ON THE $A$-LAPLACIAN

NOUREDDINE AÎSSAOUI

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We prove, for Orlicz spaces $L_A(\mathbb{R}^N)$ such that $A$ satisfies the $\Delta_2$ condition, the nonresolvability of the $A$-Laplacian equation $\Delta_A u + h = 0$ on $\mathbb{R}^N$, where $\int h \neq 0$, if $\mathbb{R}^N$ is $A$-parabolic. For a large class of Orlicz spaces including Lebesgue spaces $L^p$ ($p > 1$), we also prove that the same equation, with any bounded measurable function $h$ with compact support, has a solution with gradient in $L_A(\mathbb{R}^N)$ if $\mathbb{R}^N$ is $A$-hyperbolic.

1. Introduction

An important application of the nonlinear potential theory is the resolution of some equations involving the $p$-Laplacian operator. In [6], Gol’dshtein and Troyanov proved that the $p$-Laplace equation $\Delta_p u + h = 0$ on $\mathbb{R}^N$, $N \leq p$, has no solution if $h$ has a nonzero average. This result remains true for the same equation on any $p$-parabolic manifold. The proof is essentially based on a capacity argument. Later, Troyanov proved in [9] that the equation $\Delta_p u + h = 0$, on a $p$-hyperbolic manifold $M$, has a solution with $p$-integrable gradient for any bounded measurable function $h : M \to \mathbb{R}$ with compact support.

Since the strongly nonlinear potential theory is sufficiently developed, we propose in this paper the generalization of these two equations on $\mathbb{R}^N$ to the setting of Orlicz spaces. For this goal, we introduce, for a given $\mathcal{N}$-function $A$, the notion of $A$-parabolicity and $A$-hyperbolicity which reduces to $p$-parabolicity and $p$-hyperbolicity when $A(t) = p^{-1}|t|^p$. We also consider the so-called $A$-Laplacian $\Delta_A$, which is the $p$-Laplacian $\Delta_p$, when the Orlicz space $L_A$ is the Lebesgue space $L^p$. If the $\mathcal{N}$-function $A$ satisfies the $\Delta_2$ condition and $\mathbb{R}^N$ is $A$-parabolic, then the equation $\Delta_A u + h = 0$ has no weak solution for any function $h$ having a nonzero average.
For reflexive Orlicz spaces $L_A$, with $A$ satisfying the condition $s(A) > 0$, where
\[
s(A) := \inf \left\{ \frac{\log \int A \circ f \, d\lambda}{\log \|f\|_A} - 1, \ f \in L_A, \ \|f\|_A > 1 \right\}, \tag{1.1}\]
if the function $h$ is in $L^\infty$ and has a compact support, then the equation $\Delta_A u + h = 0$ has a weak solution when $\mathbb{R}^N$ is $A$-hyperbolic. We give large classes of Orlicz spaces $L_A$, including Lebesgue spaces $L^p$ ($p > 1$), which satisfies $s(A) > 0$.

This paper is organized as follows. In Section 2, we list the prerequisites from the Orlicz spaces and we introduce the notion of $A$-hyperbolicity. Section 3 is reserved to the resolution of the equation $\Delta_A u + h = 0$ when $h$ has a nonzero average or bounded with compact support.

2. Preliminaries

2.1. Orlicz spaces. We recall some definitions and results about Orlicz spaces. For more details, one can consult [1, 7, 8].

Let $A : \mathbb{R} \to \mathbb{R}^+$ be an $N$-function, that is, $A$ is continuous, convex, with $A(t) > 0$ for $t > 0$, $\lim_{t \to 0} A(t)/t = 0$, $\lim_{t \to +\infty} A(t)/t = +\infty$, and $A$ is even.

Equivalently, $A$ admits the representation: $A(t) = \int_0^{|t|} a(x) \, dx$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $\lim_{t \to +\infty} a(t) = +\infty$.

The $N$-function $A^*$ conjugate to $A$ is defined by $A^*(t) = \int_0^{|t|} a^*(x) \, dx$, where $a^*$ is given by $a^*(s) = \sup \{t : a(t) \leq s\}$.

