ATTRACTORS OF SEMIGROUPS ASSOCIATED WITH NONLINEAR SYSTEMS FOR DIFFUSIVE PHASE SEPARATION

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Abstract. We consider a model for diffusive phase transitions, for instance, the component separation in a binary mixture. Our model is described by two functions, the absolute temperature \( \theta := \theta(t,x) \) and the order parameter \( w := w(t,x) \), which are governed by a system of two nonlinear parabolic PDEs. The order parameter \( w \) is constrained to have double obstacles \( \sigma_* \leq w \leq \sigma^* \) (i.e., \( \sigma_* \) and \( \sigma^* \) are the threshold values of \( w \)). The objective of this paper is to discuss the semigroup \( \{S(t)\} \) associated with the phase separation model, and construct its global attractor.

1. Introduction

This paper is concerned with a system of nonlinear parabolic PDEs of the form, referred to as (PSC),

(1.1) \[ [\rho(u) + \lambda(w)]_t - \Delta u + \nu \rho(u) = f(x) \text{ in } Q := (0, +\infty) \times \Omega, \]

(1.2) \[ w_t - \Delta \{-\kappa \Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \text{ in } Q, \]

(1.3) \[ \xi \in \beta(w) \text{ in } Q, \]

(1.4) \[ \frac{\partial u}{\partial n} + n_o u = h(x) \text{ on } \Sigma := (0, +\infty) \times \Gamma, \]

(1.5) \[ \frac{\partial w}{\partial n} = 0, \quad \frac{\partial}{\partial n} \{-\kappa \Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \text{ on } \Sigma, \]

(1.6) \[ u(0,\cdot) = u_o, \quad w(0,\cdot) = w_o \text{ in } \Omega. \]

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (\( 1 \leq N \leq 3 \)) with smooth boundary \( \Gamma := \partial \Omega \); \( \rho \) is an increasing function such as \( \rho(u) = -\frac{1}{u} \) for \( -\infty < u < 0; \)

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\( \lambda, \ g \) are smooth functions on \( \mathbb{R} \) and \( \lambda' \) is the derivative of \( \lambda \); \( \beta \) is a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with bounded domain \( D(\beta) \) in \( \mathbb{R} \); \( \kappa > 0, \ n_o > 0 \) and \( \nu \geq 0 \) are constants; \( f, \ h, \ u_o, \ w_o \) are prescribed data.

This system arises in the non-isothermal diffusive phase separation in a binary mixture. In such a context, \( \theta := \rho(u) \) is the (absolute) temperature and \( w \) is the local concentration of one of the components; physically, (1.1) is the energy balance equation, where \( \rho(u) + \lambda(w) \) is the internal energy, and (1.2) is the mass balance equation with constraint (1.3) for \( w \), where \(-\kappa \Delta w + \xi + g(w) - \lambda'(w)u \) can be interpreted as the (generalized) chemical potential difference. The details of modeling are referred, for instance, to [1, 2, 7, 10, 12].

In the one-dimensional case, i.e. \( N = 1 \), the existence and uniqueness of a global solution of (PSC) was proved in [10], and in [14] for the case without constraint (1.3). In the higher dimensional case \( (N = 2 \text{ or } 3) \), any uniqueness result has not been noticed in the general setting; for a model in which the mass balance equation includes a viscosity term \(-\mu \Delta w \), the uniqueness was obtained in [10]. Recently, in [6] a uniqueness result was established in a very wide space of distributions under the additional assumption that

\[
(1.7) \quad \lambda \text{ is convex on } \overline{D(\beta)} \text{ and } D(\rho) \subset (-\infty, 0].
\]

So far as the large time behaviour of solutions is concerned, we have noticed a few papers (e.g. [5, 9, 10, 14]) including some results about the \( \omega \)-limit set of each single solution as time \( t \) goes to \(+\infty \), but no results, except [12], on attractors so far for non-isothermal phase separation models; in [12] the regular case of \( \rho \) was treated, so this result is not applicable to (PSC) including a singular function \( \rho \).

In this paper, assuming (1.7), we shall give a new existence result for problem (PSC) with initial data \([u_o, w_o]\) in a larger class than that in [10]. Also, based on our existence result, we shall consider a semigroup \( \{S(t)\}_{t \geq 0} \) consisting of operators \( S(t) \) which assign to each initial data \([u_o, w_o]\) the element \([u(t), w(t)], \{u, w]\) being the solution. Moreover we shall construct the global attractor for \( \{S(t)\} \) in the product space \( L^2(\Omega) \times H^1(\Omega) \). Unfortunately, the mapping \( t \to S(t)[u_o, w_o] \) lacks the continuity at \( t = 0 \) in \( L^2(\Omega) \times H^1(\Omega) \) for bad initial data \([u_o, w_o]\), which comes from the singularity of \( \rho(u) \). Therefore the general theory on attractors (cf. [4, 17]) cannot be directly applied to our case. However, the construction of the global attractor will be done by introducing a Lyapunov-like functional and by appropriately modified versions of some results in [4, 17]. Especially, the term \( \nu \rho(u) \) with positive \( \nu \) in (1.1) is very important in order to find an absorbing set.

**Notation.** In general, for a (real) Banach space \( W \) we denote by \( | \cdot |_W \) the norm and by \( W^* \) its dual space endowed with the dual norm. For any compact time interval \([t_o, t_1]\) we denote by \( C_w([t_o, t_1]; W) \) the space of all weakly continuous functions from \([t_o, t_1]\) into \( W \), and mean by \( "u_n \to u \) in \( C_w([t_o, t_1]; W) \) as \( n \to +\infty \) that for each \( z^* \in W^*, \ \langle z^*, u_n(t) - u(t) \rangle_{W^*, W} \to 0 \) uniformly on \([t_o, t_1]\) as \( n \to +\infty \), where \( \langle \cdot, \cdot \rangle_{W^*, W} \) stands for the duality pairing between \( W^* \) and \( W \).
For two real valued functions \( u, v \) we define
\[
    u \land v := \min\{u, v\}, \quad u \lor v := \max\{u, v\}.
\]

Throughout this paper, let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) \((1 \leq N \leq 3)\) with smooth boundary \( \Gamma := \partial \Omega \), and for simplicity fix some notation as follows:

\[
    H := L^2(\Omega), \quad H_o := \left\{ z \in L^2(\Omega) ; \int_\Omega z\, dx = 0 \right\},
\]
\[
    V := H^1(\Omega), \quad V_o := \left\{ z \in H^1(\Omega) ; \int_\Omega z\, dx = 0 \right\}.
\]

\((\cdot, \cdot)\) : inner product in \( H \).
\(<\cdot, \cdot>_o\) : duality pairing between \( V^*_o \) and \( V_o \).
\(<\cdot, \cdot>_{\Gamma}\) : inner product in \( L^2(\Gamma) \).

\[
    a(v, w) := \int_\Omega \nabla v \cdot \nabla w\, dx \quad \text{for} \quad v, w \in V.
\]

\( \pi_o \) : projection from \( H \) onto \( H_o \), i.e. \( [\pi_o z](x) := z(x) - \frac{1}{|\Omega|} \int_\Omega z(y)\, dy, \quad z \in H. \)

Also, we define \( |\cdot|_V \) and \( |\cdot|_{V_o} \) by
\[
    |v|_V := \left\{ \int_\Omega |\nabla v|^2\, dx + n_o \int_{\Gamma} v^2\, d\Gamma \right\}^{1/2}, \quad v \in V,
\]
and
\[
    |v|_{V_o} := \left\{ \int_\Omega |\nabla v|^2\, dx \right\}^{1/2}, \quad v \in V_o;
\]

clearly we have standard relations
\[
    V \subset H \subset V^*, \quad V_o \subset H_o \subset V^*_o,
\]
in which all the injections are compact and densely defined. Associated with the above norms, the duality mappings \( F : V \to V^* \) and \( F_o : V_o \to V^*_o \) are defined in the following manner:

\[
    \langle Fv, z \rangle = a(v, z) + n_o(v, z)_{\Gamma} \quad \text{for} \quad v, z \in V,
\]
\[
    \langle F_o v, z \rangle_o = a(v, z) \quad \text{for} \quad v, z \in V_o.
\]

Clearly, if \( \ell := Fv \in H \), then \( v \in H^2(\Omega) \) and it is a unique solution of
\[
    \begin{cases}
        -\Delta v = \ell \text{ in } \Omega, \\
        \frac{\partial v}{\partial n} + n_o v = 0 \text{ on } \Gamma;
    \end{cases}
\]
if \( \ell := F_{o}v \in H_{o} \), then \( v \in H^{2}(\Omega) \) and it is a unique solution of
\[
\begin{aligned}
-\Delta v &= \ell \text{ in } \Omega, \\
\frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma, \quad \int_{\Omega} v dx = 0.
\end{aligned}
\]

2. Existence and Uniqueness Result for (PSC)

Throughout this paper we suppose that \( \rho, \beta, \lambda, g, \kappa, n_{o} \) and \( \nu \) satisfy the following hypotheses (H1) - (H5):

: (H1) \( \rho \) is a single-valued maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \), its domain \( D(\rho) \) and range \( R(\rho) \) are open in \( \mathbb{R} \) and it is locally bi-Lipschitz continuous as a function from \( D(\rho) \) onto \( R(\rho) \); we denote by \( \rho^{-1} \) the inverse of \( \rho \) and by \( \rho^{-1} \) a proper l.s.c. convex function on \( \mathbb{R} \) whose subdifferential coincides with \( \rho^{-1} \) in \( \mathbb{R} \).

: (H2) \( \beta \) is a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) which is the subdifferential of a non-negative proper l.s.c. convex function \( \hat{\beta} \) on \( \mathbb{R} \) such that
\[
D(\hat{\beta}) = [\sigma_{s}, \sigma^{*}]
\]
for finite numbers \( \sigma_{s}, \sigma^{*} \) with \( \sigma_{s} < \sigma^{*} \).

: (H3) \( \lambda : \mathbb{R} \rightarrow \mathbb{R} \) is of \( C^{2} \)-class, convex on \( [\sigma_{s}, \sigma^{*}] \) and
\[
\lambda''(w)u \leq 0 \text{ for all } w \in [\sigma_{s}, \sigma^{*}] \text{ and } u \in D(\rho).
\]

: (H4) \( g : \mathbb{R} \rightarrow \mathbb{R} \) is of \( C^{2} \)-class; we denote a primitive of \( g \), which is non-negative on \( [\sigma_{s}, \sigma^{*}] \), by \( \hat{g} \).

: (H5) \( \kappa > 0, n_{o} > 0 \) and \( \nu \geq 0 \) are constants.

Now we give a variational formulation for (PSC).

