ON THE MANN AND ISHIKAWA ITERATION PROCESSES

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Abstract. It is shown that a result of Chidume, involving the strong convergence of the Mann iteration process for continuous strongly accretive operators, is actually a corollary to a result by Nevanlinna and Reich. It is then shown that the Nevanlinna and Reich result can be extended to the case of an Ishikawa iteration process.

1. Introduction and preliminaries

In [4, Theorem 1] Chidume gave a strong convergence theorem on the Mann iterative process for a class of continuous strongly accretive maps. We are going to show that Chidume’s theorem is a corollary of a result by Nevanlinna and Reich [5, Theorem 3].

Recently, the authors have proved in [9, Theorem 2.1] a considerably more general strong convergence theorem for the Ishikawa iterative process for a class of strongly quasi-accretive operators. Theorem 2.1 of [9] is closely related to those strong convergence theorems of [5], [3]. We shall discuss the relations between Theorem 2.1 of [9] and the corresponding results in [5], [3].

Let $X$ be a real Banach space with a dual $X^*$, and let $J : X \to 2^{X^*}$ be the normalized duality mapping defined by

$$Jx = \{f \in X^* : \langle f, x \rangle = \|f\|\|x\|, \|f\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is well known that if $X^*$ is strictly convex, then $J$ is single-valued and such that $J(tx) = tJx$ for all $t \geq 0$, $x \in X$. If $X$ is uniformly smooth, then $J$ is uniformly continuous on bounded subsets of $X$.

An operator $T$ with domain $D(T)$ and range $R(T)$ in $X$ is said to be “accretive” if for every $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such

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that
\begin{equation}
<Tx - Ty, j(x - y) > \geq 0.
\end{equation}

The operator \(T\) is said to be “strongly accretive” if for each \(x, y \in D(T)\) there exists \(j(x - y) \in J(x - y)\) such that
\begin{equation}
<Tx - Ty, j(x - y) > \geq k\|x - y\|^2,
\end{equation}
for some fixed real constant \(k > 0\).

An accretive operator \(T\) is “\(m\)-accretive” if \(R(I + rT) = X\) for all \(r > 0\), where \(I\) denotes the identity operator.

We let \(N(T) = \{x \in D(T) | Tx = 0\}\). If \(N(T) \neq \phi\) and the inequality (1.1) ((1.2)) holds for all \(x \in D(T)\) and \(y \in N(T)\), then the corresponding operator \(T\) is said to be “quasi-accretive” (“strongly quasi-accretive”).

We need the following Lemmas.

**Lemma 1.1.** Let \(X\) be a reflexive Banach space and let \(C\) be a closed convex subset of \(X\). Then \(C\) is proximinal.

**Proof.** The proof is straightforward.

**Lemma 1.2.** Let \(\{a_n\}\) be a nonnegative sequence satisfying
\begin{equation}
a_{n+1} \leq a_n + \sigma_n
\end{equation}
with \(\sum_{n=1}^{\infty} \sigma_n < +\infty\). Then \(\lim_{n \to \infty} a_n\) exists.

**Proof.** Note that, for all \(m \geq 1\),
\[
a_{n+m} \leq a_n + \sum_{k=n}^{n+m-1} \sigma_k,
\]
which implies \(\lim_{n \to \infty} \sup a_n \leq \lim_{n \to \infty} \inf a_n\).

**Lemma 1.3.** (Xu and Roach [8]) Let \(X\) be a real uniformly smooth Banach space. Then
\begin{equation}
\|x + y\|^2 \leq \|x\|^2 + 2 < y, Jx > + K\max\{\|x\| + \|y\|, \frac{c}{2}\}\rho_X(\|y\|),
\end{equation}
for all \(x, y \in X\), where \(K\) and \(c\) are positive constants, and \(\rho(\tau)\) is the modulus of smoothness of \(X\) (defined by
\[
\rho_X(\tau) = \sup\{\frac{1}{2}\|x + y\| + \frac{1}{2}\|x - y\| - 1\|x\| = 1, \|y\| \leq \tau\},
\]
and satisfying
\[ \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0. \]

**Lemma 1.4.** If \( X \) is uniformly smooth and \( T : X \ni D(T) \to X \) is \( m \)-accretive, then \( T \) is demiclosed, i.e., for any sequence \( \{x_n\} \subset D(T) \), with \( x_n \to x \) strongly and \( Tx_n \to y \) weakly as \( n \to \infty \), we have \( x \in D(T) \) and \( Tx = y \).

