ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR PERIODIC PARABOLIC SUBLINEAR PROBLEMS

T. GODOY AND U. KAUFMANN

Received 26 May 2003

We give necessary and sufficient conditions for the existence of positive solutions for sublinear Dirichlet periodic parabolic problems \(Lu = g(x, t, u)\) in \(\Omega \times \mathbb{R}\) (where \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain) for a wide class of Carathéodory functions \(g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}\) satisfying some integrability and positivity conditions.

1. Introduction

Let \(\Omega\) be a smooth bounded domain in \(\mathbb{R}^N\), \(N \geq 2\). For \(T > 0\), \(1 \leq p \leq \infty\), and \(1 \leq q \leq \infty\), let \(L^p(L^q)\) be the Banach space of \(T\)-periodic functions \(f\) on \(\Omega \times \mathbb{R}\) (i.e., satisfying \(f(x, t) = f(x, t + T)\) a.e. \((x, t) \in \Omega \times \mathbb{R}\)) such that
\[
\|f\|_{L^p(L^q)} := \left\|\|f(\cdot, t)\|_{L^q(\Omega)}\right\|_{L^p(0, T)} < \infty.
\]
(1.1)

Similarly, let \(L^p_T\) be the Banach space of \(T\)-periodic functions \(f\) such that \(f|_{\Omega \times (0, T)} \in L^p(\Omega \times (0, T))\), equipped with the norm \(\|f\|_{L^p_T} := \|f|_{\Omega \times (0, T)}\|_{L^p(\Omega \times (0, T))}\). Finally, let \(C_T\) be the space of continuous and \(T\)-periodic functions on \(\Omega \times \mathbb{R}\) provided with the \(L^\infty\)-norm.

For the whole paper, we fix \(\nu, s \in (1, \infty]\) such that \(N/2\nu + 1/s < 1\), \(s > 2\). Let \(\{a_{ij}\}\) and \(\{b_j\}\), \(1 \leq i, j \leq N\), be two families of functions satisfying \(a_{ij}, b_j \in L^\infty_T\) and \(a_{ij} = a_{ji}\). Assume that \(\sum a_{ij}(x, t)\xi_i\xi_j \geq \alpha_0|\xi|^2\) for some \(\alpha_0 > 0\) and all \((x, t) \in \Omega \times \mathbb{R}, \xi \in \mathbb{R}^N\). Let \(A\) be the \(N \times N\) matrix whose \(i, j\) entry is \(a_{ij}\), let \(b = (b_1, \ldots, b_N)\), let \(0 \leq c_0 \in L^s(L^\nu)\), and let \(L\) be the parabolic operator given by
\[
Lu = u_t - \text{div}(A\nabla u) + \langle b, \nabla u \rangle + c_0u.
\]
(1.2)

Let \(W = \{u \in L^2((0, T), H^1_0(\Omega)) : u_t \in L^2((0, T), H^{-1}(\Omega))\}\). Given \(f \in L^1_{T,\text{loc}}(\Omega \times \mathbb{R})\), we say that \(u\) is a (weak) solution of the Dirichlet periodic problem \(Lu = f\).
Periodic parabolic sublinear problems

in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial \Omega \times \mathbb{R}$, if $u$ is $T$-periodic, $u_{|\Omega \times (0,T)} \in W$, and

$$\int_{\Omega \times (0,T)} \left[ - u \frac{\partial h}{\partial t} + A \nabla u \cdot \nabla h + \langle b, \nabla u \rangle h + c_0 u h \right] = \int_{\Omega \times (0,T)} fh$$  \hspace{1cm} (1.3)

for all $h \in C_c^\infty (\Omega \times \mathbb{R})$ (and so for all $h \in L^\infty_T$ such that $h_{|\Omega \times (0,T)} \in V_0$, where $V_0 := L^2((0,T), H^1_0(\Omega))$). For $u \in W$, the inequality $Lu \geq f$ (resp., $\leq$) will be understood in the same sense.

Let $\tilde{W} = \{ u \in L^2((0,T), H^1(\Omega)) : u_t \in L^2((0,T), H^{-1}(\Omega)) \}$. Following [6], we say that $v$ is a supersolution of the above problem if $v_{|\Omega \times (0,T)} \in \tilde{W}$, $v_t \in L^2((0,T), H^{-1}(\Omega)) + L^1+\eta(\Omega \times (0,T))$ for $\eta > 0$ small enough, $v_{|\partial \Omega \times (0,T)} \geq 0$, $v(\cdot,0) \geq v(\cdot, T)$ a.e. in $\Omega$, and

$$\int_{\Omega \times (0,T)} \left[ - v \frac{\partial h}{\partial t} + A \nabla v \cdot \nabla h + \langle b, \nabla v \rangle h + c_0 v h \right] \geq \int_{\Omega \times (0,T)} fh$$  \hspace{1cm} (1.4)

for all $0 \leq h \in C_c^\infty (\Omega \times (0,T))$ (and so for all $h \in L^\infty_T$ such that $h_{|\Omega \times (0,T)} \in V_0$ with $V_0$ as above). A subsolution is similarly defined by reversing the above inequalities.