Let $A$ be an $N$-function, let $\lambda$ be the Lebesgue measure on $\mathbb{R}^N$, and let $\Omega$ be an open set in $\mathbb{R}^N$. We denote by $L_A(\Omega)$ the set, called an Orlicz class, of measurable functions $f$, on $\Omega$, such that
\[
\rho(f, A, \Omega) = \int_\Omega A(f(x)) \, d\lambda(x) < \infty. \tag{2.1}\]

Let $A$ and $A^*$ be two conjugate $N$-functions and let $f$ be a measurable function defined almost everywhere in $\Omega$. The Orlicz norm of $f$, $\|f\|_{A,\Omega}$, or $\|f\|_A$, if there is no confusion, is defined by
\[
\|f\|_A = \sup \left\{ \int_\Omega |f(x)g(x)| \, d\lambda(x) : g \in L_{A^*}(\Omega), \ \rho(g, A^*, \Omega) \leq 1 \right\}. \tag{2.2}\]

The set $L_A(\Omega)$ of measurable functions $f$ such that $\|f\|_A < \infty$ is called an Orlicz space. When $\Omega = \mathbb{R}^N$, we set $L_A$ in place of $L_A(\mathbb{R}^N)$.

If $f \in L_A(\Omega)$, then
\[
\|f\|_A = \inf \left\{ k^{-1} \left[ 1 + \int_\Omega A(k|f(x)|) \, d\lambda(x) \right] : k > 0 \right\}. \tag{2.3}\]
The **Luxemburg norm** \( \| f \|_{A, \Omega} \) or \( \| f \|_A \), if there is no confusion, is defined in \( L_A(\Omega) \) by

\[
\| f \|_A = \inf \left\{ r > 0 : \int_\Omega A \left( \frac{f(x)}{r} \right) d\lambda(x) \leq 1 \right\}.
\]  

(2.4)

Orlicz and Luxemburg norms are equivalent. More precisely, if \( f \in L_A(\Omega) \), then

\[
\| f \|_A \leq \| f \|_A \leq 2 \| f \|_A.
\]  

(2.5)

It is well known that we can suppose that \( a \) and \( a^* \) are continuous and strictly increasing. Hence the \( \mathcal{N} \)-functions \( A \) and \( A^* \) are strictly convex and \( a^* = a^{-1} \).

Let \( A \) be an \( \mathcal{N} \)-function. We say that \( A \) verifies the \( \Delta_2 \) condition if there exists a constant \( C > 0 \) such that \( A(2t) \leq CA(t) \) for all \( t \geq 0 \).

Recall that \( A \) verifies the \( \Delta_2 \) condition if and only if \( L_A = L_A \). Moreover, \( L_A \) is reflexive if and only if \( A \) and \( A^* \) satisfy the \( \Delta_2 \) condition.

Hölder inequality in Orlicz spaces is expressed in the following way:

\[
\int |f \cdot g|d\lambda \leq \| f \|_A \cdot \| g \|_{A^*}, \quad f \in L_A, \ g \in L_{A^*}.
\]  

(2.6)

We recall the following results. Let \( A \) be an \( \mathcal{N} \)-function and \( a \) its derivative. Then the following occurs.

1. The \( \mathcal{N} \)-function \( A \) verifies the \( \Delta_2 \) condition if and only if one of the following holds:
   (i) for all \( r > 1 \), there exists \( k = k(r) \) (for all \( t \geq 0 \), \( A(rt) \leq kA(t) \));
   (ii) there exists \( \alpha > 1 \) (for all \( t \geq 0 \), \( ta(t) \leq \alpha A(t) \));
   (iii) there exists \( \beta > 1 \) (for all \( t \geq 0 \), \( ta^*(t) \geq \beta A^*(t) \));
   (iv) there exists \( d > 0 \) (for all \( t \geq 0 \), \( (A^*(t)/t')^d \geq d(a^*(t)/t) \)).
   Moreover, \( \alpha \) in (ii) and \( \beta \) in (iii) can be chosen such that \( \alpha^{-1} + \beta^{-1} = 1 \).
   We note that \( \alpha(A) \) is the smallest \( \alpha \) such that (ii) holds.

2. If \( A \) verifies the \( \Delta_2 \) condition, then

\[
A(t) \leq A(1)t^\alpha, \quad \forall t \geq 1, \quad A(t) \geq A(1)t^\alpha, \quad \forall t \leq 1,
\]
\[
A^*(t) \geq A^*(1)t^\beta, \quad \forall t \geq 1, \quad A^*(t) \leq A^*(1)t^\beta, \quad \forall t \leq 1.
\]  

(2.7)

We set \( \alpha^* = \alpha(A^*) \).

