Some Growth Properties of Entire Functions
Represented by Vector Valued Dirichlet Series in Two Variables

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Abstract

In the present paper, we study the entire functions represented by vector valued Dirichlet series of several complex variables. The characterizations of their order and type have been obtained. For the sake of simplicity, we have considered the functions of two variables only.

Keywords: Entire function, Dirichlet Series, Order, Type.

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1 Introduction

Consider

$$f(s_1, s_2) = \sum_{m,n=1} \alpha_m e^{\sigma_1 + it_j}, \quad (s_j = \sigma_j + it_j, \ j=1,2)$$

where \(\alpha_m\)'s belong to the Banach space \((E, \|\cdot\|)\); \(0 < \lambda_1 < \cdots < \lambda_m \to \infty \text{ as } m \to \infty\), \(0 < \mu_l < \cdots < \mu_n \to \infty \text{ as } n \to \infty\),

and

$$\limsup_{m+n \to \infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < +\infty.$$

Such a series is called a vector valued Dirichlet series in two complex variables. The concepts of order and type of an entire function (also for analytic function) represented by vector valued Dirichlet series of one complex variable were first introduced in 1983 by O.P.Juneja and B.L.Srivastava. They also obtained the coefficient characterizations of order and type. In this paper we have extended the results to the entire functions represented by vector valued Dirichlet series of several complex variables. For simplicity, we consider here functions of two variables only, though these results can easily be extended to functions of several complex variables.

Let \(f(s_1, s_2)\) defined by (1) represent an entire function. We define

$$M(\sigma_1, \sigma_2) = \sup \{\|f(\sigma_1 + it_1, \sigma_2 + it_2)\|; \ -\infty < t_j < \infty, \ j=1,2\}$$

to be the maximum modulus of \(f(s_1, s_2)\). Then \(M(\sigma_1, \sigma_2) \to \infty \text{ as } \sigma_1, \sigma_2 \to \infty\). We define the order \(\rho\) \((0 \leq \rho \leq \infty)\) of \(f(s_1, s_2)\) as follows.

**Class A**: An entire function \(f(s_1, s_2)\) defined by vector valued Dirichlet series of finite order belongs to class A, if there exist positive constants \(K_1, \gamma_1, K_2, \gamma_2\) such that

(i) For any fixed value of \(\sigma_2 > 0\), there exists a number \(\sigma_1^{(1)}(\gamma_1, \sigma_2) = \sigma_1^{(1)}(K_1, \gamma_1, \sigma_2)\) such that

$$M(\sigma_1, \sigma_2) < \exp\{K_1 \exp(\sigma_1 \gamma_1)\} \text{ for } \sigma_1 \geq \sigma_1^{(1)}.$$

(ii) For any fixed value of \(\sigma_1 > 0\), there exists a number \(\sigma_1^{(2)}(K_2, \gamma_2, \sigma_1) = \sigma_1^{(2)}(K_2, \gamma_2, \sigma_1)\) such that

$$M(\sigma_1, \sigma_2) < \exp\{K_2 \exp(\sigma_2 \gamma_2)\} \text{ for } \sigma_2 \geq \sigma_1^{(2)}.$$

Therefore, there exists a number \(\sigma = \sigma(K_1, K_2, \gamma_1, \gamma_2)\) such that

$$M(\sigma_1, \sigma_2) < \exp\{K_1 \exp(\sigma \gamma_1) + K_2 \exp(\sigma \gamma_2)\} \text{ for } \sigma_1, \sigma_2 \geq \sigma.$$
Definition 1.1 An entire function \( f(s_1, s_2) \) defined by the vector valued Dirichlet series (1.1) has finite orders \( \rho_1 \) and \( \rho_2 \) with respect to \( s_1 \) and \( s_2 \) respectively if

(i) For any arbitrarily small \( \epsilon > 0 \), and any \( \sigma_2 > 0 \), there exists a number \( \sigma^{(1)} = \sigma^{(1)}(\epsilon, \sigma_2) \) such that

\[
M(\sigma_1, \sigma_2) < \exp[\exp\{\sigma_1(\rho_1 + \epsilon)\}] \quad \text{for} \quad \sigma_1 \geq \sigma^{(1)}.
\]

