambda| \tag{9.3}
\]

for all \( n = 2, 3, \ldots \), contradicting (9.2). Hence, \( AA_2(Ax,Ax) = 0 \) for all \( x \in \mathbb{C} \), and therefore

\[
C_2^{(n)}(x,x) = AA_2(x,x) + A_2(Ax,Ax) \tag{9.4}
\]

for all \( n = 2, 3, \ldots \), and all \( x \in \mathbb{C} \).

Thus, \( C_2^{(n)}(x,x) \) does not depend on \( n \geq 2 \). Proceeding by induction on \( N \), we show that \( C_N^{(n)}(x,...,x) \) is independent of \( n \geq N \) for all \( N \).

Assuming this fact to hold for \( C_2, \ldots, C_N \), then

\[
f^N(x) = Ax + C_2(x,x) + \cdots + C_N(x,...,x) + F_{N+1}(x,...,x) + \cdots \tag{9.5}
\]
Periodicity of holomorphic maps

for all \( x \in B(0, r) \), where \( F_{N+1} \) is a homogeneous, continuous polynomial of degree \( N + 1 \) from \( \mathcal{E} \) to \( \mathcal{E} \).

Then, setting \( A_1 = A \),

\[
\begin{align*}
f^{N+1}(x) &= Ax + \sum_{q=2}^{N} C_q(x, \ldots, x) + AA_{N+1}(x, \ldots, x) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(x, \ldots, x), \ldots, A_{p_q}(x, \ldots, x)) + F_{N+1}(Ax, \ldots, Ax) + \cdots, \\
f^{N+2}(x) &= Ax + \sum_{q=2}^{N} C_q(x, \ldots, x) + AA_{N+1}(x, \ldots, x) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(x, \ldots, x), \ldots, A_{p_q}(x, \ldots, x)) + F_{N+1}(Ax, \ldots, Ax) + AA_{N+1}(Ax, \ldots, Ax) + \cdots, \\
&\vdots \\
f^{N+\ell}(x) &= Ax + \sum_{q=2}^{N} C_q(x, \ldots, x) + AA_{N+1}(x, \ldots, x) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(x, \ldots, x), \ldots, A_{p_q}(x, \ldots, x)) + F_{N+1}(Ax, \ldots, Ax) + (\ell - 1) [AA_{N+1}(Ax, \ldots, Ax) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(Ax, \ldots, Ax), \ldots, A_{p_q}(Ax, \ldots, Ax))] + \cdots
\end{align*}
\]

for all \( x \in B(0, r) \) and all \( \ell = 2, 3, \ldots \).

A similar argument to that devised for \( C_2 \) implies that

\[
AA_{N+1}(Ax, \ldots, Ax) + \sum_{q=2}^{N} \sum_{(q,N)} C_q(A_{p_1}(Ax, \ldots, Ax), \ldots, A_{p_q}(Ax, \ldots, Ax)) = 0
\]

for all \( x \in \mathcal{E} \).
The inductive argument is now complete, showing that

\[ f^n(x) = Ax + C_2(x, x) + \cdots + C_N(x, \ldots, x) + O(\|x\|^{N+1}) \]  

(9.8)

for all \( x \in B(0, r) \) and all \( n \geq N = 1, 2, \ldots \), with

\[ C_{N+1}(x, \ldots, x) = AA_{N+1}(x, \ldots, x) \]

\[ + \sum_{q=2}^{N} \sum_{(q, N)} C_q(A_{p_1}(x, \ldots, x), \ldots, A_{p_q}(x, \ldots, x)) + F_{N+1}(Ax, \ldots, Ax). \]

(9.9)

Since, by the Cauchy inequalities,

\[ \| (d^N f^n)(0) \| \leq \frac{R}{r^N} N! \]  

(9.10)

for all \( N \geq 0, n > 0 \), and therefore

\[ \limsup_N \left\| \frac{1}{n!} (d^N f^n)(0) \right\|^{1/N} \leq \frac{1}{r}, \]  

(9.11)

the Cauchy-Hadamard formula implies that the power series

\[ Ax + \sum_{N=2}^{+\infty} B_N(x, \ldots, x) \]  

(9.12)

converges uniformly on \( B(0, s) \) whenever \( 0 < s < r \). Let \( g \) be the holomorphic function on \( B(0, r) \) represented by this power series.