Recall also that if \( A \) verifies the \( \Delta_2 \) condition, then

\[
\int A \left( \frac{f}{\| f \|_A} \right) (x) d\lambda(x) = 1.
\]  

(2.8)
2.2. A-hyperbolicity

Definition 2.1. Let $A$ be an $N$-function and $K$ a compact set in $\mathbb{R}^N$. The $A$-capacity of $K$ is defined by

$$\Gamma_A(K) = \inf \{ \| \nabla u \|_A : u \in C_0^\infty (\mathbb{R}^N), u = 1 \text{ in a neighborhood of } K \}. \quad (2.9)$$

The space $\mathbb{R}^N$ is said to be $A$-parabolic if $\Gamma_A(K) = 0$ for all compact subsets $K \subset \mathbb{R}^N$ and $A$-hyperbolic otherwise.

Remark 2.2. In the definition of $\Gamma_A$, a simple truncation argument shows that we may restrict ourselves to functions $u \in C_0^\infty (\mathbb{R}^N)$ such that $0 \leq u \leq 1$.

For $m < N$, the Riesz kernel is defined on $\mathbb{R}^N$ by $R_m(x) = |x|^{m-N}$.

For $X \subset \mathbb{R}^N$, we define $R_m,A(X)$ by

$$R_m,A(X) = \inf \{ ||f||_A : f \in L_A, f \geq 0, R_m \ast f \geq 1 \text{ on } X \}. \quad (2.10)$$

The following lemma is proved in [3, Lemma 3.6].

Lemma 2.3. Let $L_A$ be a reflexive Orlicz space. Then there is a positive constant $C$ such that

$$C^{-1} R_{1,A}(K) \leq \Gamma_A(K) \leq CR_{1,A}(K), \quad (2.11)$$

for all compact $K$, $C$ independent of $K$.

We recall the following result proved in [4, Theorem 3.1].

Lemma 2.4. Let $A$ be an $N$-function such that $\| R_m \|_{A^*, \{|x| > 1\}} = \infty$. Then for all $X$, $R_m,A(X) = 0$.

We will need the following lemma in the sequel.

Lemma 2.5. Let $A$ be any $N$-function such that $A^*$ verifies the $\Delta_2$ condition and let $m$ be a positive integer such that $m < N$ and $\alpha^* \leq N/(N-m)$. Then $R_m,A(X) = 0$ for all $X$.

Proof. From Lemma 2.4, it suffices to prove that $\| R_m \|_{A^*, \{|x| > 1\}} = \infty$. Since $A^*$ verifies the $\Delta_2$ condition, we must establish that

$$\int_{\{|x| > 1\}} A^*(|x|^{m-N}) d\lambda(x) = \infty. \quad (2.12)$$

By a change of variable, there is a positive constant $C$ such that

$$\int_{\{|x| > 1\}} A^* (|x|^{m-N}) d\lambda(x) = C \int_1^\infty A^* (t^{m-N}) \cdot t^{N-1} dt. \quad (2.13)$$
From the inequality \( A^*(t^{m-N}) \geq A^*(1) \cdot t^{a^*(m-N)} \), we get

\[
\int_{|x| > 1} A^*(|x|^{m-N}) d\lambda(x) \geq CA^*(1) \cdot \int_1^\infty t^{a^*(m-N)+N-1} dt.
\]

(2.14)

Now, the inequality

\[
a^*(m-N) + N - 1 \geq \frac{N}{N-m}(m-N) + N - 1 = -1
\]

(2.15)
gives the desired result. □

3. On the \( A \)-Laplacian

The Orlicz-Sobolev space \( W^1 L^A(\mathbb{R}^N) \) is defined as the space of functions \( u \) such that \( u \) and its derivatives, in a distributional sense, of order less or equal to one are in \( L^A \). The space \( W^1 L^A(\mathbb{R}^N) \) is a Banach space when equipped with the norm

\[
\|u\|_{1,A} = \sum_{|\gamma| \leq 1} \left\| D^\gamma u \right\|_A.
\]

(3.1)

Recall that \( W^1 L^A(\mathbb{R}^N) \) is reflexive if and only if \( A \) and \( A^* \) satisfy the \( \Delta_2 \) condition.

The \( A \)-Dirichlet space \( L^A(\mathbb{R}^N) \) is the space of functions \( u \in W^1_{A,loc}(\mathbb{R}^N) \) (i.e., \( u \) is locally in \( W^1 L^A(\mathbb{R}^N) \)) admitting a weak gradient such that \( \|\nabla u\|_A < \infty \).

Let \( A \) be any \( \mathcal{N} \)-function and let \( a \) be its derivative. For \( x \in \mathbb{R}^N \), we define

\[
M_A(x) = \frac{a(|x|)}{|x|} \cdot x \quad \text{if} \ x \neq 0, \ M_A(0) = 0.
\]

(3.2)

The \( A \)-Laplacian of a function \( f \) on \( \mathbb{R}^N \) is defined by \( \Delta_A f = \text{div} M_A(\nabla f) \).