In addition, there exists at least one value of \( \sigma_1 \), say \( \sigma_1^0(\epsilon) \) and correspondingly arbitrary large values of \( \sigma_2 : \{\sigma_2\} \) such that

\[
M(\sigma_1^0(\epsilon), \sigma_2) > \exp\{\sigma_1^0(\rho_1 - \epsilon)\}.
\]

Hence, (i) is equivalent to

\[
\limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1} \right\} = \rho_1.
\]

(ii) For any arbitrarily small \( \epsilon > 0 \), and any \( \sigma_1 > 0 \), there exists a number \( \sigma^{(2)} = \sigma^{(2)}(\epsilon, \sigma_1) \) such that

\[
M(\sigma_1, \sigma_2) < \exp[\exp\{\sigma_2(\rho_2 + \epsilon)\}] \quad \text{for} \quad \sigma_2 \geq \sigma^{(2)}.
\]

In addition, there exists at least one value of \( \sigma_1 \), say \( \sigma_1^0(\epsilon) \) and correspondingly arbitrarily large values of \( \sigma_2 : \{\sigma_2\} \) such that

\[
M(\sigma_1^0(\epsilon), \sigma_2) > \exp\{\sigma_1^0(\rho_2 - \epsilon)\}.
\]

So, (ii) is equivalent to

\[
\limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_2} \right\} = \rho_2.
\]

Definition 1.2 An entire function \( f(s_1, s_2) \) defined by vector valued Dirichlet series has a finite order \( (\rho_1, \rho_2) \) if

(i) \( f(s_1, s_2) \in A \)

(ii) \( f(s_1, s_2) \) has finite orders \( \rho_1 \) and \( \rho_2 \) with respect to \( s_1 \) and \( s_2 \) respectively as above.

(iii) For \( \epsilon > 0 \), there exists a number \( \sigma = \sigma(\epsilon) \) such that

\[
M(\sigma_1, \sigma_2) < \exp\{\exp \sigma_1(\rho_1 + \epsilon) + \exp \sigma_2(\rho_2 + \epsilon)\} \quad \text{for} \quad \sigma_1, \sigma_2 \geq \sigma.
\]

Thus, we define the order of vector valued Dirichlet series as
\[
\lim_{\sigma_1, \sigma_2 \to \infty} \sup \frac{\log(\log M(\sigma_1, \sigma_2))}{\log(e^{\sigma_1} + e^{\sigma_2})} = \rho.
\]

Similarly, if \(0 < \rho < \infty\), then the type \(T(0 \leq T \leq \infty)\) of \(f(s_1, s_2)\) is defined as:

\[
\lim_{\sigma_1, \sigma_2 \to \infty} \sup \frac{\log M(\sigma_1, \sigma_2)}{(e^{\rho \sigma_1} + e^{\rho \sigma_2})} = T.
\]

2. Basic Results.

We now prove

**Theorem 2.1** The necessary and sufficient condition for the series (1) satisfying the condition (2) to be entire is that

\[
\lim_{m,n \to \infty} \sup \frac{\log(\|a_{m,n}\|)}{\lambda_m + \mu_n} = -\infty.
\]

For proving our result, we need the following result:

**Lemma 2.2.** The following conditions are equivalent:

(i) \(\lim_{m,n \to \infty} \sup \frac{\log(m+n)}{\lambda_m + \mu_n} = D < +\infty\)

(ii) \(\lim_{m \to \infty} \sup \frac{\log m}{\lambda_m} = D_1 < +\infty, \lim_{n \to \infty} \sup \frac{\log n}{\mu_n} = D_2 < +\infty\),

(iii) There exists \(\alpha, 0 < \alpha < \infty\), such that the series \(\sum_{m,n=0}^{\infty} \exp[-\alpha(\lambda_m + \mu_n)]\) converges.

For the proof of this Lemma we refer to [2].

**Proof of Theorem 2.1** We suppose that (1) defines an entire function. Then it converges absolutely for all \((s_1, s_2)\). Now take the points with coordinates \((p, p)\) with \(p > 0\). Then it follows that \(\sum_{m,n=0}^{\infty} \|a_{m,n}\| \exp(\lambda_{m,n} + \mu_{m,n}) p < \infty\)

Therefore \(\|a_{m,n}\| \exp(\lambda_{m,n} + \mu_{m,n}) p \leq M(p, p)\)

and \(\lim_{m,n \to \infty} \sup \frac{\log \|a_{m,n}\|}{\lambda_m + \mu_n} < -p\).