Let $m \in L^s(L^v)$ and let

$$P(m) := \int_0^T \text{ess sup}_{x \in \Omega} m(x,t) dt$$  \hspace{1cm} (1.5)

(with the value “$+\infty$” allowed). For such $m$ (cf. [8, Theorem 3.6]), $P(m) > 0$ is necessary and sufficient for the existence of a positive principal eigenvalue for the periodic parabolic Dirichlet problem with weight function $m$ (i.e., an eigenvalue with a positive $T$-periodic eigenfunction associated to the problem $Lu = \lambda mu$ in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial \Omega \times \mathbb{R}$). Moreover, this positive principal eigenvalue denoted by $\lambda_1(L,m)$ (or $\lambda_1(m)$), if exists, is unique.

We are interested in the existence of positive solutions for the semilinear periodic parabolic problem

$$Lu = g(x,t,u) \quad \text{in } \Omega \times \mathbb{R},$$

$$u = 0 \quad \text{on } \partial \Omega \times \mathbb{R},$$

$$u_{|\partial \Omega \times (0,T)} \text{ $T$-periodic},$$

where $g$ is a given function on $\Omega \times \mathbb{R} \times [0,\infty)$.

In [9, Theorem 3.7], it is proved that

$$\lambda_1 \left( \sup_{\xi > 0} \frac{g(\cdot,\xi)}{\xi} \right) < 1 < \lambda_1 \left( \inf_{\xi > 0} \frac{g(\cdot,\xi)}{\xi} \right)$$  \hspace{1cm} (1.7)

is a necessary and sufficient condition for the existence of positive solutions in $C_T$ for (1.6) provided that $g$ satisfies $\xi \to g(x,t,\xi) \in C^1[0,\infty)$, $\xi \to g(x,t,\xi)/\xi$
nonincreasing in $(0, \infty)$, and some integrability and positivity conditions. In [10, Theorem 3.1], with the same monotonicity and regularity assumptions, and assuming also some integrability conditions, it is proved that if either $\inf_{\xi>0}(g(\cdot, \xi)/\xi) \in L^s(L^r)$ and $P(\inf_{\xi>0}(g(\cdot, \xi)/\xi)) \leq 0$ or $\inf_{\xi>0}(g(\cdot, \xi)/\xi) \leq 0$, then

$$\lambda_1 \left( \sup_{\xi>0} \frac{g(\cdot, \xi)}{\xi} \right) < 1$$

(1.8)

is necessary and sufficient for the existence of a positive solution $u \in C_T$ of (1.6).

Our aim in this paper is to prove, following a different approach, similar results without monotonicity and $C^1$-regularity assumptions on $g$ (see Theorems 3.1, 3.2, 3.3, and 3.4). Moreover, we will also cover some cases where $\lim_{\xi \to 0^+} (g(\cdot, \xi)/\xi) = \infty$. These theorems will be obtained using the well-known sub- and supersolutions method combined with some facts concerning linear problems with weight.

In order to relate our results to others in the literature, we mention that, for the case $\xi \to g(\cdot, \xi)/\xi$ nonincreasing, similar results to Theorem 3.1 for elliptic problems have been obtained, for example, in [4, 5, 13], assuming more regularity in the function $g$. In the periodic parabolic case, there are also well-known results if $\xi \to g(\cdot, \xi)/\xi$ is concave and Hölder-continuous, and $g(\cdot, 0) = 0$ (see [2, 3, 12] and the references therein).

On the other side, necessary and sufficient conditions for the existence of positive solutions for equations of type $Lu = a(x)u - b(x)u^p$, $p > 1$, $b \geq 0$ (logistic equation), are also known (see, e.g., [11, 12]). More general equations of the form $Lu = a(x)u - b(x)f(x, u)$, with $b \geq 0$ and $f$ superlinear, were studied, for example, in [7] for $f \in C^{1,1}_t(\Omega \times [0, \infty))$, $f$ strictly increasing, and $b > 0$, and, for the Laplacian, the case $f = f(u)$ is treated in [1] assuming $f \in C([0, \infty))$.

Theorem 3.2 generalizes the aforementioned results, while Theorems 3.3 and 3.4 also extend some well-known results, see, for example, [2, 3, 11, 12].

Some examples are also given at the end of the paper.