A function \( u \in W^1_{A,loc}(\mathbb{R}^N) \) is said to be a weak solution to the equation

\[
\Delta_A u + h = 0
\]

(3.3)

if, for all \( \varphi \in C_0^1(\mathbb{R}^N) \), we have

\[
\int \langle M_A(\nabla u), \nabla \varphi \rangle d\lambda = \int h\varphi d\lambda.
\]

(3.4)

Let \( D \subset \mathbb{R}^N \) be a nonempty bounded domain. The Banach space \( \mathcal{E}_A(D) \) is the space of functions \( u \in W^1_{A,loc}(\mathbb{R}^N) \) such that

\[
\|u\|^D_A := \|u\|_{A,D} + \|\nabla u\|_A < \infty.
\]

(3.5)

We denote by \( \mathcal{E}_A^0(D) \) the closure of \( C_0^1(\mathbb{R}^N) \) in \( \mathcal{E}_A(D) \).
3.1. A nonresolvability result

**Theorem 3.1.** Let $A$ be an $N$-function satisfying the $\Delta_2$ condition. Suppose that $\mathbb{R}^N$ is $A$-parabolic and let $h \in L_1(\mathbb{R}^N)$ be such that $\int h d\lambda \neq 0$. Then the equation

$$\Delta_A u + h = 0 \quad (3.6)$$

has no weak solution on $L_A^1(\mathbb{R}^N)$.

**Proof.** We may suppose that $\int h d\lambda > 0$. Hence there is a bounded set $D \subset \mathbb{R}^N$ such that $\lambda(D) > 0$, $s := \inf_D h > 0$, and $\int_D h d\lambda > |\int h^- d\lambda|$.

Let $0 < c < 1$ be such that $0 \leq -\int h^- d\lambda < c \int_D h d\lambda$.

By the definition of $\Gamma_{1,A}(D)$, for $\varepsilon > 0$, we can find a function $v \in C^\infty_0(\mathbb{R}^N)$ such that $0 \leq v \leq 1$, $v = 1$, on $D$ and

$$\|\nabla v\|_A \leq \Gamma_A(D) + \varepsilon. \quad (3.7)$$

On the other hand, we have $-c \int_D vh d\lambda < \int vh^- d\lambda$. Hence

$$(1 - c) \int_D vh d\lambda < \int_D vh d\lambda + \int vh^- d\lambda$$

$$< \int_D vh d\lambda + \int vh^- d\lambda + \int vh^+ d\lambda \quad (3.8)$$

$$\leq \int vh d\lambda.$$

But $s \cdot \lambda(D) \leq \int_D vh d\lambda$. Thus

$$(1 - c) \cdot s \cdot \lambda(D) \leq \int vh d\lambda. \quad (3.9)$$

Now suppose that $u \in L_A^1(\mathbb{R}^N)$ is a weak solution of (3.6) and let $\xi := -\left(\frac{a(|\nabla u|)}{|\nabla u|} \cdot \nabla u\right)$. Then $\text{div}(\xi) = -\Delta_A u = h$, and since $A$ satisfies the $\Delta_2$ condition, $|\xi| \in L_{A^*}(\mathbb{R}^N)$.

An integration by part and Hölder inequality in Orlicz spaces applied to inequality (3.9) imply that

$$(1 - c) \cdot s \cdot \lambda(D) \leq \int v \cdot \text{div}(\xi) d\lambda$$

$$= -\int \langle \nabla v, \xi \rangle d\lambda \leq \|\xi\|_{A^*} \|\nabla v\|_A. \quad (3.10)$$

From (3.7), and since $\varepsilon$ is arbitrary, we get

$$0 < \lambda(D) \leq \frac{\|\xi\|_{A^*}}{(1 - c) \cdot s} \cdot \Gamma_A(D). \quad (3.11)$$

This is impossible, and the theorem is proved. \qed
Corollary 3.2. Let $L_A$ be a reflexive Orlicz space such that $\alpha^* \leq N/(N-1)$. Let $h \in L^1(\mathbb{R}^N)$ be such that $\int h \lambda \neq 0$. Then (3.6) has no weak solution on $L^1_A(\mathbb{R}^N)$.