**Proof.** See Barbu [1].

**Lemma 1.5.** If \( X \) is uniformly smooth and \( T : X \to X \) is demi-continuous and accretive, then \( T \) is \( m \)-accretive.

**Proof.** See Browder [2].

### 2. Main results

Before we show our main results, we give a slight extension of [5, Theorem 3]. For the sake of simplicity, we only consider the following Mann iterative process

\[ x_{n+1} = x_n - \lambda_n Tx_n, \quad n \geq 0, \quad (2.1) \]

where \( x_0 \in X \) and \( \{\lambda_n\} \) is a positive sequence. We shall study the convergence of \( \{x_n\} \) under more general assumptions.

In the sequel, we always assume that \( X \) is uniformly smooth and \( N(T) \) has a nonempty convex subset \( N_0(T) \).

**Theorem 2.1.** Let \( T \) be a quasi-accretive and demiclosed operator, and let \( \{\lambda_n\} \) be a positive sequence such that \( \sum_{n=0}^{\infty} \lambda_n = \infty \) and \( \sum_{n=0}^{\infty} \rho_X(\lambda_n) < \infty \). Assume that \( \{x_n\} \) satisfies (2.1) and \( \{Tx_n\} \) is bounded. Let \( P_0 \) be an arbitrary selection of the nearest point mapping from \( X \) onto \( N_0(A) \) such that

\[ < y - P_0x, J(x - P_0x) > \leq 0 \text{ for all } y \in N_0(A). \]

If there exists a strictly increasing function \( \psi : R^+ \to R^+ \), \( \psi(0) = 0 \), such that

\[ < Tx_n, J(x_n - P_0x_n) > \geq \psi(||x_n - P_0x_n||) ||Tx_n||, \quad n \geq 0, \quad (2.2) \]

then \( \{x_n\} \) converges strongly to a zero of \( T \).

**Proof.** Since \( T \) is demiclosed, we know that \( N_0(T) \) is closed. By Lemma 1.1 we see that \( N_0(T) \) is proximinal. Thus we can choose a section \( P_0 : X \to N_0(T) \) of the nearest point operator such that

\[ < y - P_0x, J(x - P_0x) > \leq 0 \text{ for all } y \in N_0(T). \]
Let \( j_n = J(x_n - P_0x_n) \) and \( M = \sup \{ \|Tx_n\| | n \geq 0 \} \). By (2.1) and (1.4) we have
\[
\|x_{n+1} - P_0x_{n+1}\|^2 \leq \|x_{n+1} - P_0x_n\|^2 = \|x_n - P_0x_n - \lambda_n Tx_n\|^2 \\
\leq \|x_n - P_0x_n\|^2 - 2\lambda_n < Tx_n, j_n > + K \max\{\|x_n - P_0x_n\| + \lambda_n\|Tx_n\|, \frac{c}{2}\} \rho_X (\lambda_n\|Tx_n\|) \\
\leq \|x_n - P_0x_n\|^2 - 2\lambda_n < Tx_n, j_n > + M_1 \max\{\|x_n - P_0x_n\| + \lambda_n M, \frac{c}{2}\} \rho_X (\lambda_n),
\]
for some \( M_1 > 0 \). Here we have used the fact that \( \rho_X (\tau) \) is nondecreasing and that there exists a constant \( c_0 > 0 \) such that \( \frac{\rho_X (\eta)}{\eta^2} \leq c_0 \frac{\rho_X (\tau)}{\tau^2} \) for any \( \eta \geq \tau > 0 \). The condition \( \sum_{n=0}^{\infty} \rho_X (\lambda_n) < +\infty \) implies \( \lambda_n \to 0 \) as \( n \to \infty \).