2. Preliminaries and auxiliary results

As usual, for $\xi \in [0, \infty)$ and $u : \Omega \times \mathbb{R} \to [0, \infty)$, we write $g(\xi)$ and $g(u)$ for the functions $(x, t) \to g(x, t, \xi)$ and $(x, t) \to g(x, t, u(x, t))$, $(x, t) \in \Omega \times \mathbb{R}$. We assume, from now on, that $g : \Omega \times \mathbb{R} \to [0, \infty)$ is a Carathéodory function (i.e., $(x, t) \to g(x, t, \xi)$ is measurable for all $\xi \in [0, \infty)$, and $\xi \to g(x, t, \xi)$ is continuous in $[0, \infty)$ a.e. $(x, t) \in \Omega \times \mathbb{R}$) such that $\sup_{\sigma \in \xi} g(\xi)/\sigma)$ and $\inf_{\sigma < \xi}(g(\sigma)/\sigma)$ are measurable functions for all $\xi > 0$, and $\inf_{\xi>0}(g(\xi)/\xi) \neq \sup_{\xi>0}(g(\xi)/\xi)$, that is, (1.6) is not a linear problem.

We start recalling some facts about periodic parabolic problems with weight.

Remark 2.1. (a) Let $D = \{ m \in L^s(L^r) : P(m) > 0 \}$. Then $D$ is open in $L^s(L^r)$ and the map $m \to \lambda_1(m)$ is continuous from $D$ into $\mathbb{R}$ (cf. [8, Theorem 3.9]). Also,
the following comparison principle holds: if \( m_1, m_2 \in L^s(L^r) \) and \( m_1 \leq m_2 \) in \( \Omega \times \mathbb{R} \), then \( \lambda_1(m_1) \geq \lambda_1(m_2) \); and if, in addition, \( m_1 < m_2 \) in a set of positive measure, then \( \lambda_1(m_1) > \lambda_1(m_2) \) (cf. [8, Remark 3.7]).

(b) For \( \lambda \in \mathbb{R} \) and \( m \in L^s(L^r) \), let \( \mu_m(\lambda) \) be defined as the unique \( \mu \in \mathbb{R} \) such that the Dirichlet periodic problem
\[
Lu = \lambda mu + \mu_m(\lambda)u
\]
in \( \Omega \times \mathbb{R} \) has a positive solution \( u \). We recall that \( \mu_m(\lambda) \) is well defined and that the map \( (\lambda, m) \to \mu_m(\lambda) \) is continuous from \( \mathbb{R} \times L^s(L^r) \) into \( \mathbb{R} \) (cf. [9, Proposition 2.7]). Moreover, \( \mu_m(0) > 0 \), \( \mu_m \) is concave and continuous, and a given \( \lambda \in \mathbb{R} \) is a principal eigenvalue associated to the weight \( m \) if and only if \( \mu_m(\lambda) = 0 \) (cf. [8, Lemma 3.2]). Also, if \( \lambda_1(m) \) exists, then for \( \lambda > 0 \), \( \mu_m(\lambda) > 0 \) if and only if \( \lambda < \lambda_1(m) \), and if \( \lambda_1(m) \) does not exist, \( \mu_m(\lambda) > 0 \) for all \( \lambda > 0 \).

(c) Let \( m \in L^s(L^r) \) such that \( P(m) > 0 \) and let \( m_j \) be a sequence such that \( m_j \) converges to \( m \) in \( L^s(L^r) \). Then it follows from [9, Remark 2.5] that \( P(m_j) > 0 \) for \( j \) large enough.

Remark 2.2. If \( u \in L^s_\infty \) is a positive solution of (1.6) and
\[
\inf_{0 < \xi \leq M} \left( \frac{g(\xi)}{\xi} \right) \in L^s(L^r),
\]
\[
\sup_{0 < \xi \leq M} \left( \frac{g(\xi)}{\xi} \right) \in L^s(L^r),
\]
for all \( M > 0 \), then \( u \in \mathcal{C}_\infty \) and \( u(x, t) > 0 \) for all \( (x, t) \in \Omega \times \mathbb{R} \). Indeed, this follows from [9, Remark 2.2 and Corollary 2.12].