**Definition 2.1.** Let \( f \in H, h \in L^{2}(\Gamma), u_{o} \) be a measurable function on \( \Omega \) with \( \rho(u_{o}) \in H \) and \( w_{o} \in V \) with \( \hat{\beta}(w_{o}) \in L^{1}(\Omega) \). Then, for any finite \( T > 0 \), a couple \{ \( u, w \) \} of functions \( u : [0, T] \rightarrow V \) and \( w : [0, T] \rightarrow H^{2}(\Omega) \) is called a (weak) solution of (PSC):=(PSC;\( f, h, u_{o}, w_{o} \)) on \([0, T]\), if the following conditions (w1) - (w4) are satisfied.

: (w1) \( u \in L^{2}(0, T; V), \rho(u) \in C_{w}([0, T]; H), \rho(u)' \in L^{1}(0, T; V^{*}), \)
\[
w \in L^{2}(0, T; H^{2}(\Omega)) \cap C_{w}([0, T]; V) \text{ with } \hat{\beta}(w) \in L^{\infty}(0, T; L^{1}(\Omega)), \ w' \in L^{2}(0, T; V^{*}) \text{ and } \lambda(w)' \in L^{1}(0, T; V^{*}).
\]

: (w2) \( \rho(u)(0) = \rho(u_{o}) \) and \( w(0) = w_{o} \).

: (w3) For a.e. \( t \in [0, T] \) and all \( z \in V, \)
\[
\frac{d}{dt}(\rho(u(t)) + \lambda(w(t)), z) + a(u(t), z)
\]
\[
+ (n_{o}u(t) - h, z)_{\Gamma} + \nu(\rho(u(t)), z)
\]
\[
= (f, z).
\]

: (w4) \( \frac{\partial w(t)}{\partial n} = 0 \) a.e. on \( \Gamma \) for a.e. \( t \in [0, T] \), and there is a function \( \xi \in L^{2}(0, T; H) \) such that \( \xi(t) \in \hat{\beta}(w(t)) \) a.e. on \( \Omega \) for a.e. \( t \in [0, T] \).
and
\begin{equation}
\frac{d}{dt}(w(t,\eta) + \kappa(\Delta w(t),\Delta \eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t),\Delta \eta) = 0
\end{equation}
for all $\eta \in H^2(\Omega)$ with $\frac{\partial \eta}{\partial n} = 0$ a.e. on $\Gamma$ and for a.e. $t \in [0,T]$.

A couple $\{u,w\}$ of functions $u: \mathbb{R}_+ \to V$ and $w: \mathbb{R}_+ \to H^2(\Omega)$ is called a global solution of (PSC) (or a solution of (PSC) on $\mathbb{R}_+$), if it is a solution of (PSC) on $[0,T]$ for every finite $T > 0$.

As easily understood from the above definition, since $\sigma_* \leq w \leq \sigma^*$ for any solution $\{u,w\}$ of (PSC), the behaviour of $g$, $\lambda$ on the outside of $[\sigma_*,\sigma^*]$ gives no influence to the solution and we may assume without loss of generality that the support of $g$ is compact in $\mathbb{R}$ and $\lambda$ is linear on the outside of $[\sigma_*,\sigma^*]$.

**Remark 2.1.** From the above definition we easily observe (1)-(3) below.

1. For any solution $\{u,w\}$ of (PSC) on $[0,T]$ we see that
$$\frac{d}{dt} \int_\Omega w(t)dx = 0$$
for a.e. $t \in [0,T]$, so that
$$\frac{1}{|\Omega|} \int_\Omega w(t)dx = \frac{1}{|\Omega|} \int_\Omega w_0dx =: m_o$$
for all $t \in [0,T]$.

This implies that $w - m_o \in C_w([0,T];V_0)$ and $w' \in L^2(0,T;V_o^*)$.

2. In terms of the duality mapping $F: V \to V^*$ the variational identity (2.2) is written in the form
$$\frac{d}{dt}(\rho(u(t)) + \lambda(w(t))) + Fu(t) + \nu\rho(u(t)) = f^*$$
in $V^*$ for a.e. $t \in [0,T]$, where $f^* \in V^*$ is given by
$$\langle f^*, z \rangle = (f, z) + (h, z)_\Gamma$$
for all $z \in V$.

3. In terms of the duality mapping $F_o: V_o \to V_o^*$ variational identity (2.3) is written in the form
$$F_o^{-1}w'(t) + \kappa F_o(\pi_o w(t)) + \pi_o[\xi(t) + g(w(t)) - \lambda'(w(t))u(t)] = 0$$
in $H_o$ for a.e. $t \in [0,T]$.

We now introduce some functions and spaces in order to formulate an existence-uniqueness result. Let $u^\infty$ be the unique solution of
\begin{equation}
\begin{cases}
u \in V; \\
ap(u^\infty,z) + (n_o u^\infty - h, z)_\Gamma + \nu(\rho(u^\infty),z) = (f, z) \text{ for all } z \in V;
\end{cases}
\end{equation}
clearly (2.7) has one and only one solution $u^\infty \in V$ for given $f \in H$ and $h \in L^2(\Gamma)$. If $\nu > 0$, then
\begin{equation}
\rho(u^\infty) \in H.
\end{equation}
In case of $\nu = 0$ we suppose (2.8) holds.
Next we define a functional $J(\cdot, \cdot)$ on the set $\rho^{-1}(H) \times V := \{[z, v]; \rho(z) \in H, v \in V\}$, by putting

$$J(z, v) := J_o(z, v) + J_1(z, v)$$

with

$$J_o(z, v) := \varepsilon_o|\rho(z) + \lambda(v)|_H^2$$

and

$$J_1(z, v) := \int_\Omega \rho^{-1}(\rho(z))dx - (\rho(z) + \lambda(v), u^\infty)$$

$$+ \frac{k}{2} |v|_H^2 + \int_\Omega (\hat{\beta}(v) + \hat{g}(v))dx + C_o,$$

where $\varepsilon_o$ is a (small) positive number determined later and $C_o$ is a constant so that $J_1(\cdot, \cdot)$ is non-negative; in fact, such a constant $C_o$ exists, since

$$\rho^{-1}(r) - ru^\infty(x) \geq \rho^{-1}(\rho(u^\infty(x))) - \rho(u^\infty(x))u^\infty(x)$$

for all $r \in \mathbb{R}$ and a.e. $x \in \Omega$. With the functional $J_1$ and a number $m_o$ with $\sigma_* \leq m_o \leq \sigma^*$, we put

$$(2.11) \quad D(m_o) := \{[z, v] \in \rho^{-1}(H) \times V; J_1(z, v) < +\infty, \frac{1}{|\Omega|} \int_\Omega vx = m_o\}$$

and

$$(2.12) \quad D^M(m_o) := \{[z, v] \in D(m_o); J_1(z, v) \leq M\} \text{ for each } M > 0.$$ 

Also, for a number $m_o$ with $\sigma_* \leq m_o \leq \sigma^*$, we put

$$(2.13) \quad D_o(m_o) := \left\{ \begin{array}{l} z \in V, \rho(z) \in H, v \in H^2(\Omega), v - m_o \in V_o, \\
[z, v]; \frac{\partial v}{\partial n} = 0 \text{ a.e. on } \Gamma, \text{ there is } \xi \in H \text{ such that} \\
\xi \in \beta(v) \text{ a.e. on } \Omega, -\kappa \Delta v + \xi \in V \end{array} \right\},$$

$$(2.14) \quad D^M_o(m_o) := \{[z, v] \in D_o(m_o); J_1(z, v) \leq M\} \text{ for each } M > 0.$$ 

Clearly, $D^M_o(m_o) \subset D^M(m_o)$, $D(m_o) = \cup_{M > 0} D^M(m_o)$ and $D_o(m_o) = \cup_{M > 0} D^M_o(m_o)$.

First we recall the following theorem which guarantees the uniqueness of the solution of (PSC).

**Theorem 2.1.** ([5; Theorem 2.1]) Assume that (H1)-(H5) hold, and let $f \in H$, $h \in L^2(\Gamma)$ and $m_o$ be a number with $\sigma_* \leq m_o \leq \sigma^*$. Let $[u_{o_i}, w_{o_i}], i = 1, 2$, be initial data in $D(m_o)$, and $\{u_i, w_i\}$ be any solution of (PSC)$_i := (PSC; f, h, u_{o_i}, w_{o_i})$ on $[0, T]$ for $i = 1, 2$. Then, with notation $e_i(t) := \rho(u_i(t)) + \lambda(w_i(t))$ for $t \in [0, T]$, and for all $s$, $t \in [0, T], s \leq t$,

$$|e_1(t) - e_2(t)|_{\nu_s}^2 + |w_1(t) - w_2(t)|_{\nu_s}^2$$

$$\leq e^{R_o(t-s)}(|e_1(s) - e_2(s)|_{\nu_s}^2 + |w_1(s) - w_2(s)|_{\nu_s}^2),$$

where $R_o := R_o(\kappa, n_o, \lambda, g)$ is a positive constant dependent only on $\kappa, n_o$ and the Lipschitz constants of $\lambda$ and $g$.  


Hypothesis (2.1) of (H3) is essential for the proof of inequality (2.15).

An existence result is stated as follows.

**Theorem 2.2.** Assume that (H1)-(H5) hold as well as (2.8), and let \( m_o \) be any number with \( \sigma_* < m_o < \sigma^* \). Also, let \( f \in H \) and \( h \in L^\infty(\Gamma) \). Assume that

\begin{align}
&n_o \sup D(\rho) \geq h(x) \geq n_o \inf D(\rho) \text{ for a.e. } x \in \Gamma \\
&\text{and there are constants } A_1, A_1' \text{ such that}
\end{align}

\begin{align}
&\rho(r)(n_o r - h(x)) \geq -A_1|r| - A_1' \text{ for a.e. } x \in \Gamma \text{ and all } r \in D(\rho).
\end{align}

Then, for each \([u_o, w_o] \in D(m_o)\), problem (PSC) admits a (unique) global solution \([u, w]\). Moreover, the following inequalities (2.18) - (2.20) hold.

\begin{align}
&J_1(u(t), w(t)) + \int_s^t |u(\tau) - u^\infty|_{V}^2 d\tau + \int_s^t |w'(\tau)|_{V_o}^2 d\tau \leq J_1(u(s), w(s))
\end{align}

for \( 0 \leq s \leq t \);

\begin{align}
&|\rho(u(t)) + \lambda(w(t))|_{H}^2 \leq M_1(T) \{|\rho(u(s)) + \lambda(w(s))|_{H}^2 + J_1(u(s), w(s)) + 1\}
\end{align}

for \( 0 \leq s \leq t \leq s + T \), where \( M_1(T) \) is an increasing function of \( T \in \mathbb{R}_+ \), independent of initial data \([u_o, w_o] \in D(m_o)\);

\begin{align}
&(t - s)\{|u(t) - u^\infty|_{V}^2 + |w'(t)|_{V_o}^2 + \nu|\int_{\Omega} \rho(u(t))dx\| \\
&\quad + \kappa \int_s^t (\tau - s)|w'(\tau)|_{V_o}^2 d\tau
\end{align}

\begin{align}
\quad \leq M_2(T) \{J_1(u(s), w(s)) + |\rho(u(s)) + \lambda(w(s)) - \rho(u^\infty)|_{V}^2 + 1\}
\end{align}

for all \( s \geq 0 \) and a.e. \( t \in [s, s + T] \), where \( M_2(T) \) is an increasing function of \( T \in \mathbb{R}_+ \) independent of initial data \([u_o, w_o] \in D(m_o)\).