We claim that \( \|x_n - P_0x_n\| \) is bounded. Assume the contrary and let \( d_n = \|x_n - P_0x_n\| \). We may also assume that \( d_n + \lambda_n M \geq \frac{c}{2} \) for all \( n \geq 0 \). Then (2.2) leads to
\[
d_{n+1}^2 \leq d_n^2 + M_1 (d_n + \lambda_n M) \rho_X (\lambda_n)
\]
and, consequently, for \( \lambda_n \in (0, \frac{c}{M}) \),
\[
d_{n+1} \leq d_n + M_1 \rho_X (\lambda_n).
\]
In view of Lemma 1.2 we see that \( \lim_{n \to \infty} d_n \) exists, which contradicts with the assumption that \( \{d_n\} \) is unbounded. Hence, from (2.3) we get
\[
\|x_{n+1} - P_0x_{n+1}\|^2 \leq \|x_n - P_0x_n\|^2 + M_2 \rho_X (\lambda_n),
\]
for some constant \( M_2 > 0 \), since \( \|x_n - P_0x_n\| \) is bounded. By Lemma 1.2 we see that \( \lim_{n \to +\infty} \|x_n - P_0x_n\| \) exists. Furthermore,
\[
2 \sum_{n=0}^{\infty} \lambda_n < Tx_n, j_n > \leq \|x_0 - P_0x_0\|^2 + M_2 \sum_{n=0}^{\infty} \rho_X (\lambda_n) < +\infty.
\]

Multiplying by \( \lambda_n \) both sides of (2.2), we obtain
\[
\lambda_n < Tx_n, j_n > \geq \psi (\|x_n - P_0x_n\|) \lambda_n \|Tx_n\|.
\]
Using (2.1), we have
\[
\lambda_n < Tx_n, j_n > \geq \psi (\|x_n - P_0x_n\|) \|x_{n+1} - x_n\|.
\]
It follows that
\[
\sum_{n=0}^{\infty} \lambda_n < Tx_n, j_n > \geq \sum_{n=0}^{\infty} \psi (\|x_n - P_0x_n\|) \|x_{n+1} - x_n\|.
\]
Hence
\[
\sum_{n=0}^{\infty} \psi (\|x_n - P_0x_n\|) \|x_{n+1} - x_n\| < \infty.
\]

Now we consider the following two possible cases:
Case 1. \( \lim \inf_{n \to \infty} \psi(\|x_n - P_0x_n\|) = 0. \)

Since \( \psi : R^+ \to R^+ \) is strictly increasing, we have \( \lim \inf_{n \to \infty} \|x_n - P_0x_n\| = 0. \) Since \( \lim_{n \to \infty} \|x_n - P_0x_n\| \) exists, we get \( \lim_{n \to \infty} \|x_n - P_0x_n\| = 0. \) On the other hand, by (2.3), we have

\[
\|x_n - P_0x_k\|^2 \leq \|x_k - P_0x_k\|^2 + M_2 \sum_{i=k}^{\infty} \rho_X(\lambda_i), \text{ for all } n > k.
\]

Consequently, \( \|x_n - x_k\| \leq \|x_n - P_0x_k\| + \|P_0x_k - x_k\| \to 0, \) as \( k \to \infty, n \to \infty. \) Hence \( \{x_n\} \) must be a Cauchy sequence. Let \( \lim_{n \to \infty} x_n = z. \) Then \( P_0x_n \to z \)
as \( n \to \infty \) because \( \|x_n - P_0x_n\| \to 0 \) as \( n \to \infty. \)

Note that the closedness of \( N_0(T) \) and \( \{P_0x_n\} \subset N_0(T) \) imply \( z \in N_0(T). \)

Case 2. \( \sum_{n=0}^{\infty} \|x_{n+1} - x_n\| < \infty. \)

Observing that \( \|x_{n+m} - x_n\| \leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \to 0 \) as \( n, m \to \infty, \) we assert that \( \{x_n\} \) is a Cauchy sequence. Assume that \( x_n \to x \) as \( n \to \infty. \)

By (2.5) we know that \( Tx_n \to 0 \) as \( n \to \infty. \) Hence \( x \in N_0(T), \) since \( T \) is demiclosed.

The proof is complete. \( \blacksquare \)

Remark 1. In [6, p. 89] Reich considered a continuous nondecreasing function \( b : [0, \infty) \to [0, \infty) \) such that \( b(0) = 0, b(ct) \leq cb(t) \) for \( c \geq 1. \) In [7, p. 337] he established a relationship between the function \( b \) and the modulus of smoothness of the Banach space \( X. \) Since any map satisfying the convergence condition introduced in [5] certainly satisfies condition (2.2), and the condition \( \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty \) in [5] implies the condition \( \sum_{n=0}^{\infty} \rho_X(\lambda_n) < \infty, \) we see that our Theorem 2.1 generalizes the strong convergence results in [5, Theorem 3], [3, Theorem 3.1] and others.