Since \(p > 0\) is arbitrary, the necessity part of the theorem is proved.
Conversely, let the given condition (3) be satisfied. It suffices to prove that (1) converges absolutely for all \((s_1, s_2)\). Let us consider \((s_1, s_2)\in C^2, \quad \sigma > 0\) such that \(\text{Re } s_1 < \sigma, \text{Re } s_2 < \sigma\). Then by (3) and for some \(\delta > 0\),

\[
\log \frac{\|a_{m,n}\|}{\hat{\lambda}_m + \mu_n} < -\sigma - \delta \quad \text{for } m + n \geq N_0(\delta)
\]

and so

\[
\|a_{m,n}\| \exp\{(\hat{\lambda}_m + \mu_n)\sigma\} < \exp(-\delta(\hat{\lambda}_m + \mu_n)) .
\]

But the double series \(\sum_{m,n=0}^{\infty} \exp[-\delta(\hat{\lambda}_m + \mu_n)]\) being convergent in view of Lemma 2.2, it follows from (4) that the series (1) converges absolutely for all \((s_1, s_2)\), given by \(\text{Re } s_1 < \sigma, \text{Re } s_2 < \sigma\). Now, when the series (1) converges at \((s_1, s_2)\) it also converges at \((s_1', s_2')\) where \(\text{Re } s_1' < \text{Re } s_1, \text{Re } s_2' < \text{Re } s_2\), hence the result follows.

Next we prove

**Theorem 2.3** If

\[
f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} e^{s_1 \hat{\lambda}_m + s_2 \mu_n}
\]

is an entire function of order \((\rho_1, \rho_2), (0 < \rho_1, \rho_2 < \infty)\), then

\[
\lim_{\sigma_1, \sigma_2 \to \infty} \sup \left\{ \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 \rho_1 + \sigma_2 \rho_2} \right\} = 1 .
\]

**Proof.** The proof of this theorem follows on the lines of the proof of Theorem 1 in [2].

### 3 Main Results

**Theorem 3.1** If \(f(s_1, s_2)\) is an entire function of order \(\rho \quad (0 < \rho < \infty)\), then

\[
\rho = \lim_{m,n \to \infty} \sup \frac{\log(\hat{\lambda}_m \mu_n)}{\log \|a_{m,n}\|^{\frac{1}{n}}} .
\]

**Proof.** Let
\[
\mu = \lim_{m,n \to \infty} \sup \frac{\log(\lambda_m^\lambda \mu_n^\mu)}{\log \|a_{m,n}\|^{1}}
\]

First we show that \( \rho \geq \mu \). Let us assume that \( \mu > 0 \), for otherwise the result is trivially true. Then for a given \( \varepsilon > 0 \), we have two sequences \( \{\lambda_m^\rho\} \) and \( \{\mu_n^\mu\} \) with \( M_p \to \infty \) as \( p \to \infty \) and \( N_q \to \infty \) as \( q \to \infty \) such that

\[
\log \|a_{m,n}\| > - (\mu - \varepsilon)^{-1} (\lambda_m^\lambda \mu_n^\mu) \quad \text{for} \quad m = M_p \quad \text{and} \quad n = N_q.
\]

Since the inequality:

\[
M(\sigma_1, \sigma_2) \geq \|a_{m,n}\| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \quad \text{holds for all} \quad \sigma_1, \sigma_2 \quad \text{and} \quad m, n,
\]

it follows that for all \( \sigma_1 \) and \( \sigma_2 \) and \( m = M_p \) and \( n = N_q \)

\[
\log M(\sigma_1, \sigma_2) > \lambda_m^\lambda \{\sigma_1 - (\mu - \varepsilon)^{-1} \log \lambda_m^\lambda \} + \mu_n^\mu \{\sigma_2 - (\mu - \varepsilon)^{-1} \log \mu_n^\mu \}.
\]

Taking \( \sigma_1, \rho = (\mu - \varepsilon)^{-1} \log(e \lambda_m^\rho) \) and \( \sigma_2, \rho = (\mu - \varepsilon)^{-1} \log(e \mu_n^\rho) \) in the above we find that

\[
\log M(\sigma_1, \sigma_2) > \frac{\exp(\sigma_1 (\mu - \varepsilon)) + \exp(\sigma_2 (\mu - \varepsilon))}{e(\mu - \varepsilon)}
\]

for \( \sigma_1 = \sigma_1, \rho \) and \( \sigma_2 = \sigma_2, \rho \). Proceeding to limits as \( \sigma_1, \sigma_2 \to \infty \), we get \( \rho \geq \mu \).