**Remark 2.2.** From (2.18) and (2.20) of Theorem 2.2 we further derive an estimate of the form

\begin{align}
&(t - s)\{|w(t)|_{H^2(\Omega)}^2 + |\xi(t)|_{H}^2 \}
\end{align}

\begin{align}
\quad \leq M_3(T) \{J_1(u(s), w(s)) + |\rho(u(s)) + \lambda(w(s)) - \rho(u^\infty)|_{V}^2 + 1\}
\end{align}

for \( 0 \leq s < t \leq s + T \), where \( \xi \in L^2_{loc}(\mathbb{R}_+; H) \) is the function as in condition (w4) of Definition 2.1 and \( M_3(\cdot), i = 1, 2 \) is a function having the same properties as \( M_i(\cdot), i = 1, 2 \). In fact, since

\begin{align}
\kappa F_o(\pi_o w(t)) + \pi_o \xi(t) = -F_o^{-1}w'(t) + \pi_o [\lambda'(w(t))u(t) - g(w(t))] =: \ell(t),
\end{align}

it follows from a regularity result in [3] that

\begin{align}
|w(t)|_{H^2(\Omega)}^2 + |\xi(t)|_{H}^2 \leq C_1(|\ell(t)|_{H}^2 + 1) \text{ for a.e. } t \geq 0
\end{align}

with a constant \( C_1 \) independent of initial data \([u_o, w_o] \in D(m_o)\) and \( \ell \). Combining the above inequality with (2.18) and (2.20) we conclude an estimate of the form (2.21) for all \( 0 \leq s < t \leq s + T \).
Remark 2.3. Let \( \{u, w\} \) be the global solution of (PSC) which is given by Theorem 2.2. Then, we have by estimates (2.18) - (2.21) that

(i) \( u \) is a bounded and weakly continuous function from \( [\delta, +\infty) \) into \( V \) for each \( \delta > 0 \);

(ii) \( w \) is a bounded and weakly continuous function from \( [\delta, +\infty) \) into \( H^2(\Omega) \) for each \( \delta > 0 \);

(iii) \( \xi \) is a bounded function from \( [\delta, +\infty) \) into \( H \) for each \( \delta > 0 \), and satisfies that

\[ \xi(t) \in \beta(w(t)) \text{ a.e. on } \Omega \text{ for all } t > 0. \]

Also, we have

\[ (2.23) \quad [u(t), w(t)] \in D_o(m_o) \text{ for all } t > 0, \]

which is nothing else but the smoothing effect for solutions.

Remark 2.4. In case \( m_o = \sigma^* \) (resp. \( m_o = \sigma^- \)), it follows that \( w \equiv m_o \) for any solution \( \{u, w\} \) of (PSC) on \([0, T]\), since \( w \geq m_o \) (resp. \( w \leq m_o \)) and \( \int_\Omega w(t)dx = |\Omega|m_o \), and moreover \( u \) satisfies

\[ \langle \rho(u)'(t), z \rangle + a(u(t), z) + (n_o u(t) - h(t), z)_\Gamma + \nu(\rho(u(t)), z) = (f(t), z) \]

for all \( z \in H^1(\Omega) \) and a.e. \( t \in [0, T] \)

and \( u(0) = u_o \).

3. Approximate Problems and Estimates for Their Solutions

The solution of (PSC) will be constructed as a limit of solutions \( \{u_{\mu\varepsilon\eta}, w_{\mu\varepsilon\eta}\} \) of approximate problems (PSC)\(\mu\varepsilon\eta\), defined below, as \( \mu, \varepsilon, \eta \to 0 \); parameters \( \varepsilon, \eta \) concern with approximation \( \rho_{\varepsilon \eta} \) of function \( \rho \), while parameter \( \mu \) concerns with the coefficient of viscosity term, i.e. \( -\mu \Delta w_t \), added in the mass balance equation.

The main idea for approximation is found in [7, 10, 15], and uniform estimates for approximate solutions with respect to parameters are quite similar to those in the above cited papers. Therefore, we mention very briefly some estimates for approximate solutions. In the rest of this section, we make all the assumptions of Theorem 2.2 as well as (2.4).

If \( \lambda \) is linear, i.e. \( \lambda'' = 0 \) on \([\sigma^*, \sigma^-]\), then the proof of Theorem 2.2 is very simple. Therefore, in the rest of this section, we assume that

\[ (3.1) \quad \lambda'' > 0 \text{ somewhere on } [\sigma^*, \sigma^-], \quad D(\rho) \subset (-\infty, 0). \]

Given real parameters \( \varepsilon, \eta \in (0, 1] \), we consider approximations \( \rho_{\varepsilon} \) and \( \rho_{\varepsilon \eta} \) of \( \rho \) as follows. Putting

\[ D(\rho) = (r_*, r^*) \text{ for } -\infty \leq r_* < r^* \leq 0, \]

choose two families of numbers \( \{a_{\varepsilon}; 0 \leq \varepsilon \leq 1\} \) with \( a_o = r_* \) and \( \{b_{\eta}; 0 \leq \eta \leq 1\} \) with \( b_o = r^* \) such that

\[ r_* < a_{\varepsilon} < a_{\varepsilon'} < a_1 < r_o < b_1 < b_{\eta} < b_{\eta'} < r^* \]
if $0 < \varepsilon < \varepsilon' < 1$ and $0 < \eta' < \eta < 1$, where $r_o$ is a fixed number in $D(\rho)$, and

$$a_{\varepsilon} \downarrow r_* \text{ as } \varepsilon \to 0, \ b_{\eta} \uparrow r^* \text{ as } \eta \to 0.$$  

For each $\varepsilon$ and $\eta$ we define

$$\rho_{\varepsilon}(r) = \begin{cases} 
\rho(r) & \text{for } r \geq a_{\varepsilon}, \\
\rho(a_{\varepsilon}) + r - a_{\varepsilon} & \text{for } r < a_{\varepsilon}, 
\end{cases}$$

and

$$\rho_{\varepsilon\eta}(r) = \begin{cases} 
\rho(b_{\eta}) + r - b_{\eta} & \text{for } r > b_{\eta}, \\
\rho(r) & \text{for } a_{\varepsilon} \leq r \leq b_{\eta}, \\
\rho(a_{\varepsilon}) + r - a_{\varepsilon} & \text{for } r < a_{\varepsilon}. 
\end{cases}$$

Note that $\rho_{\varepsilon\eta}$ is bi-Lipschitz continuous on $\mathbb{R}$ and

$$\rho_{\varepsilon\eta} \to \rho_{\varepsilon} \text{ in the graph sense as } \eta \to 0 \text{ for each fixed } \varepsilon,$$

and moreover there is a positive constant $C(\varepsilon)$ for each $\varepsilon \in (0, 1]$ such that

$$\frac{d}{dr}\rho_{\varepsilon\eta}(r) \geq C(\varepsilon), \quad \frac{d}{dr}\rho_{\varepsilon}(r) \geq C(\varepsilon).$$

We write sometimes $\rho_o$ or $\rho_{oo}$ for $\rho$ and $\rho_{\varepsilon\eta}$ for $\rho_{\varepsilon}$.

Besides, for $\varepsilon, \eta \in [0, 1]$, let $u_{\varepsilon\eta}^\infty$ be the solution of

$$\begin{cases} 
\rho_{\varepsilon\eta}^\infty \in V; \\
\rho(u_{\varepsilon\eta}^\infty, z) + (n_o u_{\varepsilon\eta}^\infty - h, z)_\Gamma + \nu(\rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty), z) = (f, z) \text{ for all } z \in V.
\end{cases}$$

Clearly, $\{u_{\varepsilon\eta}^\infty\}$ is bounded in $V$, $\{\rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)\}$ is bounded in $H$,

$$u_{\varepsilon\eta}^\infty \to u_{\varepsilon0}^\infty \text{ in } V \text{ as } \eta \to 0 \text{ for each fixed } \varepsilon \in [0, 1],$$

and

$$\rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty) \to \rho(u_{\varepsilon0}^\infty) \text{ weakly in } H \text{ as } \eta \to 0 \text{ for each fixed } \varepsilon \in [0, 1], \text{ if } \nu > 0.$$  

Also, we define functionals $J_{1_{\varepsilon\eta}}(\cdot, \cdot)$ by

$$J_{1_{\varepsilon\eta}}(z, v) := \int_{\Omega} \rho_{\varepsilon\eta}^{-1}(\rho_{\varepsilon\eta}(z)) \, dx - (\rho_{\varepsilon\eta}(z) + \lambda(v), u_{\varepsilon\eta}^\infty)$$

$$+ \frac{\kappa}{2} |\nabla v|_H^2 + \int_{\Omega} \{\tilde{\beta}(v) + \tilde{\gamma}(v)\} \, dx + C_o$$

for $[z, v] \in H \times V$, where $\rho_{\varepsilon\eta}^{-1}$ is the primitive of $\rho_{\varepsilon\eta}^{-1}$ with $\rho_{\varepsilon\eta}^{-1}(r_o) = \rho^{-1}(r_o)$ and $C_o$ is a sufficiently large positive constant so that $J_{1_{\varepsilon\eta}} \geq 0$ on $H \times V$ for all $\varepsilon, \eta \in [0, 1]$; of course, $J_{1_{oo}} = J_1$, $u_{oo}^\infty = u^\infty$ and $C_o$ is supposed to be the same constant as in expression (2.9) of $J_1$.

Now, let us consider approximate problems including $\rho_{\varepsilon\eta}, 0 < \varepsilon \leq 1, 0 \leq \eta \leq 1$, and the viscosity term $-\mu \Delta w_t, 0 < \mu \leq 1$, which is formulated below and referred as (PSC)$_{\mu_{\varepsilon\eta}}$. 