We introduce some additional notation. For \( (x, t, \xi) \in \Omega \times \mathbb{R} \times (0, \infty) \), let
\[
\bar{g}(x, t, \xi) = \xi \sup_{0 < \xi \leq \sigma} \left( \frac{g(x, t, \sigma)}{\sigma} \right),
\]
\[
\underline{g}(x, t, \xi) = \xi \inf_{0 < \sigma \leq \xi} \left( \frac{g(x, t, \sigma)}{\sigma} \right)
\]
(with the values “\( \pm \infty \)” allowed). It is easy to check that if \( g(\xi) \) is finite for \( \xi \leq \xi_0 \), then \( \xi \to \underline{g}(\xi) \) is continuous in \( (0, \xi_0) \) a.e. in \( \Omega \times \mathbb{R} \), and that if \( \bar{g}(\xi) \) is finite for \( \xi_0 \leq \xi \), then \( \xi \to \bar{g}(\xi) \) is continuous in \( (\xi_0, \infty) \) a.e. in \( \Omega \times \mathbb{R} \). We also set
\[
\underline{m}_\infty(x, t) = \inf_{\xi > 0} \left( \frac{g(x, t, \xi)}{\xi} \right), \quad \bar{m}(x, t) = \sup_{\xi > 0} \left( \frac{g(x, t, \xi)}{\xi} \right),
\]
\[
\underline{m}_0(x, t) = \liminf_{\xi \to 0^+} \left( \frac{g(x, t, \xi)}{\xi} \right), \quad \bar{m}(x, t) = \limsup_{\xi \to \infty} \left( \frac{g(x, t, \xi)}{\xi} \right).
\]
Note that
\[
\begin{align*}
m_\infty &= \lim_{\xi \to \infty} \left( \frac{g(\xi)}{\xi} \right), \quad m_0 = \lim_{\xi \to 0^+} \left( \frac{g(\xi)}{\xi} \right), \\
m_0 &= \lim_{\xi \to 0^-} \left( \frac{g(\xi)}{\xi} \right), \quad m_\infty = \lim_{\xi \to \infty} \left( \frac{\overline{g}(\xi)}{\xi} \right). \tag{2.4}
\end{align*}
\]

**Lemma 2.3.** Let \( \xi_0 > 0 \). Assume that \( \overline{g}(\xi) \in L^s(L^r) \) for all \( \xi \geq \xi_0 \), and that either \( m_\infty \in L^s(L^r) \) with \( \lambda_1(m_\infty) > 1 \) if \( \lambda_1(m_\infty) \) exists or \( m_\infty \leq 0 \). Then, for all \( c > 0 \), there exists a supersolution \( w \in C_T \) of (1.6) such that \( w \geq c \).

**Proof.** We first study the case \( m_\infty \in L^s(L^r) \). Let \( c > 0 \). We claim that there exists \( \xi \geq c \) such that \( \mu_{\overline{g}(\xi)/\xi}(1) > 0 \). Indeed, for \( \xi \geq \xi_0 \), we have \( m_\infty \leq \overline{g}(\xi)/\xi \leq \overline{g}(\xi_0)/\xi_0 \) and also \( \lim_{\xi \to \infty} \overline{g}(\xi)/\xi = m_\infty \) with convergence a.e. Thus, by dominated convergence, \( \lim_{\xi \to \infty} \overline{g}(\xi)/\xi = m_\infty \) in \( L^s(L^r) \) and then Remark 2.1(b) implies \( \lim_{\xi \to \infty} \mu_{\overline{g}(\xi)/\xi}(\lambda) = \mu_{m_\infty}(\lambda) \) for all \( \lambda \). Moreover, either if \( P(m_\infty) > 0 \) and \( \lambda_1(m_\infty) > 1 \) or if \( P(m_\infty) \leq 0 \), the last statement in Remark 2.1(b) also gives \( \mu_{m_\infty}(1) > 0 \). Thus, it follows that \( \mu_{\overline{g}(\xi)/\xi}(1) > 0 \) for \( \xi \) large enough.

We fix \( \xi^* \geq \max(\xi_0, c) \) such that \( \mu_{\overline{g}(\xi^*)/\xi^*}(1) > 0 \). Let \( k \) be a function defined by \( k(x, t) = \sup_{\xi \geq \xi^*} |\overline{g}(\xi)/\xi| \). Since \( m_\infty \leq k \leq \overline{g}(\xi^*)/\xi^* \), we get \( k \in L^s(L^r) \). For \( \xi \in [0, \infty) \), let \( g^*(x, t, \xi) = \overline{g}(x, t, \xi) + k(x, t) \xi \). Then \( g^*(x, t, \xi) \geq 0 \) and \( \overline{g}(\xi)/\xi \in L^s(L^r) \) for \( \xi \geq \xi^* \). Also, \( \mu_{L + k \overline{g}(\xi^*)/\xi^*}(\lambda) = \mu_{L \overline{g}(\xi^*)/\xi^*}(\lambda) \) for all \( \lambda \). In particular, \( \mu_{L + k \overline{g}(\xi^*)/\xi^*}(1) = \mu_{L \overline{g}(\xi^*)/\xi^*}(1) > 0 \). Thus, Lemma 2.9 in [9] says that the Dirichlet periodic problem \( (L + k - g^*(\xi^*)/\xi^*) \Phi = g^*(\xi^*) \) in \( \Omega \times \mathbb{R} \) has a solution \( \Phi \in C_T \) satisfying \( \Phi(x, t) > 0 \) a.e. \( (x, t) \in \Omega \times \mathbb{R} \). Now,
\[
\begin{align*}
g(\xi^* + \Phi) &\leq \overline{g}(\xi^* + \Phi) \\
&\leq \frac{\overline{g}(\xi^*)}{\xi^*}(\xi^* + \Phi) \\
&\leq \overline{g}(\xi^*) + k\xi^* + \frac{\overline{g}(\xi^*)}{\xi^*}\Phi \\
&= g^*(\xi^*) + \frac{\overline{g}(\xi^*)}{\xi^*}\Phi - k\Phi \\
&= L\Phi \leq L(\xi^* + \Phi), \tag{2.5}
\end{align*}
\]
and therefore \( \xi^* + \Phi \) is a supersolution for (1.6).