Proof. By Lemmas 2.5 and 2.3, $\mathbb{R}^N$ is then $A$-parabolic. We apply Theorem 3.1 to get the result. □

Remark 3.3. When $A(t) = p^{-1}|t|^p$, $L_A = L^p$ is the usual Lebesgue space and $\alpha^* = p/(p-1)$. Hence the condition $\alpha^* \leq N/(N-1)$ is exactly the condition $N \leq p$. Thus our result recovers the one in [6, Théorème 1].

3.2. A resolvability result. In this section, we resolve the equation $\Delta_A u + h = 0$ under some assumptions on the $\mathcal{N}$-function $A$ and on the function $h$.

We begin by recalling the following Poincaré inequality for Orlicz-Sobolev functions, which is a combination of [5, Theorem 3.3] and [5, Proposition 3.9].

Lemma 3.4. Let $A$ be an $\mathcal{N}$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Let $E$ be any measurable set in $\mathbb{R}^N$ such that $0 < \lambda(E) < \infty$. Then there exists a positive constant $C$ such that

$$\|u - u_E\|_{A,E} \leq C \|\nabla u\|_{A,E},$$

(3.12)

for all $u \in W^1_{A,\text{loc}}(\mathbb{R}^N)$, where $u_E = (1/\lambda(E)) \int_E u d\lambda$ is the mean value of $u$ on $E$.

An application of Hölder inequality in Orlicz spaces gives

$$\int_E |u - u_E| d\lambda \leq \|\chi_E\|_{A^*} \|u - u_E\|_{A,E},$$

(3.13)

where $\chi_E$ is the characteristic function of $E$.

Recall that

$$\|\chi_E\|_{A^*} = \lambda(E) \cdot A^{-1}\left(\frac{1}{\lambda(E)}\right),$$

$$\|1\|_{A,E} = \|\chi_E\|_A = \frac{1}{A^{-1}(1/\lambda(E))}.$$

(3.14)

Hence we obtain the following proposition.

Proposition 3.5. Let $A$ be an $\mathcal{N}$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Let $E$ be any measurable set in $\mathbb{R}^N$ such that $0 < \lambda(E) < \infty$. Then there exists a positive constant $C$ such that

$$\int_E |u - u_E| d\lambda \leq C \|\nabla u\|_{A,E},$$

(3.15)

for all $u \in W^1_{A,\text{loc}}(\mathbb{R}^N)$.

We will need the following proposition in what follows.
Proposition 3.6. Let $A$ be an $\mathcal{N}$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Suppose that $\mathbb{R}^N$ is $A$-hyperbolic. Let $E$ be any nonempty bounded domain in $\mathbb{R}^N$. Then there exists a positive constant $C$ such that, for all $u \in \mathcal{E}_A^0(E)$,

$$\int_E |u|d\lambda \leq C||\nabla u||_A. \quad (3.16)$$

Proof. Suppose that such constant does not exist. Then for all $\varepsilon > 0$, we can find a function $u \in \mathcal{E}_A^0(E)$ such that

$$\int_E |u|d\lambda = \lambda(E), \quad ||\nabla u||_A \leq \varepsilon. \quad (3.17)$$

We may assume that $u \geq 0$. Proposition 3.5 implies that

$$\int_E |u|d\lambda \leq C\varepsilon. \quad (3.18)$$

We now choose a ball $B \Subset E$ and a function $\phi \in C_0^1$ such that $0 \leq \phi \leq 2^{-1}$, $\text{supp}(\phi) \subset E$, and $\phi = 2^{-1}$ on $B$. Define the function $v \in \mathcal{E}_A^0(E)$ by $v = 2\max(u, \phi)$. Then $v \geq 1$ on $B$. Now, define the sets

$$S = \{x \in E: \phi(x) \geq u(x)\}, \quad S' = \{x \in E: |u(x) - 1| \geq 2^{-1}\}. \quad (3.19)$$

We have $S \subset S'$ and, by (3.18), $2^{-1}\lambda(S') \leq C\varepsilon$. Thus

$$\lambda(S) \leq 2C\varepsilon. \quad (3.20)$$

On the other hand, we have almost everywhere

$$\nabla v = \begin{cases} 2\nabla u & \text{on } cS, \\ 2\nabla \phi & \text{on } S. \end{cases} \quad (3.21)$$

This implies that

$$|\nabla v| \leq 2|\nabla u| + 2\chi_S|\nabla \phi| \quad \text{a.e.} \quad (3.22)$$

Since $v \geq 1$ on $B$ and $\varepsilon$ is arbitrary, we deduce that $\Gamma_A(B) = 0$. This contradicts the fact that $\mathbb{R}^N$ is $A$-hyperbolic. The proof is complete. □

Lemma 3.7. Let $A$ be an $\mathcal{N}$-function. If $\mathbb{R}^N$ is $A$-parabolic, then $1 \in \mathcal{E}_A^0(D)$ for any nonempty bounded domain $D$.