By the Cauchy inequalities, if \( \|x\| \leq s < r \),

\[ \|g(x) - f^n(x)\| \leq \sum_{N=n+1}^{+\infty} \|C_N(x, \ldots, x) - C_N^{(n)}(x, \ldots, x)\| \]

\[ \leq \sum_{N=n+1}^{+\infty} (\|C_N(x, \ldots, x)\| + \|C_N^{(n)}(x, \ldots, x)\|) \]

\[ \leq \sum_{N=n+1}^{+\infty} \left( \|C_N\| + \|C_N^{(n)}\| \right) \|x\|^N \]

\[ \leq 2R \sum_{N=n+1}^{+\infty} \left( \frac{\|x\|}{r} \right)^N \]

\[ \leq 2R \sum_{N=n+1}^{+\infty} \left( \frac{s}{r} \right)^N \]

\[ = 2R \left( \frac{s}{r} \right)^{N+1} \frac{1}{1 - s/r}. \]  

(9.13)
Hence, the sequence \( \{ f^n \} \) converges to \( g \) uniformly on \( B(0,s) \). By Vitali’s theorem [8, Theorem 3.18.1], the sequence \( \{ f^n(x) \} \) converges for all \( x \in D \), and the limit is a holomorphic map \( h : D \to \mathcal{E} \). Clearly, \( h|_{B(0,r)} = g \).

The convergence being uniform on \( B(0,s) \), the sequence \( \{ f^n \} \) tends to \( h \) for the topology of local uniform convergence.

In conclusion, the following theorem has been established.

**Theorem 9.1.** Let \( f \) be a holomorphic map of a bounded domain \( D \) into itself. If \( f \) fixes a point \( x_0 \in D \), and if \( df(x_0) \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \), the sequence \( \{ f^n \} \) converges for the topology of local uniform convergence to a holomorphic map \( h : D \to \mathcal{E} \).

Obviously, \( h(D) \subset \overline{D} \), \( h(x_0) = x_0 \),

\[
    dh(x_0) = df(x_0),
\]

and \( h \circ f = h \). Furthermore,

\[
    f \circ h = h,
\]

and therefore \( \text{Fix } f = h(D) \), provided that \( h(D) \subset D \). This latter condition is fulfilled if \( D \) satisfies the following principle.

**Maximum principle.** Whenever a holomorphic function \( h : D \to \mathcal{E} \) is such that \( h(D) \subset \overline{D} \) and \( h(D) \cap \partial D \neq \emptyset \), then \( h(D) \subset \partial D \).

**Example 9.2.** If the bounded domain \( D \) is convex, its support function is plurisubharmonic [14]. Thus, \( D \) satisfies the maximum principle.

Summing up, the following proposition holds.

**Proposition 9.3.** Under the hypotheses of Theorem 9.1, and if moreover \( D \) satisfies the maximum principle, \( h \) is an idempotent of the semigroup of all holomorphic maps of \( D \) into \( D \) which commute with \( f \) and is such that \( h(D) = \text{Fix } f \).

If \( df(x_0) \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \), then

\[
    \sigma(df(x_0)) = p\sigma(df(x_0)) \subset \{0,1\},
\]

\[
    \sigma(df(x_0)) = \{0\} \implies df(x_0) = 0,
\]

\[
    \sigma(df(x_0)) = \{1\} \implies df(x_0) = I.
\]

Since \( D \) is bounded, by Cartan’s identity theorem, (9.18) holds if, and only if, \( f = \text{id} \).

Theorem 8.1 and (9.17) yield the following proposition.
Proposition 9.4. If $D$ is bounded, if $f(x_0) = x_0$, and if $df(x_0)$ is an idempotent of $\mathcal{L}(\mathbb{C})$ with $\sigma(df(x_0)) = \{0\}$, then the sequence $\{f^n\}$ converges to the constant map $x \mapsto x_0$ for the topology of local uniform convergence on $D$.

Theorem 9.5 [16]. Let $D$ be a bounded, open, convex neighbourhood of 0, and let $f \in \text{Hol}(D)$ be such that $f(0) = 0$ and $df(0)$ is an idempotent of $\mathcal{L}(\mathbb{C})$. If $\partial D \cap \text{Ran} df(0)$ consists of complex extreme points of $D$, then $h(D) = D \cap \text{Ran} df(0)$.