Consider now the case \( m_\infty \leq 0 \). In this case, we have \( \lim_{\xi \to \infty} (\overline{g}^+(\xi)/\xi) = 0 \) a.e. in \( \Omega \times \mathbb{R} \), where, as usual, we write \( f = f^+ - f^- \). Also, \( 0 \leq \overline{g}^+(\xi)/\xi \leq \overline{g}^+(\xi_0)/\xi_0 \) for all \( \xi \geq \xi_0 \), and thus \( \lim_{\xi \to \infty} (\overline{g}^+(\xi)/\xi) = 0 \) in \( L^s(L^r) \). So, \( \lim_{\xi \to \infty} \mu_{\overline{g}^+(\xi)/\xi}(\lambda) = \lambda_1 \) for all \( \lambda \), where \( \lambda_1 \) is the (positive) principal eigenvalue for \( L \) associated to the weight 1 (because for \( m \equiv 1, \mu_m \equiv \lambda_1 \)). Thus, we can choose \( \xi^* \geq \max(\xi_0, c) \) such that \( \mu_{\overline{g}^+(\xi^*)/\xi^*}(1) > 0 \), and then, as above, the Dirichlet periodic problem \( (L - \overline{g}^+(\xi^*)/\xi^*) \Phi = \overline{g}^+(\xi^*) \) in \( \Omega \times \mathbb{R} \) has a solution \( \Phi \in C_T \) satisfying \( \Phi(x, t) > 0 \) a.e.
(x,t) in \( \Omega \times \mathbb{R} \). Also,

\[
g(\xi^* + \Phi) \leq \overline{g}^+(\xi^* + \Phi) \\
\leq \frac{\overline{g}^+(\xi^*)}{\xi^*}(\xi^* + \Phi) \\
= \overline{g}^+(\xi^*) + \frac{\overline{g}^+(\xi^*)}{\xi^*}\Phi \\
= L\Phi \leq L(\Phi + \xi^*),
\]

and this concludes the proof. \( \square \)

**Lemma 2.4.** Let \( \xi_0 > 0 \). Assume that \( g(\xi_0) \in L'(L^r), \ P(g(\xi_0)/\xi_0) > 0, \) and \( \lambda_1(g(\xi_0)/\xi_0) \leq 1 \). Then there exists a subsolution \( v \in C_T \) of (1.6) such that \( v(x,t) > 0 \) for all \( (x,t) \in \Omega \times \mathbb{R} \).

**Proof.** Let \( \Phi \) be the positive eigenfunction of

\[
\left( L + \frac{g^-(\xi_0)}{\xi_0} \right) \Phi = \lambda_1 \left( \frac{g^+(\xi_0)}{\xi_0} \right) \left( \frac{g^+(\xi_0)}{\xi_0} \right) \Phi \quad \text{in } \Omega \times \mathbb{R},
\Phi = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]

\( \Phi \) \( T \)-periodic.

Then \( \Phi \in C_T \) and \( \Phi(x,t) > 0 \) for all \( (x,t) \in \Omega \times \mathbb{R} \). Now, \( \lambda_1(L,g(\xi_0)/\xi_0) < 1 \) implies \( \mu_{L,g(\xi_0)/\xi_0}(1) \leq 0 \). Thus, since \( \mu_{L,g(\xi_0)/\xi_0}(1) = \mu_{L,g^-(\xi_0)/\xi_0,g^+(\xi_0)/\xi_0}(1) \), we get \( \lambda_1(g^+(\xi_0)/\xi_0) \leq 1 \).