---
Definition 3.1. For $0 < T < +\infty$ we say that a couple of functions $u := u_{\mu \varepsilon \eta} : [0, T] \to V$ and $w := w_{\mu \varepsilon \eta} : [0, T] \to H^2(\Omega)$ is a “solution” of (PSC)$_{\mu \varepsilon \eta}$ on $[0, T]$, if the following properties $(w_{1})_{\mu \varepsilon \eta} - (w_{4})_{\mu \varepsilon \eta}$ are fulfilled:

\[ (w_{1})_{\mu \varepsilon \eta} \quad u \in W^{1,2}(0, T; H) \cap C_{w}([0, T]; V), \]
\[ w \in W^{1,2}(0, T; H) \cap C([0, T]; V) \cap L^{2}(0, T; H^2(\Omega)); \]

\[ (w_{2})_{\mu \varepsilon \eta} \quad u(0) = u_{\omega \varepsilon \eta} := \{ u_{\omega} \lor a_{\varepsilon} \} \land b_{\eta} \text{ and } w(0) = w_{\omega}; \]

\[ (w_{3})_{\mu \varepsilon \eta} \text{ for a.e. } t \in [0, T] \text{ and } z \in V, \]
\[ \begin{align*}
\rho_{\varepsilon \eta}(u')(t) &+ \lambda(w')(t, z) + a(u(t), z) \\
+ (n_{o}u(t) - h, z)_{\Gamma} + \nu(\rho_{\varepsilon \eta}(u(t)), z) &= (f(t), z); \\
\end{align*} \]

\[ (w_{4})_{\mu \varepsilon \eta} \quad \frac{\partial w(t)}{\partial n} = 0 \text{ a.e. on } \Gamma \text{ for a.e. } t \in [0, T], \text{ and there is a function } \xi := \xi_{\mu \varepsilon \eta} \in L^{2}(0, T; H) \text{ such that } \xi(t) \in \beta(w(t)) \text{ a.e. in } \Omega \text{ for a.e. } t \in [0, T] \text{ and all } z \in H^{2}(\Omega) \text{ with } \frac{\partial z}{\partial n} = 0 \text{ a.e. on } \Gamma. \]

Clearly, (3.4) and (3.5) are respectively written in the forms (cf. (2.5), (2.6) in Remark 2.1).

\[ (w_{5})_{\mu \varepsilon \eta} \quad \rho_{\varepsilon \eta}(u')(t) + \lambda(w')(t) + Fu(t) + \nu \rho_{\varepsilon \eta}(u(t)) = f^{*}(t) \text{ in } V^{*} \]

for a.e. $t \in [0, T]$, and

\[ (w_{6})_{\mu \varepsilon \eta} \quad (F_{\omega}^{-1} + \mu I)w'(t) + \kappa F_{\omega}(\pi_{\omega}w(t)) + \pi_{\omega}[\xi(t) + g(w(t)) - \lambda'(w(t))u(t)] = 0 \]

in $H_{\omega}$ for a.e. $t \in [0, T]$.

With functions $u_{\varepsilon \eta}^{\infty}$, (3.6) is also written in the form

\[ (w_{7})_{\mu \varepsilon \eta} \quad \rho_{\varepsilon \eta}(u')(t) + \lambda(w')(t) + F(u(t) - u_{\varepsilon \eta}^{\infty}) + \nu(\rho_{\varepsilon \eta}(u(t)) - \rho_{\varepsilon \eta}(u_{\varepsilon \eta}^{\infty})) = 0 \]

in $V^{*}$ for a.e. $t \in [0, T]$.

According to an existence-uniqueness result in [8, Theorem 2.2], for each $\mu, \varepsilon, \eta \in (0, 1]$ problem (PSC)$_{\mu \varepsilon \eta}$ has one and only one solution $\{u_{\mu \varepsilon \eta}, w_{\mu \varepsilon \eta}\}$ on $[0, T]$, if the initial data $u_{\omega}$ and $w_{\omega}$ are given so that $[u_{\omega}, w_{\omega}] \in D_{\omega}(m_{\omega})$ (hence $[u_{\omega \varepsilon \eta}, w_{\omega \varepsilon \eta}] \in D_{\omega}(m_{\omega \varepsilon \eta})$ for every $\varepsilon, \eta \in (0, 1]$); moreover, $w_{\mu \varepsilon \eta}$ has regularity properties (cf. [10; Lemmas 5.2, 6.2])

\[ (w_{8})_{\mu \varepsilon \eta} \quad w_{\mu \varepsilon \eta} \in C_{w}([0, T]; H^{2}(\Omega)), \quad w_{\mu \varepsilon \eta}' \in L^{\infty}(0, T; H) \cap L^{2}(0, T; V), \]

\[ \xi_{\mu \varepsilon \eta} \in L^{\infty}(0, T; H). \]

Now we give some estimates for $\{u_{\mu \varepsilon \eta}, w_{\mu \varepsilon \eta}\}$.

Estimate (I)
By regularity (3.9) we can compute rigorously
\[
(3.8) \times (u_{\mu\varepsilon\eta} - u_{\varepsilon\eta}^\infty) + (3.7) \times w'_{\mu\varepsilon\eta}
\]
to get
\[
\frac{d}{d\tau} J_{1\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau), w_{\mu\varepsilon\eta}(\tau)) + |u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty|_V^2
\]
\[
+ |w'_{\mu\varepsilon\eta}(\tau)|_{V'}^2 + \mu |w'_{\mu\varepsilon\eta}(\tau)|_H^2 \leq 0
\]
for a.e. \( \tau \in [0, T] \). For details, see [10, Lemma 5.1]. The integration of (3.10) over \([0, t]\) yields
\[
J_{1\varepsilon\eta}(u_{\mu\varepsilon\eta}(t), w_{\mu\varepsilon\eta}(t)) + \int_0^t |u_{\mu\varepsilon\eta} - u_{\varepsilon\eta}^\infty|_V^2 d\tau
\]
\[
+ \int_0^t (|w'_{\mu\varepsilon\eta}|_{V'}^2 + \mu |w'_{\mu\varepsilon\eta}|_H^2) d\tau \leq J_{1\varepsilon\eta}(u_{\varepsilon\eta}, w_0) \text{ for all } t \in [0, T].
\]

Estimates (II)

We observe from hypothesis (2.17) that (cf. [7; Lemma 3.1])
\[
\rho_{\varepsilon\eta}(r)(n_o r - h(x)) \geq -A_2 |r| - A_2' \text{ for all } r \in \mathbb{R} \text{ and a.e. } x \in \Gamma,
\]
where \( A_2, A_2' \) are positive constants independent of \( \varepsilon, \eta \in (0, 1] \). By using this inequality we compute
\[
(3.6) \times \{\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}) + \lambda(w_{\mu\varepsilon\eta})\}
\]
to get
\[
\frac{d}{d\tau} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H
\]
\[
+ (\nu - \nu_1)|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H
\]
\[
\leq k_1(\nu_1)\{|u_{\mu\varepsilon\eta}(\tau)|_V^2 + |w_{\mu\varepsilon\eta}(\tau)|_{V'}^2 + 1\}
\]
for a.e. \( \tau \in [0, T] \), where \( \nu_1 \) is an arbitrary positive number and \( k_1(\nu_1) \) is a positive constant depending on \( \nu_1 \) but neither of \( \mu, \varepsilon, \eta \in (0, 1] \) nor initial data \( [u_o, w_o] \in D_o(m_o) \). From (3.12) with (3.11) it follows that
\[
|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t)) + \lambda(w_{\mu\varepsilon\eta}(t))|^2_H
\]
\[
\leq R_1(T)\{||\rho(u_{\varepsilon\eta}) + \lambda(w_o)||_H^2 + J_{1\varepsilon\eta}(u_{\varepsilon\eta}, w_0) + 1\}
\]
for all \( t \in [0, T] \), where \( R_1(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing function independent of \( \mu, \varepsilon, \eta \in (0, 1] \) and initial data \( [u_o, w_o] \in D_o(m_o) \).

In particular, if \( \nu > 0 \), then we obtain, by taking \( \nu_1 = \frac{\nu}{2} \) in (3.12),
\[
\frac{d}{d\tau} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H + \frac{\nu}{2} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H
\]
\[
\leq k_2(\nu)\{||u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty||_V^2 + |w_{\mu\varepsilon\eta}(\tau)|_{V'}^2 + 1\}
\]
for a.e. $\tau \in [0, T]$, where $k_2(\nu)$ is a positive constant depending on $\nu$ but neither of $\mu, \varepsilon, \eta \in (0, 1)$ nor initial data $[u_o, w_o] \in D_0(m_o)$.

**Estimates (III)**

Next, compute
\[
(3.8) \times F^{-1}(\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}) + \lambda(w_{\mu\varepsilon}) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty))
\]
to obtain
\[
\frac{1}{2} \frac{d}{d\tau} \rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2
\]
\[
+ (u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty, \rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty))
\]
\[
+ (u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty, \lambda(w_{\mu\varepsilon\eta}(\tau))) + \frac{\nu}{2} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))|^2_{V^*}
\]
\[
+ \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2
\]
\[
\leq \frac{\nu}{2} |\lambda(w_{\mu\varepsilon\eta}(\tau))|^2_{H^*}
\]
for a.e. $\tau \in [0, T]$. Since
\[
(3.15) \quad (u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty, \rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))) \geq \int_{\Omega} \hat{\rho}_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) dx - \int_{\Omega} \hat{\rho}_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty) dx
\]
\[
\geq (u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty, \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)),
\]
it follows from the above inequality that
\[
\frac{d}{d\tau} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2
\]
\[
+ \nu |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2
\]
\[
+ 2|\int_{\Omega} \hat{\rho}_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) dx|
\]
\[
\leq k_3 \{|u_{\mu\varepsilon\eta}(\tau) - u_{\varepsilon\eta}^\infty|_H^2 + 1\}
\]
for a.e. $\tau \in [0, T]$, where $k_3$ is a positive constant independent of $\mu, \varepsilon, \eta \in (0, 1]$ and initial data $[u_o, w_o] \in D_o(m_o)$. Therefore the integration of (3.16) over $[0, t]$ yields
\[
(3.17) \quad |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t)) + \lambda(w_{\mu\varepsilon\eta}(t)) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2 + |\int_0^t \int_{\Omega} \hat{\rho}_{\varepsilon\eta}(u_{\mu\varepsilon\eta}) dx d\tau|
\]
\[
\leq R_2(T) \{J_{1\varepsilon\eta}(u_{\varepsilon\eta}, w_o) + |\rho_{\varepsilon\eta} + \lambda(w_o) - \rho_{\varepsilon\eta}(u_{\varepsilon\eta}^\infty)|_{V^*}^2 + 1\}
\]
for all $t \in [0, T]$, where $R_2(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function independent of $\mu, \varepsilon, \eta \in (0, 1]$ and initial data $[u_o, w_o] \in D_o(m_o)$.

**Estimate (IV)**

Finally, compute the following items (1) - (4):

(1): Multiply (3.8) by $u'_{\mu\varepsilon\eta}$ and integrate over $[0, \tau] \times \Omega$ for each $0 \leq \tau \leq t$. 

(2): Multiply $\frac{d}{dt}(3.7)$ by $w'_{\mu \eta}$ and integrate over $[0, \tau] \times \Omega$ for each $0 \leq \tau \leq t$.

(3): Add the results of (1) and (2).

(4): Multiply the result of (3) by $\tau$ and integrate in $\tau$ over $[0, t]$.