Proof. Since $\mathbb{R}^N$ is $A$-parabolic, $\Gamma_A(\overline{D}) = 0$. Hence for all $\varepsilon > 0$, there exists a function $u \in C_0^1$ such that $u = 1$ on $D$ and $|||\nabla u||_A \leq \varepsilon$. Thus

$$||1 - u||_A = ||1 - u||_{A,D} + ||\nabla u||_A = ||\nabla u||_A \leq \varepsilon. \quad (3.23)$$

This means that $1 \in \mathcal{E}_A^0(D)$. □
Theorem 3.8. Let $A$ be an $N$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Let $D$ be nonempty bounded domain in $\mathbb{R}^N$. Then the following assertions are equivalent

(i) $\mathbb{R}^N$ is $A$-hyperbolic;
(ii) there exists a constant $C$ such that, for all $u \in \mathcal{C}^0_A(D)$,
\[ \| u \|_{A,D} \leq C \| \nabla u \|_A; \]  
(3.24)
(iii) $1 \notin \mathcal{C}^0_A(D)$.

Proof. It is easy to verify that (ii) implies (iii). The implication (iii) $\Rightarrow$ (i) is Lemma 3.7. It remains to prove that (i) implies (ii).

Write $u = (u - u_D) + u_D$. Proposition 3.6 and Lemma 3.4 give
\[ \| u \|_{A,D} \leq \| u - u_D \|_{A,D} + \| u_D \|_{A,D} \]
\[ \leq C \| \nabla u \|_{A,D} + \| u_D \| \cdot \| 1 \|_{A,D} \]
\[ \leq C \| \nabla u \|_{A,D} + \frac{1}{A^{-1}(1/\lambda(D))} \cdot \lambda(D)^{-1} \int_D |u|d\lambda \]
\[ \leq C \| \nabla u \|_{A,D} + \frac{1}{A^{-1}(1/\lambda(D))} \cdot \lambda(D)^{-1} C' \| \nabla u \|_A \]
\[ \leq C' \| \nabla u \|_A. \]  
(3.25)

The proof is complete. \qed

Recall that for all $f \in L_A$ such that $\| f \|_A > 1$, we have $\int A \circ f d\lambda > \| f \|_A$. We set
\[ s(A) = \inf \left\{ \frac{\log \| A \circ f \|_A}{\log \| f \|_A} - 1, \ f \in L_A, \ \| f \|_A > 1 \right\}. \]  
(3.26)

Hence $s(A) \geq 0$.

Now we are ready to solve the $A$-Laplace equation.

Theorem 3.9. Let $L_A$ be a reflexive Orlicz space such that $s(A) > 0$. Let $h \in L^\infty(\mathbb{R}^N)$ have compact support. Then the equation $\Delta_A u + h = 0$ has a weak solution $u \in L^1_A(\mathbb{R}^N)$ if $\mathbb{R}^N$ is $A$-hyperbolic.

Proof. Let $D$ be a bounded domain such that $\text{supp}(h) \subset D$. Define the functional $\mathcal{F} : \mathcal{C}^0_A(D) \to \mathbb{R}$ by
\[ \mathcal{F}(u) = \int A(|\nabla u|)d\lambda - \int hu d\lambda. \]  
(3.27)
Hence
\[ \mathcal{F}(u) \geq \int A(|\nabla u|) d\lambda - \left| \int hu d\lambda \right| \geq \int A(|\nabla u|) d\lambda - \|h\|_\infty \cdot \|u\|_{L^1(D)}. \] (3.28)

Since \( \mathbb{R}^N \) is \( A \)-hyperbolic, by Proposition 3.6, we get
\[ \mathcal{F}(u) \geq \int A(|\nabla u|) d\lambda - C\|h\|_\infty \cdot \|\nabla u\|_A. \] (3.29)

Hence there is a constant \( C_1 \) such that
\[ \mathcal{F}(u) \geq \int A(|\nabla u|) d\lambda - C_1 \cdot \|\nabla u\|_A. \] (3.30)

By (2.3) and (2.5), there is a constant \( C_2 \) such that, for all \( k > 0 \),
\[ \mathcal{F}(u) \geq \int A(|\nabla u|) d\lambda - C_2 \int A(k|\nabla u|) d\lambda - C_2 k. \] (3.31)