Let \( \varepsilon > 0 \) be such that \( \varepsilon < \xi_0/\|\Phi\|_\infty \). Taking into account the above-mentioned facts and that \( \xi - g(\xi)/\xi \) is nonincreasing, we have

\[
L(\varepsilon\Phi) + g^-(\varepsilon\Phi) \leq \left( L + \frac{g^-(\varepsilon\|\Phi\|)}{\varepsilon\|\Phi\|} \right)\varepsilon\Phi \\
\leq \left( L + \frac{g^-(\xi_0)}{\xi_0} \right)\varepsilon\Phi \\
\leq \left( \frac{g^+(\xi_0)}{\xi_0} \right)\varepsilon\Phi \\
\leq \left( \frac{g^+(\varepsilon\|\Phi\|)}{\varepsilon\|\Phi\|} \right)\varepsilon\Phi \\
\leq g^+(\varepsilon\Phi),
\]

and the lemma follows. \( \square \)
3. The main results

**Theorem 3.1.** (a) Assume that

\[(1) \quad m_0, \overline{m} \in L^1(L^v), P(m_0) > 0, \text{ and } P(\overline{m}) > 0,\]

\[(2) \quad \overline{g}(\xi) \in L^1(L^v) \text{ for some } \xi_0 > 0 \text{ and } g(\xi_1) \in L^1(L^v) \text{ for some } \xi_1 > 0.\]

Then, if \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}) \), there exists a solution \( u \in L^\infty_T \) of (1.6) satisfying \( u(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R}\).

(b) Assume (1), \( \overline{m}_0 = m_0, \overline{m} = \overline{m}_\infty \), and that for all \( \xi > 0, \)

\[m_0 \neq \overline{g}(\xi), \quad (3.1)\]

\[\overline{m}_\infty \neq \overline{g}(\xi). \quad (3.2)\]

Then there exists a positive solution \( u \in L^\infty_T \) of (1.6) if and only if \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}_\infty) \).

**Proof.** Suppose that \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}) \). Since, for \( 0 < \xi \leq \xi_1 \), we have \( g(\xi_1)/\xi_1 \leq \frac{\overline{g}(\xi)}{\xi} \leq m_0 \) and \( \lim_{\xi \to 0^+} \frac{\overline{g}(\xi)}{\xi} = 0 \) a.e. in \( \Omega \times \mathbb{R} \), taking into account (1) and (2), we get \( \frac{g(\xi_1)}{\xi_1} \in L^1(L^v) \) for such \( \xi_1 \) and so \( \lim_{\xi \to 0^+} \frac{g(\xi)}{\xi} = m_0 \) with convergence in \( L^1(L^v) \). Then, by Remark 2.1(c), we have \( \lim_{\xi \to 0^+} P(\overline{g}(\xi)/\xi) = P(m_0) > 0 \), and thus there exists \( \lambda_1(\overline{g}(\xi)/\xi) \) for \( \xi > 0 \) small enough. Moreover, Remark 2.1(a) says that \( \lim_{\xi \to 0^+} \lambda_1(\overline{g}(\xi)/\xi) = \lambda_1(m_0) < 1 \) and so \( \lambda_1(\overline{g}(\xi)/\xi) < 1 \) for such \( \xi \). Hence, Lemma 2.4 can be applied to give a subsolution \( \overline{v} \in C_T \) of (1.6) with \( \overline{v}(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R}\).

On the other hand, for all \( \xi \geq \xi_0 \), we have \( \overline{m}_\infty \leq \overline{g}(\xi)/\xi \leq \overline{g}(\xi_0)/\xi_0 \), and so \( \overline{g}(\xi)/\xi \in L^1(L^v) \). Therefore, taking \( c = \|\overline{v}\|_\infty \) in Lemma 2.3, we obtain a supersolution \( w \in C_T \) of (1.6) with \( w \geq c \geq v \). Now, [6, Theorem 1] gives a solution \( u \in L^\infty_T \) such that \( v \leq u \leq w \) and then \( u(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R} \). Thus (a) is proved.

To prove (b), suppose that \( u \in L^\infty_T \) is a positive solution of (1.6). By Remark 2.2, we have \( u(x, t) > 0 \) for all \((x, t) \). Let \( m_u : \Omega \times \mathbb{R} \to \mathbb{R} \) be defined by \( m_u = g(u)/u \). Since \( m_u \) is measurable and \( m_\infty \leq m_u \leq m_0 \), it follows that \( m_u \in L^1(L^v) \). Moreover, we have \( Lu = m_u u \) and so \( 1 = \lambda_1(m_u) \). Now, the comparison principle in Remark 2.1(a) gives \( 1 = \lambda_1(m_u) \geq \lambda_1(m_0) = \lambda_1(\overline{m}_0) \) and also \( 1 \leq \lambda_1(\overline{m}_\infty) = \lambda_1(\overline{m}_0) \). Suppose \( \lambda_1(\overline{m}_0) = 1 \). Since \( \lambda_1(m_u) = 1 \) and \( m_u \leq m_0 \), we must have \( m_u(x, t) = m_0(x, t) \) a.e. \((x, t) \in \Omega \times \mathbb{R} \) (see Remark 2.1(a)), but \( \sup_{0 \leq \xi \leq \|u\|_\infty} g(\xi)/\xi \geq g(u)/u = m_0 \) in \( \Omega \times \mathbb{R} \) contradicting (3.1). Then \( \lambda_1(\overline{m}_0) < 1 \). Suppose now that \( \lambda_1(\overline{m}_\infty) = 1 \). Reasoning as above, we get \( 1 = \lambda_1(m_u) \leq \lambda_1(\overline{m}_\infty) = 1 \) and so \( m_u = m_\infty \). Thus, \( \inf_{0 \leq \xi \leq \|u\|_\infty} g(\xi)/\xi \leq g(u)/u = \inf_{\xi > 0}(g(\xi)/\xi) \) a.e., which is again a contradiction. Then \( \lambda_1(\overline{m}_\infty) > 1 \). \( \square \)