Then we have

$$
\frac{1}{2}\{t|u_{\mu \eta}(t) - u_{\infty}^\eta|_V^2 + t|w'_{\mu \eta}(t)|_V^2 + t\mu|w'_{\mu \eta}|_H^2\}
+ \nu t \int_\Omega \hat{\rho}_{\mu \eta}(u_{\mu \eta}(t)) dx - \nu t(\rho_{\mu \eta}(u_{\infty}^\eta), u_{\mu \eta}(t))
+ \kappa \int_0^t \tau|\nabla w'_{\mu \eta}|_H^2 d\tau + \int_0^t \tau(\rho_{\mu \eta}(u_{\mu \eta}'), u'_{\mu \eta})d\tau
\leq L_g \int_0^t \tau|w'_{\mu \eta}|_V^2 d\tau + \frac{1}{2} \int_0^t \{u_{\mu \eta} - u_{\infty}^\eta\}_V^2 + |w'_{\mu \eta}|_{V^*}^2 + \mu|w'_{\mu \eta}|_H^2 d\tau
+ \nu \int_0^t \int_\Omega \hat{\rho}_{\mu \eta}(u_{\mu \eta}) dx d\tau - \nu \int_0^t (\rho_{\mu \eta}(u_{\infty}^\eta), u_{\mu \eta}) d\tau
+ \int_0^t \int_\Omega \tau \lambda''(w_{\mu \eta})|w'_{\mu \eta}|_V^2 u_{\mu \eta} dx d\tau
$$

for all $t \in [0, T]$. For the rigorous derivation of (3.18), we refer to [10, Lemma 5.2].

Here, we use the interpolation inequality

$$
|z|_H^2 \leq \frac{\kappa}{2} |\nabla z|_H^2 + C_\kappa |z|_{V^*}^2 \quad \text{for all } z \in V_o,
$$

where $C_\kappa$ is a positive constant depending only on $\kappa$. Applying (3.19) to the first term of the right hand side of (3.18), we derive from (3.18) with (3.11) and (3.17) that

$$
\frac{1}{2}\{t|u_{\mu \eta}(t) - u_{\infty}^\eta|_V^2 + t|w'_{\mu \eta}(t)|_V^2 + t\mu|w'_{\mu \eta}(t)|_H^2\}
+ \nu t \int_\Omega \hat{\rho}_{\mu \eta}(u_{\mu \eta}(t)) dx - \nu t(\rho_{\mu \eta}(u_{\infty}^\eta), u_{\mu \eta}(t))
+ \kappa \int_0^t \tau|\nabla w'_{\mu \eta}|_H^2 d\tau + \int_0^t \tau(\rho_{\mu \eta}(u_{\mu \eta}'), u'_{\mu \eta})d\tau
\leq R_3(T)\{J_{1\mu \eta}(u_{\mu \eta}, w_o) + |\rho(u_{\mu \eta}) + \lambda w_o - \rho_{\mu \eta}(u_{\infty}^\eta)|_{V^*}^2 + 1\}
+ 2 \int_0^t \int_{\Omega} \tau \lambda''(w_{\mu \eta})|w'_{\mu \eta}|_V^2 u_{\mu \eta} dx d\tau
$$

for all $t \in [0, T]$, where $R_3(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a function having the same properties as $R_i(\cdot), i = 1, 2$.

**Remark 3.1.** In Estimates (IV), if (4) is replaced by the following (4)'

...
(4)' Integrate the result of (3) in $\tau$ over $[0, t]$, then we have, instead of (3.20),

$$
|u_{\mu \varepsilon \eta}(t) - u_{\varepsilon \eta}^{\infty}|_{V}^{2} + |w'_{\mu \varepsilon \eta}(t)|_{H_{v}}^{2}
+ \mu|w'_{\mu \varepsilon \eta}(t)|_{H}^{2} + \nu|\tilde{\phi}_{\varepsilon}(u_{\mu \varepsilon \eta}(t))dx|
+ \kappa \int_{0}^{t} |w'_{\mu \varepsilon \eta}(\tau)|_{H_{v}}^{2}d\tau + 2 \int_{0}^{t} (\rho_{\varepsilon \eta}(u_{\mu \varepsilon \eta}'), u')d\tau
\leq K_{o}(T, |\rho(u_{\varepsilon \eta})|_{H}, |u_{\varepsilon \eta}|_{V}, |w_{o}|_{H^{2}(\Omega)}, \kappa\Delta w_{o} + \xi_{o}|_{V})
+ 2 \int_{0}^{t} \int_{\Omega} \lambda''(u_{\mu \varepsilon \eta})|w_{\mu \varepsilon \eta}'|^{2} dx d\tau
$$

(3.21)

for a.e. $t \in [0, T]$ and $\mu, \varepsilon, \eta \in (0, 1)$, where $\xi_{o}$ is a function in $H$ satisfying that

$$
-\kappa\Delta w_{o} + \xi_{o} \in V \text{ and } \xi_{o} \in \beta(w_{o}) \text{ a.e. } \Omega
$$

and $K_{o}(\cdot, \cdots)$ is an increasing function on $\mathbb{R}_{+}^{5}$ with respect to all the arguments. Moreover, just as in Remark 2.2, we note that $|w_{\mu \varepsilon \eta}(t)|_{H^{2}(\Omega)}^{2} + |\xi_{\mu \varepsilon \eta}(t)|_{H}^{2}$ is estimated from above by the $H$-norm of

$$
\ell_{\mu \varepsilon \eta}(t) := -(F_{o}^{-1} + \mu I)w_{\mu \varepsilon \eta}'(t) + \pi_{o}[\lambda'(u_{\mu \varepsilon \eta}(t))u_{\mu \varepsilon \eta}(t) - g(u_{\mu \varepsilon \eta}(t))],
$$

namely,

$$
|w_{\mu \varepsilon \eta}(t)|_{H^{2}(\Omega)}^{2} + |\xi_{\mu \varepsilon \eta}(t)|_{H}^{2} \leq C_{1}(|\ell_{\mu \varepsilon \eta}(t)|_{H}^{2} + 1) \text{ (cf. (2.22))}
\leq C_{2}(|w_{\mu \varepsilon \eta}(t)|_{H_{v}}^{2} + \mu^{2}|w_{\mu \varepsilon \eta}'(t)|_{H_{v}}^{2} + |u_{\mu \varepsilon \eta}(t)|_{H}^{2} + 1)
\text{ for a.e. } t \in [0, T] \text{ and all } \mu, \varepsilon, \eta \in (0, 1),
$$

where $C_{1}$ and $C_{2}$ are positive constants. Therefore, it follows from (3.21) that

$$
|w_{\mu \varepsilon \eta}(t)|_{H^{2}(\Omega)}^{2} + |\xi_{\mu \varepsilon \eta}(t)|_{H}^{2}
\leq K_{1}(T, |\rho(u_{\varepsilon \eta})|_{H}, |u_{\varepsilon \eta}|_{V}, |w_{o}|_{H^{2}(\Omega)}, |\kappa\Delta w_{o} + \xi_{o}|_{V})
+ k_{4} \int_{0}^{T} \int_{\Omega} \lambda''(u_{\mu \varepsilon \eta})|w_{\mu \varepsilon \eta}'|^{2} dx d\tau
$$

(3.22)

for all $t \in [0, T]$, where $K_{1}(\cdot, \cdots)$ is a function on $\mathbb{R}_{+}^{5}$ having the same properties as $K_{o}(\cdot)$ and $k_{4}$ is a positive constant independent of $\mu, \varepsilon, \eta \in (0, 1]$.

4. Convergence of Approximate Solutions and Proof of the Existence Result

The solution of (PSC) is constructed in two steps of limiting process as $\eta \to 0$ and $\varepsilon, \mu \to 0$.

In the first step, parameters $\mu$ and $\varepsilon$ are fixed, and parameter $\eta$ goes to 0. For each $\varepsilon \in (0, 1]$, we write $J_{1\varepsilon}$ for $J_{1\varepsilon 0}$ and $w_{\varepsilon}^{\infty}$ for $w_{\varepsilon 0}^{\infty}$.
Lemma 4.1. Let $\mu, \varepsilon \in (0, 1]$, $0 < T < +\infty$ and $[u_o, w_o] \in D_o(m_o)$. Then the problem $(PSC)_{\mu \varepsilon} := (PSC)_{\mu \varepsilon 0}$ has one and only one solution $\{u_{\mu \varepsilon}, w_{\mu \varepsilon}\}$ on $[0, T]$. Moreover it satisfies the following energy inequalities (4.1) - (4.3):

\[
J_{\varepsilon}(u_{\mu \varepsilon}(t), w_{\mu \varepsilon}(t)) + \int_0^t \left| u_{\mu \varepsilon} - u_{\varepsilon}^\infty \right|^2_{1, \varepsilon} dt \\
+ \int_0^t \left( |w_{\mu \varepsilon}|^2_{V_o} + \mu |w_{\mu \varepsilon}|^2_H \right) dt \\
\leq J_{\varepsilon}(u_{0\varepsilon}, w_o)
\]

for all $t \in [0, T]$, where $u_{0\varepsilon} := u_o \wedge u_{\varepsilon}$;

\[
|\rho_{\varepsilon}(u_{\mu \varepsilon}(t)) + \lambda(w_{\mu \varepsilon}(t))|^2_H \leq R_1(T)\{|\rho(u_{0\varepsilon}) + \lambda(w_o)|^2_H + J_{\varepsilon}(u_{0\varepsilon}, w_o) + 1\}
\]

for all $t \in [0, T]$;

\[
|\rho_{\varepsilon}(w_{\mu \varepsilon}(t))|^2_{V_o} + t|w_{\mu \varepsilon}'(t)|^2_{V_o} + t\mu|w_{\mu \varepsilon}'(t)|^2_H \\
+ \nu |\rho_{\varepsilon}(u_{\mu \varepsilon}(t))| dx | + \kappa \int_0^t \tau |w_{\mu \varepsilon}'|^2_{V_o} d\tau \\
\leq R_3(T)\{J_{\varepsilon}(u_{0\varepsilon}, w_o) + |\rho(u_{0\varepsilon}) + \lambda(w_o)|^2_H + 1\}
\]

for all $t \in [0, T]$. Here $R_1(\cdot)$ and $R_3(\cdot)$ are the same ones as in Estimates (II), (IV).