Further, to prove the converse part, we suppose that \( \mu < \infty \), for otherwise the result is obviously true, so that

\[
\|a_{m,n}\| < \lambda_m^{-\lambda_m^{\mu \varepsilon}/\mu} \mu_n^{-\mu_n^{\mu \varepsilon}/\mu} \quad \text{for} \quad m > m_0 \quad \text{and} \quad n > n_0.
\]

Now,

\[
M(\sigma_1, \sigma_2) \leq \left( \sum_{m=1}^{m_0} \sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty} \sum_{n=1}^{n_0} + \sum_{m=1}^{m_0} \sum_{n=n_0+1}^{\infty} + \sum_{m=m_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} \right) \|a_{m,n}\| \|\exp(\sigma_1 \lambda_m + \sigma_2 \mu_n)\|
\]

\[
= \sum_{1}^{1} + \sum_{2}^{1} + \sum_{3}^{1} + \sum_{4}^{1}.
\]

Now we estimate the four parts of (6). Clearly

\[
\sum_{1} = O(\exp(\sigma_1 \lambda_{n_0} + \sigma_2 \mu_{m_0})).
\]

In view of (5), we get
\[ \sum_{m>n_0} \sum_{n>n_0} \exp\{ (\sigma \lambda_m + \sigma_2 \mu_n + \log(\lambda_m^{1/(\mu+\varepsilon)}) \mu_n^{1/(\mu+\varepsilon)} ) \} \]
\[ \leq \sum_{m>n_0} \sum_{n>n_0} \exp\{ \sigma \lambda_m + \sigma_2 \mu_n - (\mu + \varepsilon)^{-1}(\lambda_m \log \lambda_m + \mu_n \log \mu_n) \} \]
\[ \leq \max\{ \exp(\sigma \lambda_m + \sigma_2 \mu_n - (\mu + 2\varepsilon)^{-1}(\lambda_m \log \lambda_m + \mu_n \log \mu_n)) \} \]
\[ \leq \sum_{m>n_0} \sum_{n>n_0} \exp\left( -\frac{\lambda_m \log \lambda_m + \mu_n \log \mu_n}{\varepsilon^{-1}(\mu + \varepsilon)(\mu + 2\varepsilon)} \right) \]

The series on the right hand side of the above inequality is convergent and the maximum of the expression
\[ \exp(\sigma \lambda_m + \sigma_2 \mu_n - (\mu + 2\varepsilon)^{-1}(\lambda_m \log \lambda_m + \mu_n \log \mu_n) \]
is attained at \( \lambda_m = e^{-1} \exp(\sigma(\mu + 2\varepsilon)) \) and \( \mu_n = e^{-1} \exp(\sigma_2(\mu + 2\varepsilon)) \), we find that
\[ \sum_4 \leq A \exp\{ e^{-1}(\mu + 2\varepsilon)^{-1}(e^{\sigma(\mu+2\varepsilon)} + e^{\sigma_2(\mu+2\varepsilon)}) \} \]
where \( A \) is an absolute constant. Further, to estimate \( \sum_2 \mu \) being finite, we have
for all values of \( m \) and \( n \)
\[ \log(\lambda_m^{\mu_n} / \mu_n \lambda_m^{\mu_n}) \leq \xi, \]
\[ \log \| a_m \|^{-1} \leq \xi, \]

Therefore,
\[ \sum_2 \leq \sum_{m>n_0} \sum_{n=1}^{n_0} \exp(\sigma \lambda_m + \sigma_2 \mu_n - \xi^{-1} \lambda_m \log \lambda_m - \xi^{-1} \mu_n \log \mu_n) \]
\[ = O(\exp(\sigma_2 \mu_n)) \sum_{m>n_0} \exp(\sigma \lambda_m - \xi^{-1} \lambda_m \log \lambda_m) \]
\[ \leq O(\exp(\sigma_2 \mu_n)) \max\{ \exp(\sigma \lambda_m - (\xi + \varepsilon)^{-1} \lambda_m \log \lambda_m) \} \sum_{m>n_0} (-\xi \frac{(\xi + \varepsilon)^{-1}}{\xi}) \lambda_m \log \lambda_m) \]
\[ = O(\exp(\sigma_2 \mu_n)) \max\{ \exp(\sigma \lambda_m - (\xi + \varepsilon)^{-1} \lambda_m \log \lambda_m) \}. \]

But the \( \max. \exp(\sigma \lambda_m - (\xi + \varepsilon)^{-1} \lambda_m \log \lambda_m) \) is attained at
\[ \lambda_m = e^{-1} \exp(\sigma(\xi + \varepsilon)). \]

Hence,
\[ \sum_2 \leq O(\exp(\sigma_2 \mu_n)) \exp\{ e^{-1}(\xi + \varepsilon)^{-1} \exp(\sigma(\xi + \varepsilon)) \} \].