Proof. Let $A = df(0)$ and $\mathcal{F} = \ker(I - A) = \text{Ran} A$. As a consequence of the strong maximum principle [15, Corollary 5.4], if $x \in \mathcal{F} \cap D$, $f(x) = Ax = x$, and with the same notations of (8.2),

$$A_2(x,x) = A_3(x,x,x) = \cdots = 0 \quad \forall x \in \mathcal{F}. \quad (9.19)$$

Therefore,

$$A_2(Ax, Ax) = A_3(Ax, Ax, Ax) = \cdots = 0 \quad \forall x \in \mathbb{C}. \quad (9.20)$$

Thus, by (9.4),

$$C_2(x,x) = AA_2(x,x) \quad \forall x \in \mathbb{C}. \quad (9.21)$$

Similarly, for any $N = 2, 3, \ldots$, if $x \in \mathcal{F} \cap D$, then $f^N(x) = Ax = x$, and

$$C_2(Ax, Ax) = \cdots = C_N(Ax, \ldots, Ax) = F_{N+1}(Ax, \ldots, Ax) = 0 \quad \forall x \in \mathbb{C}. \quad (9.22)$$

Assuming that there are continuous polynomials $x \mapsto \tilde{C}_2(x,x), \ldots, x \mapsto \tilde{C}_N(x,\ldots, x)$ such that $C_2 = A\tilde{C}_2, \ldots, C_N = A\tilde{C}_N$, (9.9) yields

$$C_{N+1} = A\tilde{C}_{N+1} \quad (9.23)$$

with

$$\tilde{C}_{N+1}(x,\ldots, x) = A_{N+1}(x,\ldots, x) + \sum_{q=2}^{N} \sum_{(q,N)} \tilde{C}_q(A_{p_1}(x,\ldots, x),\ldots, A_{p_q}(x,\ldots, x)). \quad (9.24)$$

This inductive argument shows that $h(B(0,r)) \subset \mathcal{F}$, and therefore $h(D) \subset \mathcal{F} \cap D$. Since, on the other hand, $\mathcal{F} \cap D \subset \text{Fix} f = h(D)$, the conclusion follows. \hfill \Box

10. Extensions to semiflows

In this section, we apply the results of Section 8 to the case in which $f$ is an element of a semiflow. Thus, let $x_0 \in D$ be a fixed point of a semiflow $\phi : \mathbb{R}_+ \times D \to D$ acting by holomorphic maps $\phi_t$ on a domain $D$ of $\mathbb{C}$. Denoting by $d\phi_t(x) \in \mathcal{L}(\mathbb{C})$ the Fréchet differential of $\phi_t$ at $x$, then

$$d\phi_{t_1+t_2}(x_0) = d\phi_{t_1}(x_0) d\phi_{t_2}(x_0) \quad \forall t_1, t_2 \in \mathbb{R}_+, \quad d\phi_0(x_0) = I. \quad (10.1)$$
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Lemma 10.1. If the semiflow \( \phi \) is continuous, the semigroup \( d\phi_\cdot(x_0) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E}) \) is strongly continuous.

If the domain \( D \) is bounded, the semigroup is uniformly bounded.

Proof. Choose \( r > 0 \) so small that \( B(x_0, r) \subset D \).

If \( \xi \in \mathcal{E} \), choose \( s > 0 \) in such a way that \( \phi_t(x_0 + \zeta \xi) \in B(x_0, r) \) whenever \( |\zeta| \leq s \) and for any \( t \) in a neighbourhood of 0 in \( \mathbb{R}_+ \).

If \( \lambda \in \mathcal{E}' \), the Cauchy integral formula yields

\[
\langle d\phi_t(x_0) \xi, \lambda \rangle = \frac{1}{2\pi i} \int_{\partial\Delta(0,s)} \frac{\langle \phi_t(x_0 + \zeta \xi), \lambda \rangle}{\zeta^2} d\zeta.
\] (10.2)

Since, for \( \zeta \in \partial\Delta(0,s) \),

\[
\left| \frac{\langle \phi_t(x_0 + \zeta \xi), \lambda \rangle}{\zeta^2} \right| \leq \frac{r\|\lambda\|}{s^2},
\] (10.3)

the dominated convergence theorem implies that

\[
\lim_{t \downarrow 0} \langle d\phi_t(x_0) \xi - \xi, \lambda \rangle = 0,
\] (10.4)

that is, the semigroup \( d\phi_\cdot(x_0) \) is weakly, hence strongly, continuous.