Chidume [4] proved the following theorem:

**Theorem 1 of [4].** Let \( X \) be a real Banach space with a uniformly convex dual space, \( X^*. \) Suppose that \( T : X \to X \) is a continuous strongly accretive map such that \( (I - T) \) has bounded range. For a given \( f \in X, \) define \( S : X \to X \) by \( Sx = f - Tx + x \) for each \( x \in X. \) Consider the sequence \( \{x_n\}_{n=0}^{\infty} \) defined iteratively by \( x_0 \in X \) and

\[
x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Sx_n,
\]

for \( n \geq 0, \) where \( \{\lambda_n\}_{n=0}^{\infty} \) is a real sequence satisfying the following:

(i) \( 0 < \lambda_n \leq 1 \) for all \( n \geq 0; \)

(ii) \( \sum_{n=0}^{\infty} \lambda_n = \infty; \)

(iii) \( \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty. \)

Then the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the solution of \( Tx = f. \)

We have the following theorem:
Theorem 2.2. Theorem 1 of Chidume [4] is a corollary of Theorem 2.1 above.

Proof. Set $A = T - f$, for any given $f \in X$. Under the assumptions of Chidume [4, Theorem 1], $N_0(A) = \{q\}$, where $q$ is the unique solution to $Tx = f$. Observe that

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n(f - Tx_n + x_n)$$

(2.6)

$$= x_n - \lambda_n x_n + \lambda_n f - \lambda_n T x_n + \lambda_n x_n$$

$$= x_n - \lambda_n Ax_n,$$

and

$$Ax_n = Tx_n - f = x_n - (x_n - Tx_n + f).$$

Since $\{x_n - Tx_n\} \subset R(I - T)$ is bounded, the only thing we need to do is to verify the boundedness of $\{x_n\}$. We consider the two possible cases:

Case 1. There exists an $m_0 \geq 0$ such that $\|x_{m_0} - q\| \leq 1$.

We let $M_3 = \sup\{\|f + x_n - Tx_n\| \mid n \geq 0\}$, $M_4 = \max\{1, 2M_3\}$. By (2.6) we have

$$\|x_{n+1} - q\| \leq (1 - \lambda_{m_0}) + 2\lambda_{m_0} M_3 \leq M_4,$$

and, by induction, we find

$$\|x_{m_0 + m} - q\| \leq M_4 \text{ for all } m \geq 1.$$}

This shows that $\{x_n\}$ is bounded.

Case 2. For all $n \geq 0$, $\|x_n - q\| > 1$.

We shall show that this case is impossible.

Since $T$ is strongly accretive, so is $A$. Thus there exists some constant $k \in (0, 1)$ such that

$$(2.7) \quad < Ax_n, J(x_n - q) > \geq k \|x_n - q\|^2.$$}

By using [10, Lemma 1.1] and (2.7) we have

$$\|x_{n+1} - q\|^2$$

$$\leq \|x_n - q\|^2 - 2\lambda_n < Ax_n, J(x_{n+1} - q) >$$

$$= \|x_n - q\|^2 - 2\lambda_n < Ax_n, J(x_n - q) >$$

$$- 2\lambda_n < Ax_n, J(x_{n+1} - q) - J(x_n - q) >$$

$$\leq \|x_n - q\|^2 - 2\lambda_n k \|x_n - q\|^2$$

$$- 2\lambda_n < Ax_n \frac{x_{n+1} - q}{\|x_n - q\|}, \frac{x_n - q}{\|x_n - q\|} - J \frac{x_n - q}{\|x_n - q\|} > \|x_n - q\|^2$$

$$= (1 - 2\lambda_n k - 2\lambda_n a_n) \|x_n - q\|^2,$$

where $a_n = < \frac{Ax_n}{\|x_n - q\|}, J \frac{x_{n+1} - q}{\|x_n - q\|} - J \frac{x_n - q}{\|x_n - q\|} >$.