Proof. Let $\{u_{\mu \varepsilon}, w_{\mu \varepsilon}\}$, $\mu, \varepsilon, \eta \in (0, 1]$, be the approximate solutions considered in section 3. Then we now recall such an estimate, essentially due to [16], for the integral

\[
I_{\mu \varepsilon}(t) := \int_0^t \int_\Omega \lambda''(w_{\mu \varepsilon}(t))w_{\mu \varepsilon}'^2 \mu^\varepsilon dx d\tau
\]

in [10; the proof of Lemma 5.2] that

\[
I_{\mu \varepsilon}(t) \leq \delta_1 \sup_{0 \leq \tau \leq t} |w_{\mu \varepsilon}'(\tau)|^2_H + \delta_1 \int_0^t |w_{\mu \varepsilon}'|^2_{V_o} d\tau + k_\delta(\mu, \delta_1)
\]

for all $t \in [0, T]$, where $\delta_1$ is an arbitrary (small) positive number and $k_\delta(\mu, \delta_1)$ is a positive constant depending only on $\mu, \delta_1$, but not on $\varepsilon, \eta \in (0, 1]$. It is easy to see from (3.13), (3.21), (3.22) in Remark 3.1 and (4.4) that

\[
|u_{\mu \varepsilon}|^2_{L^\infty(0,T;V)} + |\rho_{\varepsilon}(u_{\mu \varepsilon})|^2_{L^\infty(0,T;H)} \\
+ |w_{\mu \varepsilon}|^2_{L^\infty(0,T;H^2(\Omega))} + |\mu_{\varepsilon}|^2_{L^\infty(0,T;H)} \\
+ |w_{\mu \varepsilon}'|^2_{L^2(0,T;V_o)} + |w_{\mu \varepsilon}'|^2_{L^2(0,T;H)} + \int_0^T (\rho_{\varepsilon}(u_{\mu \varepsilon})', u_{\mu \varepsilon}') d\tau \\
\leq k_0 := k_0(T, \mu, u_o, w_o)
\]

for $\varepsilon, \eta \in (0, 1]$, where $k_0$ is a positive constant depending on $T, \mu$ and initial data $[u_o, w_o] \in D_o(m_o)$, but not on $\varepsilon, \eta \in (0, 1]$. Also, by (3.2) we have

\[
\int_0^T (\rho_{\varepsilon}(u_{\mu \varepsilon})', u_{\mu \varepsilon}') d\tau \geq C(\varepsilon)|w_{\mu \varepsilon}'|^2_{L^2(0,T;H)}.
\]
Besides, it is easily inferred from the above convergences (4.7) - (4.9) that
\[ u_{\mu \varepsilon \eta_n} \rightarrow u_{\mu \varepsilon} \quad \text{in} \quad C([0,T];H) \cap C_w([0,T];V) \]
and weakly in \( W^{1,2}(0,T;H) \),

\[ w_{\mu \varepsilon \eta_n} \rightarrow w_{\mu \varepsilon} \quad \text{in} \quad C([0,T];V) \cap C_w([0,T];H^2(\Omega)) \]
and weakly in \( W^{1,2}(0,T;V) \) and

\[ \xi_{\mu \varepsilon \eta_n} \rightarrow \xi_{\mu \varepsilon} \quad \text{weakly* in} \quad L^\infty(0,T;H); \]

these imply that \( \rho_{\varepsilon \eta_n}(u_{\mu \varepsilon \eta_n}) \rightarrow \rho_{\varepsilon}(u_{\mu \varepsilon}) \quad \text{weakly* in} \quad L^\infty(0,T;H) \) and that \( \xi_{\mu \varepsilon} \in \beta(w_{\mu \varepsilon}) \quad \text{a.e. on} \quad [0,T] \times \Omega \), since \( \xi_{\mu \varepsilon \eta_n} \in \beta(w_{\mu \varepsilon \eta_n}) \quad \text{a.e. on} \quad [0,T] \times \Omega. \)

Besides, it is easily inferred from the above convergences (4.7) - (4.9) that \( \{u_{\mu \varepsilon},w_{\mu \varepsilon}\} \) is a (unique) solution of (PSC)\( _{\mu \varepsilon} \) on \( [0,T] \). By the way, since \( u_{\mu \varepsilon} \leq 0 \quad \text{a.e. on} \quad [0,T] \times \Omega \), condition (H3) implies that \( \lambda''(w_{\mu \varepsilon \eta_n})|w'_{\mu \varepsilon \eta_n}|^2u_{\mu \varepsilon} \leq 0 \quad \text{a.e. on} \quad [0,T] \times \Omega \), so that

\[
\limsup_{n \to +\infty} \int_0^T \int_\Omega \tau \lambda''(w_{\mu \varepsilon \eta_n})|w'_{\mu \varepsilon \eta_n}|^2u_{\mu \varepsilon \eta_n} \, dx \, d\tau
\leq \limsup_{n \to +\infty} \int_0^T \int_\Omega \tau \lambda''(w_{\mu \varepsilon \eta_n})|w'_{\mu \varepsilon \eta_n}|^2(u_{\mu \varepsilon \eta_n} - u_{\mu \varepsilon}) \, dx \, d\tau
\leq \text{const.} \limsup_{n \to +\infty} |u_{\mu \varepsilon \eta_n} - u_{\mu \varepsilon}|_{C([0,T];H)} \cdot |w'_{\mu \varepsilon \eta_n}|_{L^2(0,T;L^4(\Omega))}^2
= 0.
\]

Now, passing to the limit in (3.11) and (3.13) and (3.20) as \( \eta_n \to 0 \), we have by (4.10) inequalities (4.1), (4.2) and (4.3). \( \blacksquare \)

**Remark 4.1.** By combining (3.21) and (3.22) in Remark 3.1 with (4.10) for the solution \( \{u_{\mu \varepsilon},w_{\mu \varepsilon}\} \) of (PSC)\( _{\mu \varepsilon} \) we have the following estimate:

\[
|u_{\mu \varepsilon}(t) - u^{\infty}_\varepsilon|^2_V + |w'_{\mu \varepsilon}(t)|^2_{V^*_o} + \mu|w'_{\mu \varepsilon}(t)|^2_H + \nu \int_\Omega \rho_{\varepsilon}(u_{\mu \varepsilon}(t)) \, dx
\]
\[
+ \kappa \int_0^t |\nabla w'_{\mu \varepsilon}|^2_H \, dt + 2 \int_0^t (\rho_{\varepsilon}(u_{\mu \varepsilon}'), u'_{\mu \varepsilon}) \, d\tau
\leq K_0(T, |\rho'(u_{o \varepsilon})|_H, |u_{o \varepsilon}|_V, |w_0|_{H^2(\Omega)}, 1 - \Delta w_o + \xi_o|_V)
\]

for a.e. \( t \in [0,T]; \)

\[
|w'_{\mu \varepsilon}(t)|^2_{H^2(\Omega)} + |\xi_{\mu \varepsilon}(t)|^2_H
\leq K_1(T, |\rho'(u_{o \varepsilon})|_H, |u_{o \varepsilon}|_V, |w_0|_{H^2(\Omega)}, 1 - \Delta w_o + \xi_o|_V)
\]

for all \( t \in [0,T] \), where \( K_i(\ldots, \ldots, \cdot) : \mathbb{R}^5_+ \to \mathbb{R}_+, \ i = 1, 2 \), are increasing functions with respect to all the arguments.
Proof of Theorem 2.2. First assume that \([u_o, w_o] \in D_o(m_o)\). In this case we can use the estimates (4.11) and (4.12) in addition to (4.1) - (4.3). Therefore, for suitable sequences \(\{\mu_n\}\) and \(\{\varepsilon_n\}\) with \(\mu_n \downarrow 0\) and \(\varepsilon_n \downarrow 0\) (as \(n \to +\infty\)) and some functions \(u \in L^\infty([0,T];V), \tilde{\rho} \in L^\infty([0,T];H), w \in C_w([0,T];H^2(\Omega))\) and \(\xi \in L^\infty([0,T];H)\), the solution \(\{u_n, w_n\}\) of \((\text{PSC})_n := (\text{PSC})_{\mu_n \varepsilon_n}\) converges to the couple \([u, w]\) in the sense that
\[
\begin{align*}
\text{(4.13)} & \quad u_n \to u \text{ weakly}^* \text{ in } L^\infty([0,T];V), \\
\text{(4.14)} & \quad \rho_n := \rho_{\varepsilon_n}(u_n) \to \tilde{\rho} \text{ in } C_w([0,T];H) \text{ and weakly in } W^{1,2}([0,T];V^*), \\
\text{(4.15)} & \quad w_n \to w \text{ in } C([0,T];H) \cap C_w([0,T];H^2(\Omega)), \\
\text{(4.16)} & \quad w_n' \to w' \text{ weakly}^* \text{ in } L^\infty((0,T];V^*_o), \\
\text{(4.17)} & \quad \xi_n := \xi_{\mu_n \varepsilon_n} \to \xi \text{ weakly}^* \text{ in } L^\infty((0,T];H).
\end{align*}
\]

By the standard techniques of the maximal monotone operator theory, it follows from (4.13) and (4.14) that \(\hat{\rho} = \rho(u)\) and \(u_n \to u\) in \(C_w([0,T];V)\) (cf. [10; the proof of Theorem 2.3] and [15; Remark 1.3]), and also from (4.15) and (4.17) that \(\xi \in \beta(w)\) a.e. on \([0,T] \times \Omega\). Now, it is easy to see that the limit \([u, w]\) is a solution of \((\text{PSC})_o\) on \([0,T]\), and estimates (2.18) - (2.20) hold for \(s = 0 < t \leq T\) and \(M_1(T) = R_1(T), M_2(T) = R_3(T)\), by Lemma 4.1.

Secondly, consider the general case of \([u_o, w_o] \in D(m_o)\). In this case, choose a sequence \([[u_{ok}, w_{ok}]\] in \(D_o(m_o)\) such that
\[
\rho(u_{ok}) \to \rho(u_o) \text{ in } H, \ w_{ok} \to w_o \text{ in } V, \ J_1(u_{ok}, w_{ok}) \to J_1(u_o, w_o)
\]
as \(k \to +\infty\). As was shown above, for each \(k\) problem \((\text{PSC}); f, h, u_{ok}, w_{ok})\) has one and only one solution \([u_k, w_k]\) on \([0,T]\) and it satisfies inequalities (2.18) - (2.20) for \(s = 0 < t \leq T\). Therefore, there is a subsequence of \([u_k, w_k]\), denoted by \([u_k, w_k]\) again, with some functions \(u, \tilde{\rho}, w, \xi\) such that
\[
\begin{align*}
\text{(4.18)} & \quad u_k \to u \text{ weakly in } L^2(0,T;V) \\
& \quad \text{and weakly}^* \text{ in } L^\infty(\delta,T;V) \text{ for every } 0 < \delta \leq T, \\
\text{(4.19)} & \quad \rho(u_k) \to \tilde{\rho} \text{ weakly}^* \text{ in } L^\infty(0,T;H), \\
\text{(4.20)} & \quad w_k \to w \text{ in } C([0,T];H) \cap C_w([0,T];V) \text{ weakly in } L^2(0,T;H^2(\Omega)) \text{ and in } C_w([\delta,T];H^2(\Omega)) \text{ for every } 0 < \delta \leq T, \\
\text{(4.21)} & \quad w_k' \to w' \text{ weakly in } L^2(0,T;V^*_o) \text{ and weakly}^* \text{ in } L^\infty(\delta,T;V^*_o) \text{ for every } 0 < \delta \leq T, \\
\text{(4.22)} & \quad \xi_k \to \xi \text{ weakly in } L^2(0,T;H) \text{ and weakly}^* \text{ in } L^\infty(\delta,T;H) \text{ for every } 0 < \delta \leq T.
\end{align*}
\]
5. The Attractor for the Semigroup Associated With (PSC)

Based on Theorem 2.2, for each $t \geq 0$ we can define a mapping $S(t) : D(m_o) \to D(m_o)$ by

\begin{equation}
(5.1) \quad S(t)[u_o, w_o] = [u(t), w(t)], \quad [u_o, w_o] \in D(m_o),
\end{equation}

where $\{u, w\}$ is the global solution of (PSC; $f, h, u_o, w_o$).