Similarly
\[ \sum_3 \leq O(\exp(\sigma \lambda_m)) \exp\{ e^{-1}(\xi + \varepsilon)^{-1} \exp(\sigma_2(\xi + \varepsilon)) \}. \]
Substituting these estimates of \( \sum_i (1 \leq i \leq 4) \), in (6), we have
\[
\log \log M(\sigma_i, \sigma_2) \leq \log \{ \exp(\sigma_i (\mu + 2 \epsilon)) + \exp(\sigma_2 (\mu + 2 \epsilon)) \} + O(1),
\]
which gives \( \rho \leq \mu \) on proceeding to limits as \( \sigma_i, \sigma_2 \to \infty \). Hence the theorem follows.

**Theorem 3.2.** Let \( f(s_1, s_2) \) be an entire function of order \( 0 < \rho < \infty \) and type \( 0 \). Then
\[
\lim_{m,n \to \infty} \sup(\lambda_n^\alpha \mu_n^{\mu}, \| a_{mn} \|^\rho)^{(1/(\lambda_n + \mu_n))} = e^{\rho T}.
\]

**Proof.** We shall only sketch the proof, since it follows on the lines of proof of Theorem 1.

For a given \( \epsilon > 0 \) we get sequences \( m = M_\rho \) and \( n = N_\rho \), such that
\[
\log M(\sigma_1, \sigma_2) > \lambda_n \left( \sigma_1 + \rho^{-1} \log \frac{\alpha - \epsilon}{\lambda_n} \right) + \mu_n \left( \sigma_2 + \rho^{-1} \log \frac{\alpha - \epsilon}{\mu_n} \right) \quad (7)
\]
where \( \alpha \) is assumed to be positive and is given by
\[
\alpha = \lim_{m,n \to \infty} \sup(\lambda_n^\alpha \mu_n^{\mu}, \| a_{mn} \|^\rho)^{(1/(\lambda_n + \mu_n))}.
\]

Choosing in (7) the sequences of the values of \( \sigma_1 \) and \( \sigma_2 \) given by
\[
\sigma_1 = \rho^{-1} \log \left( \frac{e^{\lambda_n}}{\alpha - \epsilon} \right), \quad \sigma_2 = \rho^{-1} \log \left( \frac{e^{\mu_n}}{\alpha - \epsilon} \right)
\]

it follows that \( e^{\rho T} \geq \alpha \). This assertion is trivially true for the case when \( \alpha = 0 \).

The converse part follows by using the following estimations of \( \sum_i \)'s in (6):
\[
\begin{align*}
\sum_i & \leq O(\exp(\sigma_i \lambda_n + \sigma_j \mu_n)), \\
\sum_i & \leq O(\exp(\sigma_i \lambda_n) \exp \left( \frac{\zeta + \epsilon}{e^\beta} e^{\rho \sigma_i} \right)), \\
\sum_i & \leq O(\exp(\sigma_j \mu_n) \exp \left( \frac{\zeta + \epsilon}{e^\beta} e^{\rho \sigma_j} \right)), \\
\sum_i & \leq A_i \exp \left( \frac{\alpha + 2 \epsilon}{e^\beta} (e^{\rho \sigma_i} + e^{\rho \sigma_j}) \right),
\end{align*}
\]
where \( A_i \) is an absolute constant. Hence
\[
M(\sigma_i, \sigma_2) \leq \exp(e^{\rho \sigma_i} + e^{\rho \sigma_j}) + O(1),
\]
or
\[
\log M(\sigma_i, \sigma_2) \leq (e^{\rho \sigma_i} + e^{\rho \sigma_j}) + o(1)
\]

which gives \( e^{\rho T} \leq \alpha \) on proceeding to limits as \( \sigma_i, \sigma_2 \to \infty \). This proves the result.
References