Now, we want to show that $a_n \to 0$ as $n \to \infty$. It follows from $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$ that $\lambda_n \to 0$ as $n \to \infty$, $\frac{Ax_n}{\|x_n - q\|} \leq 1 + M_3 + \|q\|$ and

$$\frac{x_{n+1} - q}{\|x_n - q\|} - \frac{x_n - q}{\|x_n - q\|} = -\lambda_n \frac{Ax_n}{\|x_n - q\|} \to 0 \text{ as } n \to \infty.$$
Hence we have
\[
J \frac{x_{n+1} - q}{\|x_n - q\|} - J \frac{x_n - q}{\|x_n - q\|} \to 0 \text{ as } n \to \infty,
\]
since \( J \) is uniformly continuous on bounded subset of \( X \). Consequently, \( a_n \to 0 \) as \( n \to \infty \).

Now, we may choose \( n_1 \geq 0 \) such that for every \( n \geq n_1, k + 2a_n > 0 \). Thus we have
\[
\|x_{n+1} - q\|^2 \leq (1 - k\lambda_n)\|x_n - q\|^2
\]
\[
\leq \exp\{-\sum_{j=0}^{n} k\lambda_j\}\|x_0 - q\|^2 \to 0 \text{ as } n \to \infty,
\]
which contradicts with the assumption that for all \( n \geq 0, \|x_n - q\| > 1 \).

**Remark 2.** In Theorems 2.1 and 2.2, all assumptions are satisfied except the boundedness of \( \{Tx_n\} \) and \( R(I - T) \) which are replaced by the boundedness of \( T \), then the conclusions of Theorems 2.1 and 2.2 hold true. See Xu and Roach [8], and authors [9].

The next result extends [5, Theorem 3] to the case of an Ishikawa iterative process. Namely, we consider the following Ishikawa process:

\[
(IS) \begin{cases} 
    x_{n+1} = x_n - \alpha_nAy_n - \alpha_n\beta_nAx_n, \\
    y_n = x_n - \beta_nAx_n, \quad n \geq 0. 
\end{cases}
\]

**Theorem 2.3.** Let \( A : X \to X \) be a demiclosed quasi-accretive operator. Assume that there exists a strictly increasing function \( \psi : R^+ \to R^+ \), \( \psi(0) = 0 \), such that
\[
(2.9) \quad < Ay_n, J(y_n - P_0y_n) > \geq \psi(\|y_n - P_0y_n\|)\|Ay_n\|, \quad n \geq 0.
\]

Furthermore, assume that the following conditions are satisfied:

(H1) \( 0 < \alpha_n < 1, 0 \leq \beta_n < 1 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \alpha_n\beta_n < \infty \);

(H2) \( \sup\{\|Ax_n\|; n \geq 0\} < \infty \) and \( \sup\{\|Ay_n\|; n \geq 0\} < \infty \);

(H3) \( \sum_{n=0}^{\infty} ((J(x_n - P_0x_n) - J(y_n - P_0y_n)) < \infty \) and \( \sum_{n=0}^{\infty} \rho X(\alpha_n) < \infty \);

(H4) \( \|P_0x_n - P_0y_n\| \to 0 \) as \( n \to \infty \).

Then \( \{x_n\} \), defined by (IS), converges strongly to an element of \( N(A) \).

**Proof.** Set \( j(x_n) = J(x_n - P_0x_n), j(y_n) = J(y_n - P_0y_n), \) \( c_1 = \sup\{\|Ax_n\|; n \geq 0\}, \) and \( c_2 = \sup\{\|Ay_n\|; n \geq 0\} \).
Using Lemma 1.3 and (IS) we have

\[
\|x_{n+1} - P_0x_{n+1}\|^2 \leq \|x_{n+1} - P_0x_n\|^2
\]

\[
= \|x_n - \alpha_0Ay_n - \alpha_0\beta_nAx_n - P_0x_n\|^2
\]

\[
\leq \|x_n - P_0x_n\|^2 - 2\alpha_0 < Ay_n, J(x_n - P_0x_n) >
\]

\[
- 2\alpha_0\beta_n < Ax_n, J(x_n - P_0x_n) >
\]

\[
+ k_1 \text{max}\{\|x_n - P_0x_n\| + \alpha_n\|Ay_n\| + \alpha_n\beta_n\|Ax_n\|, \frac{c}{2}\} \cdot \rho_X(\alpha_n\|Ay_n\| + \alpha_n\beta_n\|Ax_n\|)
\]

\[
\leq \|x_n - P_0x_n\|^2
\]

\[
- 2\alpha_0 < Ay_n, J(x_n - P_0x_n) - J(y_n - P_0y_n) >
\]