**Theorem 5.1.** Assume that (H1) - (H5), (2.16) and (2.17) are satisfied. Let $\sigma_* < m_o < \sigma^*$. Then the family $\{S(t) : t \geq 0\}$, defined by (5.1), satisfies the following properties:

- (a) $\{S(t)\}$ is a semigroup defined on $D(m_o)$, i.e.
  \[ S(0) = I \text{ on } D(m_o), \quad S(t + s) = S(t)S(s) \text{ on } D(m_o) \text{ for any } s, t \geq 0. \]
- (b) Let $0 < \delta < T < +\infty$. Then $S(\cdot)[z, v] \in C_w([\delta, T]; V \times H^2(\Omega))$ for any $[z, v] \in D(m_o)$. Moreover, if $[z_n, v_n] \in D(m_o), n = 1, 2, \ldots$, $[z, v] \in D(m_o)$, $\{J(t) [z_n, v_n] \}$ is bounded, $\rho(z_n) \to \rho(z)$ weakly in $H$ and $v_n \to v$ weakly in $V$ as $n \to +\infty$, then
  \begin{equation}
  (5.2) \quad S(\cdot)[z_n, v_n] \to S(\cdot)[z, v] \text{ in } C_w([\delta, T]; V \times H^2(\Omega))
  \end{equation}
  as $n \to +\infty$.
- (c) For each $M > 0$, $D^M(m_o)$ is positively invariant for $\{S(t)\}$ (i.e. $S(t)D^M(m_o) \subset D^M(m_o)$ for all $t \geq 0$), and in particular,
  \begin{equation}
  (5.3) \quad S(t)D^M(m_o) \subset D^M_o(m_o) \text{ for all } t > 0.
  \end{equation}

*Proof.* Assertion (a) is a direct consequence of the global existence and uniqueness result (cf. Theorems 2.1 and 2.2) for (PSC), and assertion (c) follows from (2.18) - (2.21) and (2.23). The proof of assertion (b) is exactly same as that of the second step in the proof of Theorem 2.2; see (4.20) and (4.23).  

The main result of this paper is stated in the following theorem.

**Theorem 5.2.** Assume that (H1) - (H5) with $\nu > 0$, (2.16) and (2.17) are satisfied. Let $\sigma_* < m_o < \sigma^*$ and let $\{S(t)\}$ be the semigroup on $D(m_o)$ defined by (5.1). Then there exists a subset $A$ of $D_o(m_o)$ such that

- (i) $A$ is compact and connected in $H \times V$, and is bounded in $V \times H^2(\Omega)$;
- (ii) $A$ is invariant for $\{S(t)\}$, i.e. $A = S(t)A$ for all $t \geq 0$;
proof of Theorem 5.2 is to find an absorbing set $B$

(i) There is a positive constant $N_o$ such that

\[ \inf \{ \varepsilon > 0 \mid \| w(t) \|_{V^*} < \varepsilon \} > 0 \]

for all global solutions $u(t)$ with initial data $[u_0, w_0] \in D(m_o)$.

(ii) There are positive constants $\varepsilon_1$ and $N_1$ such that

\[ \frac{d}{dt} \{ J_1(u(t), w(t)) + \epsilon \| \rho(u(t)) + \lambda(w(t)) \|^2_{H} \} \]

\[ + \epsilon \| \rho(u(t)) + \lambda(w(t)) \|^2_{H} \leq N_1, \quad \text{a.e. } t \geq 0, \]

for all global solutions $u(t)$ with initial data $[u_0, w_0] \in D(m_o)$.

Proof. (1) We multiply (2.6) by $\pi_o w(t) (= w(t) - m_o)$ to get

\[ (F_o^{-1} w'(t), \pi_o w(t)) + \kappa \| \pi_o w(t) \|^2_{V^*} + (\xi(t), w(t) - m_o) + (g(w(t)), w(t) - m_o) \]

\[ = (\lambda'(w(t))u(t), w(t) - m_o) \]

for a.e. $t \geq 0$.

This implies that (5.4) holds for some constant $N_o$ independent of initial data $[u_0, w_0] \in D(m_o)$, since

\[ (\xi(t), w(t) - m_o) \geq \int_\Omega \hat{\beta}(w(t))dx - |\Omega| \hat{\beta}(m_o), \]

\[ |(F_o^{-1} w'(t), \pi_o w(t))| \leq \text{const.}|w'(t)|_{V^*}, \]

and

\[ |(\lambda'(w(t))u(t), w(t) - m_o)| \leq \text{const.}|u(t)|_{H}. \]

(2) By the definition of subdifferential $\rho^{-1}$ of $\hat{\beta}^{-1}$, we observe that

\[ \int_\Omega \rho^{-1}(\rho(u(t)))dx - (\rho(u(t)) + \lambda(w(t)), u^\infty) \]

\[ \leq (\rho(u(t)) - \rho(u^\infty), u(t)) + \int_\Omega \rho^{-1}(\rho(u^\infty))dx \]

\[ - (\rho(u(t)) + \lambda(w(t)), u^\infty) \]

\[ = (\rho(u(t)) + \lambda(w(t)), u(t) - u^\infty) \]

\[ - (\rho(u^\infty) + \lambda(w(t)), u(t)) + \int_\Omega \rho^{-1}(\rho(u^\infty))dx \]

\[ \leq k_7 \{ |u(t) - u^\infty|^2_V + |\rho(u(t)) + \lambda(w(t))|^2_{H} + 1 \} \]

for all $t > 0$, where $\alpha_2$ and $\kappa_2$. In this paper, we say that a set $A$ is a global attractor for the semigroup $\{S(t)\}$, if it has properties (i), (ii) and (iii) of Theorem 5.2. The key for the proof of Theorem 5.2 is to find an absorbing set $B_o$ for the semigroup $\{S(t)\}$. To do so we prove a lemma under the same assumptions of Theorem 5.2.

Lemma 5.1. (1) There is a positive constant $N_o$ such that

\[ \kappa \| \pi_o w(t) \|^2_{V^*} + \int_\Omega \hat{\beta}(w(t))dx \leq N_o \| \rho(u(t)) + \lambda(w(t)) \|^2_{H} \]

\[ \text{a.e. } t \geq 0, \]

for all global solutions $u(t)$ with initial data $[u_0, w_0] \in D(m_o)$.

(2) There are positive constants $\varepsilon_1$ and $N_1$ such that

\[ \frac{d}{dt} \{ J_1(u(t), w(t)) + \varepsilon_1 \| \rho(u(t)) + \lambda(w(t)) \|^2_{H} \}

\[ + \varepsilon_1 \| \rho(u(t)) + \lambda(w(t)) \|^2_{H} \leq N_1, \quad \text{a.e. } t \geq 0, \]

for all global solutions $u(t)$ with initial data $[u_0, w_0] \in D(m_o)$.
where $k_7$ is a positive constant independent of any solution \( \{u, w\} \) of (PSC) with \([u_0, w_0] \in D(m_0)\). Combining (5.4) and (5.6), we see that
\[
J_1(u(t), w(t)) \leq k_8 \{ |u(t) - u^\infty|^2_{V^*} + |w'(t)|^2_{V^*} \\
+ |\rho(u(t)) + \lambda(w(t))|^2_H + 1 \} \text{ for a.e. } t \geq 0,
\]
where $k_8$ is a positive constant independent of any solution \( \{u, w\} \) of (PSC) with \([u_0, w_0] \in D(m_0)\). Besides, recall the inequalities
\[
\frac{d}{dt} J_1(u(t), w(t)) + |u(t) - u^\infty|^2_{V^*} + |w'(t)|^2_{V^*} \leq 0 \text{ for a.e. } t \geq 0,
\]
and
\[
\frac{d}{dt} |\rho(u(t)) + \lambda(w(t))|^2_H + \frac{\nu}{2} |\rho(u(t)) + \lambda(w(t))|^2_H \\
\leq k_2 \{ |u(t) - u^\infty|^2_{V^*} + |w'(t)|^2_{V^*} + 1 \}
\]
for a.e. $t \geq 0$, which are derived from (2.18) and (3.14), respectively. In (5.9), note from (5.4) that $|w(t)|^2_{V^*}$ can be estimated by const.\(|w'(t)|_{V^*} + |u(t)|_H + 1\), so that
\[
\frac{d}{dt} |\rho(u(t)) + \lambda(w(t))|^2_H + \frac{\nu}{2} |\rho(u(t)) + \lambda(w(t))|^2_H \\
\leq k_2 \{ |u(t) - u^\infty|^2_{V^*} + |w'(t)|^2_{V^*} + 1 \}
\]
for a.e. $t \geq 0$, where $k_9$ is a positive constant independent of all solutions \( \{u, w\} \) of (PSC) with \([u_0, w_0] \in D(m_0)\).

Now, compute (5.8) + $\varepsilon_o$ × (5.10) with $\varepsilon_o = \frac{1}{2k_9}$ to obtain
\[
\frac{d}{dt} \{ J_1(u(t), w(t)) + \varepsilon_o |\rho(u(t)) + \lambda(w(t))|^2_H \} \\
+ \frac{1}{2} \{ |u(t) - u^\infty|^2_{V^*} + |w'(t)|^2_{V^*} + 1 \} \\
+ \frac{\nu \varepsilon_o}{2} |\rho(u(t)) + \lambda(w(t))|^2_H \\
\leq 1
\]
for a.e. $t \geq 0$. Therefore, putting
\[
\varepsilon_1 = \min \left\{ \frac{1}{2k_8}, \frac{\nu \varepsilon_o}{4k_8}, \frac{\nu}{4} \right\} \text{ and } N_1 = 1,
\]
we obtain (5.5) immediately from (5.11). \(\blacksquare\)

**Remark 5.1.** In general, for a solution \( \{u, w\} \) of (PSC) we do not know the absolute continuity of $J_1(u(t), w(t))$ in time $t$. But it follows from (2.18) of Theorem 2.2 that $J_1(u(t), w(t))$ is of bounded variation (hence almost everywhere differentiable in $t$) and its derivative $\frac{d}{dt} J_1(u, w)$ is integrable on each bounded interval of $R_+$, and
\[
J_1(u(t), w(t)) - J_1(u(s), w(s)) \leq \int_s^t \frac{d}{d\tau} J_1(u, w)d\tau \text{ for any } 0 \leq s \leq t < +\infty.
\]
Therefore inequality (5.5) makes sense. Also, inequality (5.9) similarly makes sense.
Now we fix the functional
\[ J(z, v) := J_1(z, v) + \varepsilon_o|\rho(z) + \lambda(v)|^2_H \]
with \( \varepsilon_o \) in (2) of Lemma 5.1. For the simplicity of notation we write \( J(S(t)[u_0, w_0]) \) for \( J(u(t), w(t)), \{u, w\} \) being the global solution of (PSC) with initial data \([u_0, w_0] \in D(m_o)\).