The uniform boundedness of the semigroup follows from the Cauchy inequalities. \( \square \)

Let \( Z : \mathbb{D}(Z) \subset \mathcal{E} \to \mathcal{E} \) be the infinitesimal generator of the strongly continuous semigroup \( d\phi_\cdot(x_0) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E}) \).

Let \( D \) be a bounded domain in \( \mathcal{E} \), and let \( \phi : \mathbb{R}_+ \times D \to D \) be a continuous semiflow of holomorphic maps of \( D \) into \( D \) fixing a point \( x_0 \in D \).

If \( \phi_{2t_0} = \phi_{t_0} \), for some \( t_0 > 0 \), then \( d\phi_{t_0} \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \).

If \( \sigma(d\phi_{t_0}(x_0)) = \{0\} \), (9.17) applied to \( f = \phi_{t_0} \) shows that the semigroup \( d\phi_\cdot(x_0) \) is nilpotent. Theorem 8.5 implies that, as \( t \to +\infty \), \( \phi_t \) converges to the constant map \( x \mapsto x_0 \) for the topology of local uniform convergence.

If \( \sigma(d\phi_{t_0}(x_0)) = \{1\} \), (9.18) applied to \( f = \phi_{t_0} \), coupled with Cartan’s identity theorem, implies that \( \phi_{t_0} = \text{id} \), and therefore \( \phi \) is the restriction to \( \mathbb{R}_+ \) of a continuous periodic flow with period \( t_0/p \) for some positive integer \( p \).

How many values of the semigroup \( d\phi_\cdot(x_0) \) can be idempotent in \( \mathcal{L}(\mathcal{E}) \)?

Clearly, if \( d\phi_{t_0}(x_0) \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \), then \( d\phi_{nt_0}(x_0) \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \) for \( n = 1, 2, \ldots \).

If \( d\phi_{t_0}(x_0) \) is an idempotent of \( \mathcal{L}(\mathcal{E}) \) for some \( t_0 > 0 \), and if \( 1 \in \sigma(d\phi_{t_0}(x_0)) \), then \( 2n\pi i/t_0 \in p\sigma(Z) \) for some \( n \in \mathbb{Z} \). Letting

\[
V := \left\{ n \in \mathbb{Z} : \frac{2n\pi i}{t_0} \in p\sigma(Z) \right\},
\] (10.5)
then \( V \neq \emptyset \),

\[
\sigma(Z) \setminus \{0\} = p\sigma(Z) \setminus \{0\} = \frac{2\pi i}{t_0} V,
\]

\[
\ker (I - d\phi_{t_0}(x_0)) = \bigvee_{n \in \mathbb{Z}} \ker \left( \frac{2\pi i}{t_0} I - Z \right). \tag{10.6}
\]

For any \( t > 0 \) and \( n \in V \)

\[
e^{2n\pi it/t_0} \in p\sigma(d\phi_t(x_0)). \tag{10.7}
\]

Hence, if \( d\phi_{t_1}(x_0) \) is an idempotent of \( \mathcal{L} (\mathcal{E}) \) for some \( t_1 > 0 \), for any \( n \in V \),

\[
e^{2n\pi it_1/t_0} = 1, \tag{10.8}
\]

that is, there is \( m \in \mathbb{Z} \) such that

\[
\frac{2n\pi it_1}{t_0} = 2\pi im, \tag{10.9}
\]

that is,

\[
nt_1 = mt_0. \tag{10.10}
\]

As a consequence, if \( t_1/t_0 \notin \mathbb{Q} \), then \( n = m = 0 \). Hence, \( V = \{0\} \), therefore

\[
p\sigma(d\phi_t(x_0)) = \{1\}, \tag{10.11}
\]

\[
\text{Ran } d\phi_t(x_0) = \ker (I - d\phi_t(x_0)) = \ker Z \quad \forall t \in \mathbb{R}_+. \tag{10.12}
\]

Thus, since \( d\phi_{t_0}(x_0) \) is an idempotent,

\[
\mathcal{E} = \ker (d\phi_{t_0}(x_0)) \oplus \ker Z. \tag{10.13}
\]

Let \( \Pi \) and \( \Lambda = I - \Pi \) be the projectors, with ranges \( \ker d\phi_t(x_0) \) and \( \ker Z \), associated to this direct sum decomposition of \( \mathcal{E} \).