\[
- 2\alpha_0 < Ay_n, J(y_n - P_0y_n) >
\]

\[
+ k_1 \text{max}\{\|x_n - P_0x_n\| + \alpha_n(c_1 + c_2), \frac{c}{2}\}\rho_X(\alpha_n)
\]

\[
\leq \|x_n - P_0x_n\|^2 - 2\alpha_0b_n - 2\alpha_n\psi(||y_n - P_0y_n||)||Ay_n||
\]

\[
+ k_1 \text{max}\{\|x_n - P_0x_n\| + \alpha_n(c_1 + c_2), \frac{c}{2}\}\rho_X(\alpha_n)
\]

where \(k_1\) is some positive constant and

\[
b_n = < Ay_n, J(x_n - P_0x_n) - J(y_n - P_0y_n) >
\]

Here we have used the fact that \(\rho_X(\tau)\) is nondecreasing and there exists some constant \(c_0 > 0\) such that \(\frac{\rho_x(\eta)}{\eta^2} \leq \frac{\alpha\rho_X(\tau)}{\tau^2}\), for all \(\eta \geq \tau > 0\). Arguing as in the proof of Theorem 2.1, we can show that \(\|x_n - P_0x_n\|\) is bounded and \(\lim_{n \to \infty} \|x_n - P_0x_n\|\) exists. From (2.10) we see that

\[
\sum_{n=0}^{\infty} \alpha_n\psi(||y_n - P_0y_n||)||Ay_n|| < \infty.
\]

Now, we consider the following two possible cases:

Case 1. \(\lim_{n \to \infty} \inf \psi(||y_n - P_0y_n||) = 0\).

In this case, by the properties of \(\psi\), we see that \(\lim_{n \to \infty} \|y_n - P_0y_n\| = 0\). Assumption (H1) implies \(\lim_{n \to \infty} \inf \beta_n = 0\). Without any loss of generality, we assume that \(\beta_n \to 0\) as \(n \to \infty\). Then \(y_n - x_n = -\beta_nAx_n \to 0\) as \(n \to \infty\). By (H4), we have \(\lim_{n \to \infty} \|x_n - P_0x_n\| = 0\). Consequently, \(\lim_{n \to \infty} \|x_n - P_0x_n\| = 0\) since \(\lim_{n \to \infty} \|x_n - P_0x_n\|\) exists. Arguing as in the proof of Theorem 2.1, we can prove that \(x_n \to x\) as \(n \to \infty\). Hence \(x \in N(A)\).

Case 2. \(\sum_{n=0}^{\infty} \alpha_n\|Ay_n\| < \infty\).

In this case, by (H1) and (IS), we have

\[
\sum_{n=0}^{\infty} \|x_{n+1} - x_n\| \leq \sum_{n=0}^{\infty} \alpha_n\|Ay_n\| + c_1 \sum_{n=0}^{\infty} \alpha_n\beta_n < \infty,
\]
and hence \( \{x_n\} \) must be Cauchy. Assume that \( x_n \to z \) as \( n \to \infty \). Then \( y_n \to z \) as \( n \to \infty \). On the other hand, \( \sum_{n=0}^{\infty} \alpha_n \|Ay_n\| < \infty \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \) implies \( \lim_{n \to \infty} \inf \|Ay_n\| = 0 \). Therefore, \( z \in N(A) \) since \( A \) is demiclosed.

**Remark 3.** If we take \( \beta_n \equiv 0 \), then (IS) becomes \( x_{n+1} = x_n - \alpha_n Ax_n \), \( n \geq 0 \). In this case, conditions \( \sum_{n=0}^{\infty} \alpha_n \beta_n < \infty \), \( \sum_{n=0}^{\infty} (J(x_n - P_0 x_n) - J(y_n - P_0 y_n)) < \infty \) and \( \|P_0 x_n - P_0 y_n\| \to 0 \) as \( n \to \infty \) are satisfied trivially.

**Remark 4.** It is easy to see that our Theorem 2.3 works for the case that \( A \) is multi-valued.

**Remark 5.** We don’t know whether the assumptions \( \sum_{n=0}^{\infty} (J(x_n - P_0 x_n) - J(y_n - P_0 y_n)) < \infty \) and (H4) can be removed. It is also interesting to discuss the relations between Theorem 2.3 and Chidume [4, Theorem 2].

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