**Lemma 5.2.** There exists a subset \( B_o \) of \( D(m_o) \) such that

1. \( \sup_{[z, v] \in B_o} J(z, v) < +\infty \),
2. for any subset \( B \) with \( \sup_{[z, v] \in B} J(z, v) < +\infty \), there exists a time \( t_B > 0 \) such that

\[ S(t)B \subset B_o \text{ for all } t \geq t_B. \]

**Proof.** From (5.5) of Lemma 5.1 it follows that
\[ J(S(t)[z, v]) \leq e^{-ct} J(z, v) + \frac{N_1}{\varepsilon_1} \text{ for all } t \geq 0 \text{ and all } [z, v] \in D(m_o). \]

Now, take a subset
\[ B_o := \{[z, v] \in D(m_o); J(z, v) \leq 1 + \frac{N_1}{\varepsilon_1}\}. \]

Then, \( B_o \) clearly satisfies properties (a) and (b). \( \blacksquare \)

**Lemma 5.3.** Let \( B \) be any subset of \( D(m_o) \) with \( \sup_{[z, v] \in B} J(z, v) < +\infty \), and \( \delta \) be any positive number. Then \( B_\delta := \bigcup_{t \geq \delta} S(t)B^{H \times V} \) is in \( D_o(m_o) \), compact in \( H \times V \), bounded in \( V \times H^2(\Omega) \) and
\[ \sup_{[z, v] \in B_\delta} J(z, v) < +\infty. \]

**Proof.** By Lemma 5.2, there is a finite time \( t_o > 0 \) such that \( S(t)B \subset B_o \) and \( S(t)B_o \subset B_o \) for all \( t \geq t_o \). Hence, if \( t \geq 2t_o + \delta \), then \( S(t)B = S(\delta)S(t-t_o-\delta)S(t_o)B \subset S(\delta)B_o \). By (2.18) - (2.23) of Theorem 2.2 and Remarks 2.2, 2.3, \( S(\delta)B_o \) is in \( D_o(m_o) \), bounded in \( V \times H^2(\Omega) \) and \( \sup_{[z, v] \in S(\delta)B_o} J(z, v) < +\infty \). Therefore, \( B_{2t_o+\delta} := \bigcup_{t \geq 2t_o + \delta} S(t)B^{H \times V} \) is in \( D_o(m_o) \), compact in \( H \times V \), bounded in \( V \times H^2(\Omega) \) and \( \sup_{[z, v] \in B_{2t_o+\delta}} J(z, v) < +\infty \). Applying again (2.18) - (2.23) with \( s = 0 \) and \( T = 2t_o + \delta \), we see that the set
\[ B_{\delta, 2t_o+\delta} := \bigcup_{\delta \leq t \leq 2t_o + \delta} S(t)B^{H \times V} \]
has the same properties as \( B_{2t_o+\delta} \). Consequently \( B_\delta \subset B_{2t_o+\delta} \cup B_{\delta, 2t_o+\delta} \) has the required properties. \( \blacksquare \)

**Proof of Theorem 5.2.** The construction of \( A \) is quite standard. In fact, we shall show that the set
\[ A := \bigcap_{s > 0} \bigcup_{t \geq s} S(t)B_o^{H \times V} \]
(5.12)
is the required one, where \( B_o \) is the absorbing set found by Lemma 5.2. By (5.12) it is clear that

\[
\begin{align*}
[z, v] & \in A \text{ if and only if there are sequences } \{t_n\} \\
t_n & \uparrow +\infty \text{ as } n \to +\infty \\
\text{and } \{[z_n, v_n]\} & \subset B_o \text{ such that} \\
S(t_n)[z_n, v_n] & \to [z, v] \text{ in } H \times V.
\end{align*}
\]

Moreover, on account of Lemmas 5.2 and 5.3, we see that \( A \subset D_o(m_o) \cap B_o \), and in (5.13) the sequences \( \{t_n\} \) and \( \{[z_n, v_n]\} \) can be chosen so as to satisfy further that for some \( [\hat{z}, \hat{v}] \in D_o(m_o) \)

\[
\begin{align*}
S(t_n)[z_n, v_n] & \to [z, v] \text{ weakly in } V \times H^2(\Omega), \\
[z_n, v_n] & \to [\hat{z}, \hat{v}] \text{ weakly in } V \times H^2(\Omega), \\
\sup_{n \geq 1} J(z_n, v_n) & < +\infty, \quad \sup_{n \geq 1} J(S(t_n)[z_n, v_n]) < +\infty.
\end{align*}
\]

(i) By Lemma 5.3, for each \( s > 0 \), \( B_s := \bigcup_{t \geq s} S(t)B_o \) compact in \( H \times V \) and bounded in \( V \times H^2(\Omega) \), so is \( A \).

Next, by contradiction we show the connectedness of \( A \). Assume that \( A \) is not connected in \( H \times V \). Then there would exist two compact sets \( A_1 \) and \( A_2 \) in \( H \times V \) such that

\[
A_1 \cup A_2 = A, \quad A_1 \cap A_2 = \emptyset, \quad A_i \neq \emptyset, \quad i = 1, 2.
\]

Let \( X_i := [z^i, v^i] \in A_i \) for \( i = 1, 2 \), and choose by (5.13) and (5.14) sequences \( \{t_n^i\} \) with \( t_n^i \uparrow +\infty \) and \( X_n^i := [z_n^i, v_n^i] \in B_o \cap D_o(m_o) \) such that

\[
S(t_n^i)X_n^i \to X^i \text{ weakly in } V \times H^2(\Omega),
\]

\[
\sup_{n \geq 1} J(S(t_n^i)X_n^i) < +\infty, \quad i = 1, 2.
\]

Without loss of generality we may assume that

\[
t_n^1 < t_n^2 < t_{n+1}^1, \quad t_n^2 - t_n^1 \geq 1, \quad \text{for all } n = 1, 2, \ldots.
\]

Putting \( \hat{X}_n^2 := S(t_n^2 - t_n^1)X_n^1 \), we have

\[
S(t_n^1)\hat{X}_n^2 = S(t_n^1)X_n^1 \to X^2 \text{ weakly in } V \times H^2(\Omega)
\]

and by Lemma 5.3

\[
\sup_{n \geq 1} J(\hat{X}_n^2) < +\infty.
\]

Consider the segment

\[
L_n(\tau) := \tau X_n^1 + (1 - \tau)\hat{X}_n^2, \quad \tau \in [0, 1].
\]

Then, we easily observe from (5.15) - (5.17) that

\[
L_n(\tau) \subset D_o(m_o), \quad \tau \in [0, 1], \quad \sup_{\tau \in [0, 1], n \geq 1} J(L_n(\tau)) < +\infty.
\]

This implies by (b) of Theorem 5.1 that \( S(t_n^1)L_n(\tau) \), \( \tau \in [0, 1] \), is a continuous curve combining \( S(t_n^1)X_n^1 \) and \( S(t_n^1)\hat{X}_n^2 \) in \( H \times V \). Now, take \( \varepsilon \)-neighborhoods \( U^i_\varepsilon \) of \( A_i \) for \( i = 1, 2 \), for a sufficiently small number \( \varepsilon > 0 \) so
that $U_1^1 \cap U_2^2 = \emptyset$. In this case, $S(t_n^1)X_n^1 \subset U_1^1$ and $S(t_n^1)X_n^2 \subset U_2^2$ for all sufficiently large $n$ (cf. (5.15), (5.16)). Hence there is a sequence $\{\tau_n\} \subset (0,1)$ such that
\[
S(t_n^1)L_n(\tau_n) \notin U_1^1 \cup U_2^2.
\]
Besides, by Lemma 5.2, choose a positive number $t_o$ such that
\[
S(t_o)L_n(\tau) \in B_o \text{ for all } \tau \in [0,1], \; n = 1,2,\ldots.
\]
Since $S(t_n^1)L_n(\tau) = S(t_n^1 - t_o)S(t_o)L_n(\tau) \subset S(t_n^1 - t_o)B_o$, it follows from Lemma 5.3 that $\{S(t_n^1)L_n(\tau) ; \tau \in [0,1], n \geq n_o\}$ is bounded in $V \times H^2(\Omega)$ for all sufficiently large $n_o$. Therefore, by (5.13), any accumulation point of $\{S(t_n^1)L_n(\tau_n) ; n \geq n_o\}$ in $H \times V$ belongs to $A$. This contradicts (5.19).

(ii) First we prove $S(t)A \subset A$ for all $t > 0$. Let $X := [z,v]$ be any element of $A$. Then, there are sequences $\{t_n\}$ with $t_n \uparrow +\infty$ and $\{X_n := [z_n,v_n]\}$ satisfying the properties in (5.13) and (5.14). We note by (b) of Theorem 5.1 that for $t > 0$
\[
S(t + t_n)X_n = S(t)S(t_n)X_n \to S(t)X \text{ weakly in } V \times H^2(\Omega).
\]
Hence $S(t)X \in A$ by (5.13) again. Thus $S(t)A \subset A$.

Conversely, we show $A \subset S(t)A$ for $t > 0$. Let $X := [z,v]$ be any element of $A$. Then, there are sequences $\{t_n\}$ and $\{X_n := [z_n,v_n]\} \subset D_o(m_o) \cap B_o$ satisfying (5.13) and (5.14). Now, by Lemmas 5.2 and 5.3, $\{Y_n := S(t_n - t)X_n ; n \geq n_o\}$ is bounded in $V \times H^2(\Omega)$ and $\sup_{n \geq n_o} J(Y_n) < +\infty$ (if $n_o$ is sufficiently large), so we may assume that $Y_n \to Y$ weakly in $V \times H^2(\Omega)$ for some $Y \in A$. Applying (b) of Theorem 5.1 we see that
\[
S(t)Y_n \to S(t)Y \text{ weakly in } V \times H^2(\Omega),
\]
so that $X = S(t)Y$, since $S(t)Y_n = S(t_n)X_n \to X$ weakly in $V \times H^2(\Omega)$ by (5.13). Thus $A \subset S(t)A$.

(iii) By Lemma 5.2 it is enough to show that
\[
\text{dist}_{H \times V}(S(t)[z,v], A) \to 0 \text{ uniformly in } [z,v] \in B_o \text{ as } t \to +\infty.
\]
But this is evident from the definition (5.12) of $A$.  

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