Now, let \( t > 0 \) and consider the continuous function \( \psi_t \) defined on \( \mathbb{R}^+ \) by
\[ \psi_t(k) = (C_2/k)A(kt) - A(t). \]
\[ x \leq A(x), \quad \forall x \geq 0, \]
\[ \lim_{t \to 0} \frac{A(t)}{t} = 0, \quad \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty, \] (3.32)
the function \( \psi_t \) increases from \( -A(t) \) to \( +\infty \). Hence there is a \( k_0 \) such that \( \psi_t(k_0) = 0 \). Thus
\[ \mathcal{F}(u) \geq -\frac{C_2}{k_0}. \] (3.33)

We conclude that the functional \( \mathcal{F} \) is bounded below on the space \( \mathcal{E}^0_A(D) \).

Now \( \mathcal{E}^0_A(D) \) is a reflexive Banach space and \( \mathcal{E}^0_A(D) \) is a closed convex subspace of \( \mathcal{E}^0_A(D) \). We first prove that \( \mathcal{F} \) is lower semicontinuous. Let \( t \in \mathbb{R} \), and consider the set \( \mathcal{T}_t = \{ u \in \mathcal{E}^0_A(D) : \mathcal{F}(u) \leq t \} \). Let \( (u_i)_i \subset \mathcal{E}^0_A(D) \) be such that \( \mathcal{F}(u_i) \leq t \), for all \( i \), and \( (u_i)_i \) converges to \( u \) in \( \mathcal{E}^0_A(D) \). By the compactness of the imbedding \( \mathcal{E}^0_A(D) \subset L^1(D) \), we may assume that \( (u_i)_i \) converges strongly in \( L^1(D) \). Hence
\[ \int_D hu_i d\lambda \longrightarrow \int_D hu d\lambda. \] (3.34)

Theorem 3.8 implies that \( u \to \|\nabla u\|_A \) is an equivalent norm on \( \mathcal{E}^0_A(D) \).

Hence \( \|\nabla u - \nabla u_i\|_A \to 0 \). Since \( A \) verifies the \( \Delta_2 \) condition, \( \int A(|\nabla u - \nabla u_i|) d\lambda \to 0 \). Hence there is a subsequence of the sequence \( (A(|\nabla u - \nabla u_i|))_i \), still denoted by \( (A(|\nabla u - \nabla u_i|))_i \), which converges \( \lambda \)-almost everywhere to 0.
Thus \(|\nabla u_i|_i\) converges \(\lambda\)-almost everywhere to \(|\nabla u|\). By the continuity of \(A\), Fatou’s lemma, and (3.34), we get

\[
\mathcal{F}(u) = \int \lim_{i \to \infty} A(|\nabla u_i|) d\lambda - \lim_{i \to \infty} \int h u_i d\lambda \\
\leq \liminf_{i \to \infty} \int A(|\nabla u_i|) d\lambda - \lim_{i \to \infty} \int h u_i d\lambda \leq t. \tag{3.35}
\]

Hence \(\mathcal{F}\) is lower semicontinuous.

Now, \(s(A) > 0\) implies that \(\int A(|\nabla u|) d\lambda \geq \|\nabla u\| s(A) + 1\) for \(\|\nabla u\| > 1\).

Hence

\[
\mathcal{F}(u) \geq \|\nabla u\| s(A) + 1 - C_1 \cdot \|\nabla u\| \quad \text{for} \quad \|\nabla u\| > 1. \tag{3.36}
\]

This proves that \(\mathcal{F}\) is coercive.

Thus \(\mathcal{F}\) attains its minimum on \(\mathcal{E}_A^0(D)\); that is, there is \(u^* \in \mathcal{E}_A^0(D)\) such that \(\mathcal{F}(u^*) = \min\{\mathcal{F}(u) : u \in \mathcal{E}_A^0(D)\}\). By the usual arguments from variational calculus, we deduce that \(u^*\) is a weak solution to the equation \(\Delta_A u + h = 0\). The proof is complete. \(\square\)

**Remark 3.10.** We have in fact solved the equation in the space \(\mathcal{E}_A^0(D) \subset L_1^A(\mathbb{R}^N)\).

**Remark 3.11.** When \(A(t) = p^{-1} |t|^p, p > 1\), and \(L_A = L^p\) is the usual Lebesgue space, we have \(s(A) = p - 1 > 0\). Thus we recover the result in [9, Theorem 2] when the manifold \(M\) is \(\mathbb{R}^N\).

Recall the following result in [2, Lemma 3].