**Theorem 3.2.** (a) Assume that

\[(3) \quad m_0 \in L^1(L^v), P(m_0) > 0, \]
(4) \( g(\xi_0) \in L^1(L^\nu) \) for some \( \xi_0 > 0 \) and \( g(\xi) \in L^1(L^\nu) \) for all \( \xi > 0 \),
(5) either \( \overline{m}_\infty \in L^1(L^\nu) \) and \( P(\overline{m}_\infty) \leq 0 \) or \( \overline{m}_\infty \leq 0 \).

Then, if \( \lambda_1(m_0) < 1 \), there exists a solution \( u \in L^\nu_T \) of \( (1.6) \) satisfying \( u(x,t) > 0 \) for all \((x,t) \in \Omega \times \mathbb{R} \).

(b) Assume, in addition, \( (3.1) \) and \( \overline{m}_0 = m_0 \). Then there exists a positive solution \( u \in L^\nu_T \) of \( (1.6) \) if and only if \( \lambda_1(m_0) < 1 \).

Proof. As in the above theorem, we have \( g(\xi)/\xi \in L^1(L^\nu) \) and \( \lambda_1(g(\xi)/\xi) < 1 \) for \( \xi > 0 \) small enough, and so Lemma 2.4 gives a subsolution \( \nu \in C_T \) satisfying \( \nu(x,t) > 0 \) for all \((x,t) \). On the other side, since \( g(\xi)/\xi \leq g(\xi)/\xi \leq \overline{g}(\xi_0)/\xi_0 \) for \( \xi \geq \xi_0 \), from \( (4) \), we have \( \overline{g}(\xi)/\xi \in L^1(L^\nu) \) for such \( \xi \). Therefore, (a) follows as in Theorem 3.1 taking \( c = \|\nu\|_\infty \) in Lemma 2.3, and the proof of (b) follows similarly to part (b) of Theorem 3.1.

**Theorem 3.3.** (a) Assume \( (2) \) and that

\[
\begin{align*}
(6) & \quad \overline{m}_\infty \in L^1(L^\nu) \quad \text{and} \quad P(\overline{m}_\infty) > 0, \\
(7) & \quad P(g(\xi)/\xi) > 0 \quad \text{for} \quad \xi > 0 \quad \text{small and} \quad \lim_{\xi \to 0^+} \lambda_1(g(\xi)/\xi) = 0.
\end{align*}
\]

Then, if \( \lambda_1(\overline{m}_\infty) > 1 \), there exists a solution \( u \in L^\nu_T \) of \( (1.6) \) satisfying \( u(x,t) > 0 \) for all \((x,t) \in \Omega \times \mathbb{R} \).

(b) Assume, in addition, \( (3.2) \) and \( \overline{m}_\infty = m_\infty \). Then there exists a positive solution \( u \in L^\nu_T \) of \( (1.6) \) if and only if \( \lambda_1(\overline{m}_\infty) > 1 \).

Proof. Reasoning as above, (a) follows from Lemmas 2.3, 2.4, and \cite[Theorem 1]{6}. Suppose now that \( u \in L^\nu_T \) is a positive solution of \( (1.6) \). Let \( \varepsilon > 0 \) such that \( \varepsilon < \|u\|_\infty \). Let \( g_\varepsilon \) be defined by \( g_\varepsilon(\xi) = \overline{g}(\xi) \) if \( \xi > 0 \) and \( g_\varepsilon(\xi) = \overline{g}(\xi) \) if \( \xi < \varepsilon \). We have \( Lu = g(u) \geq g_\varepsilon(u) \geq g_\varepsilon(u) \) and also \( g_\varepsilon(u)/u \in L^1(L^\nu) \). Thus, \( 1 \leq \lambda_1(g_\varepsilon(u)/u) \).

Moreover, since \( g_\varepsilon(u)/u \geq \overline{m}_\infty \), the comparison principle in Remark 2.1(a) gives \( 1 \leq \lambda_1(m_\infty) \). Suppose \( 1 = \lambda_1(m_\infty) \). Then \( g_\varepsilon(u)/u = m_\infty \). But \( g_\varepsilon(u)/u \geq g_\varepsilon(||u||)/||u|| = g(||u||)/||u|| \), and therefore \( m_\infty = g(||u||)/||u|| \) in contradiction with \( (3.2) \).