Since, for any \( x \in \mathcal{E} \) and any \( t \geq t_0 \),

\[
d\phi_t(x_0)\Pi x = d\phi_{t-t_0}(x_0)(d\phi_{t_0}(x_0)\Pi x) = 0, \tag{10.14}
\]

then, by (10.12),

\[
d\phi_t(x_0)x = d\phi_t(x_0)\Lambda x = \Lambda x, \tag{10.15}
\]

and therefore

\[
d\phi_{2t}(x_0)x = d\phi_t(x_0)\Lambda x = \Lambda x = d\phi_t(x_0)x. \tag{10.16}
\]
Hence, if $d\phi_{t_0}(x_0)$ and $d\phi_{t_1}(x_0)$ are idempotents of $\mathcal{L}(\mathcal{E})$, and if $t_1/t_0 \notin \mathbb{Q}$, then

$$d\phi_t(x_0) = d\phi_{t_0}(x_0) \quad \forall t \geq \min\{t_0, t_1\}. \quad (10.17)$$

Let $0 < t < t_0$. If $x \in \ker d\phi_{t_0}(x_0)$ and $d\phi_t(x_0)x \neq 0$, then

$$\Lambda d\phi_t(x_0)x \in \ker \mathcal{Z}\setminus \{0\}, \quad (10.18)$$

and therefore

$$0 = d\phi_{t_0+t}(x_0)x = d\phi_{t_0}(x_0)(\Lambda d\phi_t(x_0)x) = \Lambda d\phi_t(x_0)x \neq 0. \quad (10.19)$$

This contradiction proves that if $x \in \ker d\phi_{t_0}(x_0)$, then $x \in d\phi_t(x_0)$ for all $t \in (0, t_0]$.

Summing up, if $1 \in \sigma(d\phi_{t_0}(x_0))$ and if $t_1/t_0 \notin \mathbb{Q}$, then $d\phi_t(x_0)$ is an idempotent of $\mathcal{L}(\mathcal{E})$ which is independent of $t > 0$. The strong continuity of the semigroup $d\phi_t(x_0)$ implies then that $d\phi_t(x_0) = I$ for all $t \geq 0$.

Since $D$ is a bounded domain, Cartan’s identity theorem yields the following theorem.

**Theorem 10.2.** If $d\phi_{t_0}(x_0)$ and $d\phi_{t_1}(x_0)$, with $t_1/t_0 \notin \mathbb{Q}$, are idempotents of $\mathcal{L}(\mathcal{E})$, and if $1 \in \sigma(d\phi_{t_0}(x_0))$, then $\phi_t = \text{id}$ for all $t \in \mathbb{R}_+$.

As in Section 4, and with the same notations, let $D$ be the open unit ball $B$ of the complex Hilbert space $\mathcal{H}$, and let $\phi$ be the periodic continuous semiflow, with period $\tau$, of holomorphic automorphisms of $B$, defined by the group $T$.

If $0 \in \text{Fix}\phi$, (4.8) shows that $\phi$ is (the restriction to $B$ of) a strongly continuous group of linear operators on $\mathcal{H},$

$$\phi_t = d\phi_t(0)|_B. \quad (10.20)$$

and $Z = X_{11} - iX_{22}I_\mathcal{H}$.

If $0 \in p\sigma(Z)$ and $x \in \ker Z\setminus \{0\}$, then

$$\phi_t(x) = d\phi_t(0)x = x \quad \forall t \in \mathbb{R}. \quad (10.21)$$

Vice versa, if $\phi_t(x) = x$ for some $x \in B\setminus \{0\}$ and all $t \in \mathbb{R}$, Bart’s theorem in [1] implies that $0 \in p\sigma(Z)$. That proves the following lemma.

**Lemma 10.3.** Let $0 \in \text{Fix}\phi$. Then $\{0\} = \text{Fix}\phi$ if, and only if, $0 \notin p\sigma(Z)$.

**References**


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