**Lemma 3.12.** Let \(A\) be an \(N\)-function satisfying the \(\Delta_2\) condition. If \(\alpha < N\), then \(R_1(A(B(x, r))) > 0\), where \(B(x, r)\) is the open ball of radius \(r > 0\), with center at \(x\).

**Corollary 3.13.** Let \(L_A\) be a reflexive Orlicz space such that \(s(A) > 0\) and \(\alpha < N\). Suppose that \(h \in L^\infty(\mathbb{R}^N)\) has compact support. Then the equation \(\Delta_A u + h = 0\) has a weak solution \(u \in L_A^1(\mathbb{R}^N)\).

**Proof.** By Lemmas 3.12 and 2.3, we deduce that \(\mathbb{R}^N\) is \(A\)-hyperbolic, and we apply Theorem 3.9 to get the result. \(\square\)

### 3.3. Some examples

In addition to the \(L^p\) Lebesgue case corresponding to \(A(t) = p^{-1} |t|^p, p > 1\), we consider the following \(N\)-functions:

\[
A_1(t) = \begin{cases} 
  t^p & \text{for } 0 \leq |t| \leq 1, \\
  t^q & \text{for } 1 < |t|,
\end{cases} \quad 1 < p < q < \infty, \tag{3.37}
\]

\[
A_2(t) = |t|^p \log(1 + |t|), \quad p > 1,
\]

\[
A_3(t) = |t|^p \log(1 + |t|^p), \quad p > 1,
\]
(4) \( A_i(t) = |t|^p \log^p(1 + |t|), \ p > 1, \)
(5) \( A_{p,q,r}(t) = |t|^p \log^q(1 + |t|^r), \ p > 1, \ q > 0, \) and \( r > 0. \)

All these \( N \)-functions and their conjugates satisfy the \( \Delta_2 \) condition. We show that \( s(A_i) > 0, \ i = 1, 2, 3, 4, \) and \( s(A_{p,q,r}) > 0. \)

First remark that \( A_2 = A_{p,1,1} \) and \( A_3 = A_{p,1,p}. \) Thus it suffices to show that \( s(A_{p,q,r}) > 0 \) and for all \( p > 1, \ q > 0, \ r > 0. \)

(1) Let \( f \in L_{A_1} \) be such that \( \|f\|_{A_1} > 1. \) Then, by (2.8),

\[
1 = \int A_1 \left( \frac{f}{\|f\|_{A_1}} \right)(x) d\lambda(x) \\
\leq \frac{1}{\|f\|_{A_1}^p} \int_{\{|f| \leq \|f\|_{A_1}\}} |f|^p d\lambda + \frac{1}{\|f\|_{A_1}^q} \int_{\{|f| > \|f\|_{A_1}\}} |f|^q d\lambda \\
\leq \frac{1}{\|f\|_{A_1}^p} \left[ \int_{\{|f| \leq \|f\|_{A_1}\}} |f|^p d\lambda + \int_{\{|f| > \|f\|_{A_1}\}} |f|^q d\lambda \right] \\
\leq \frac{1}{\|f\|_{A_1}^p} \int A_1(f)(x) d\lambda(x).
\]

Hence \( \|f\|_{A_1}^p \leq \int A_1(f)(x) d\lambda(x). \) This implies that \( s(A_1) > 0. \)

(2) Let \( p > 1, \ q > 0, \) and \( r > 0 \) and set \( A = A_{p,q,r}. \) Let \( f \in L_A \) be such that \( \|f\|_A > 1. \) Then by (2.8),

\[
1 = \int A \left( \frac{f}{\|f\|_A} \right)(x) d\lambda(x) \\
\leq \frac{1}{\|f\|_A^p} \int |f|^p \log^q \left( 1 + \frac{|f|'}{\|f\|_A} \right) d\lambda \\
\leq \frac{1}{\|f\|_A^p} \int |f|^p \log^q (1 + |f'|) d\lambda \\
\leq \frac{1}{\|f\|_A^p} \int A(f)(x) d\lambda(x).
\]

Thus \( \|f\|_A^p \leq \int A(f)(x) d\lambda(x) \) and hence \( s(A) > 0. \)
Remark 3.14. Although Theorem 3.9 gives a solution for large classes of Orlicz spaces $L_A$, including $L^p$ Lebesgue spaces, $p > 1$, it would be sharp if we can drop the condition $s(A) > 0$. This question is open.

References


Noureddine Aïssaoui: Département de Mathématiques École Normale Supérieure, BP 5206, Ben Souda, Fès, Morocco

E-mail address: n.aissaoui@caramail.com