**Theorem 3.4.** Assume \( (4), (5), \) and \( (7) \). Then \( (1.6) \) has a positive solution \( u \in L^\nu_T \) satisfying \( u(x,t) > 0 \) for all \((x,t) \in \Omega \times \mathbb{R} \).

Proof. The theorem follows again from Lemmas 2.3, 2.4, and \cite[Theorem 1]{6}.

**3.1. Examples.** (a) Suppose there exist \( \lim_{\xi \to 0^+} (g(\xi)/\xi) \) and \( \lim_{\xi \to \infty} (g(\xi)/\xi) \) and assume \( \inf_{\xi > 0} (g(\xi)/\xi) \), \( \sup_{\xi > 0} (g(\xi)/\xi) \in L^1(L^\nu) \), with \( P(\inf_{\xi > 0} (g(\xi)/\xi)) > 0 \). If \( \lim_{\xi \to 0^+} (g(\xi)/\xi) = \sup_{\xi > 0} (g(\xi)/\xi) \) and \( \lim_{\xi \to \infty} (g(\xi)/\xi) = \inf_{\xi > 0} (g(\xi)/\xi) \), from Theorem 3.1, we conclude that \( (1.6) \) has a positive solution \( u \in L^\nu_T \) if and only if \( \lambda_1(\lim_{\xi \to 0^+} (g(\xi)/\xi)) < 1 < \lambda_1(\lim_{\xi \to \infty} (g(\xi)/\xi)) \).

(b) Consider the Dirichlet periodic problem \( Lu = \sin u \) in \( \Omega \times \mathbb{R} \). Theorem 3.2 says that this problem has a positive \( T \)-periodic solution if and only if \( \lambda_1 < 1 \), where \( \lambda_1 \) is the positive principal eigenvalue corresponding to the weight 1.
Consider the problem
\[ Lu = a(x,t)u' - f(x,t,u) \quad \text{in } \Omega \times \mathbb{R}, \]
\[ u = 0 \quad \text{on } \partial \Omega \times \mathbb{R}, \]
\[ uT\text{-periodic}, \] (3.3)

where \( 0 < \gamma \leq 1 \) and \( f \) is a Carathéodory function such that \( f(\xi) \in L^s(L^r) \) for all \( \xi > 0 \) and \( f(0) = 0 \). Assume that \( \gamma = 1 \), \( a \in L^s(L^r) \), \( P(a) > 0 \), \( a \leq \lim_{\xi \to \infty} f(\xi) \leq \infty \), \( \inf_{0 < \xi \leq \xi_0} f(\xi) \in L^s(L^r) \) for some \( \xi_0 > 0 \), and \( \inf_{0 < \xi \leq \xi_0} f(\xi) \in L^s(L^r) \) for all \( \xi_0 > 0 \). From Theorem 3.2, it follows that (3.3) has a positive solution \( u \in L^\infty_T \) if and only if \( \lambda_1(a) < 1 \).

Consider now the case \( 0 < \gamma < 1 \) and \( a(x,t) \geq 0 \) a.e. \((x,t) \in \Omega \times \mathbb{R}\). If \( f(\xi) = -b \) with \( b \in L^s(L^r) \) and \( P(b) > 0 \), then Theorem 3.3 says that (3.3) has a positive solution \( u \in L^\infty_T \) if and only if \( 1 \leq \lambda_1(b) \). On the other hand, suppose \( \lim_{\xi \to \infty} f(\xi) = \infty \), \( \inf_{0 < \xi \leq \xi_0} f(\xi) \in L^s(L^r) \) for some \( \xi_0 > 0 \), and \( \sup_{0 < \xi \leq \xi_0} f(\xi) \in L^s(L^r) \) for all \( \xi_0 > 0 \). Then Theorem 3.4 gives a positive solution \( u \in L^\infty_T \) for (3.3).

We note that in all the cases, the positive solution \( u \) satisfies \( u(x,t) > 0 \) for all \((x,t)\). Moreover, recalling Remark 2.2, we also have that in (a), (b), and (c1) \( u \in C_T \).

Remark 3.5. An inspection of the proofs shows that all the above results remain true for the corresponding elliptic problem, replacing \( L^s(L^r) \) by \( L^r(\Omega) \) with \( r > N/2 \), and \( P(m) \) by \( \text{ess}\sup_{x \in \Omega} m(x) \).

Acknowledgment
This work was partially supported by CONICET, Secyt-UNC, and Agencia Córdoba Ciencia.

References
984 Periodic parabolic sublinear problems


T. Godoy: Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

E-mail address: godoy@mate.uncor.edu

U. Kaufmann: Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

E-mail address: kaufmann@mate.uncor.edu