Hyers-Ulam-Rassias Stability of Orthogonal Quadratic Functional Equation in Modular Spaces

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Abstract

In this paper, we study the Hyers-Ulam-Rassias stability of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$, $x \perp y$ in which $\perp$ is orthogonality in the sense of Rätz in modular spaces.

Keywords: Hyers-Ulam-Rassias stability, Orthogonality, Orthogonally quadratic equation, Modular space.

1 Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam. In 1940, he posed the following problem: Give conditions in order for a linear mapping near an approximately additive mapping to exist (see [27]).

In 1941, this problem was solved by D. H. Hyers [7] for the first time. Subsequently, the result of Hyers was generalized by T. Aoki [2] for additive mappings and Th. M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [20] has provided a lot of influences in the development of the Hyers-Ulam-Rassias stability of functional equations (see [16]). During the last decades several stability problems of functional equations have been investigated by a number of mathematicians in various spaces, such as fuzzy normed spaces, orthogonal normed spaces and
random normed spaces; see [3, 5, 8, 9, 15, 22, 30] and reference therein. Recently, Gh. Sadeghi [23] proved the Hyers-Ulam stability of the generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(x)$ in modular space, using the fixed point method. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by H. Nakano [18] and were intensively developed by his mathematical school: S. Koshi, T. Shimogaki, S. Yamamuro [10, 29] and others. Further and the most complete development of these theories are due to W. Orlicz, S. Mazur, J. Musielak, W. A. Luxemburg, Ph. Turpin [12, 14, 17, 26] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various W. Orlicz spaces [19] and interpolation theory [11], which in their turn have broad applications [13, 17]. The importance for applications consists in the richness of the structure of modular spaces, that-besides being Banach spaces (or $F$-spaces in more general setting)- are equipped with modular equivalent of norm or metric notions.

There are several orthogonality notions on a real normed spaces as Birkhoff-James, semi-inner product, Carlsson, Singer, Roberts, Pythagorean, isosceles and Diminnie (see, e.g., [1]). Let us recall the orthogonality space in the sense of Rätz; cf. [21].

Suppose $E$ is a real vector space with $\dim E \geq 2$ and $\perp$ is a binary relation on $E$ with the following properties:

- (O1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in E$;
- (O2) independence: if $x, y \in E - \{0\}$, $x \perp y$, then, $x, y$ are linearly independent;
- (O3) homogeneity: if $x, y \in E$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (O4) the Thalesian property: if $P$ is a 2-dimensional subspace of $E$. If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(E, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure. Some interesting examples of orthogonality spaces are:

(i) The trivial orthogonality on a vector space $E$ defined by (O1), and for nonzero elements $x, y \in E$, $x \perp y$ if and only if $x, y$ are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(E, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(E, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in E$.  

Clearly examples (i) and (ii) are symmetric but example (iii) is not. However, it is remarkable to note, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

Let \((E, \perp)\) be an orthogonality space and \((G, +)\) be an Abelian group. A mapping \(f : E \to G\) is said to be (orthogonally) quadratic if it satisfies

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y
\]

for all \(x, y \in E\). The orthogonally quadratic functional equation (1), was first investigated by Vajzović [28] when \(E\) is a Hilbert space, \(G\) is equal to \(\mathbb{C}\), \(f\) is continuous and \(\perp\) means the Hilbert space orthogonality. Later Drlijević, Fochi and Szabó generalized this result [4, 6, 25]. J. Sikorska [24] obtained the generalized orthogonal stability of some functional equations.

In the present paper, we establish the Hyers-Ulam-Rassias Stability of Orthogonal Quadratic Functional Equation (1) in Modular spaces. Therefore, we generalized the main of theorem 5 of [24].

2 Preliminary

In this section, we give the definitions that are important in the following.

**Definition 2.1.** Let \(X\) be an arbitrary vector space.

(a) A functional \(\rho : X \to [0, \infty]\) is called a modular if for arbitrary \(x, y \in X\),

(i) \(\rho(x) = 0\) if and only if \(x = 0\),

(ii) \(\rho(\alpha x) = \rho(x)\) for every scaler \(\alpha\) with \(|\alpha| = 1\),

(iii) \(\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)\) if and only if \(\alpha + \beta = 1\) and \(\alpha, \beta \geq 0\),

(b) if (iii) is replaced by

(iii') \(\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)\) if and only if \(\alpha + \beta = 1\) and \(\alpha, \beta \geq 0\),

then we say that \(\rho\) is a convex modular.

A modular \(\rho\) defines a corresponding modular space, i.e., the vector space \(X_\rho\) given by

\[X_\rho = \{x \in X : \rho(\lambda x) \to 0\text{ as } \lambda \to 0\}\].

Let \(\rho\) be a convex modular, the modular space \(X_\rho\) can be equipped with a norm called the Luxemburg norm, defined by

\[\|x\|_\rho = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}\].

A function modular is said to satisfy the \(\Delta_2\)–condition if there exists \(k > 0\) such that \(\rho(2x) \leq k\rho(x)\) for all \(x \in X_\rho\).
Definition 2.2. Let \( \{x_n\} \) and \( x \) be in \( X_\rho \). Then

(i) we say \( \{x_n\} \) is \( \rho \)-convergent to \( x \) and write \( x_n \xrightarrow{\rho} x \) if and only if \( \rho(x_n - x) \to 0 \) as \( n \to \infty \),

(ii) the sequence \( \{x_n\} \), with \( x_n \in X_\rho \), is called \( \rho \)-Cauchy if \( \rho(x_n - x_m) \to 0 \) as \( m,n \to \infty \),

(iii) a subset \( S \) of \( X_\rho \) is called \( \rho \)-complete if and only if any \( \rho \)-Cauchy sequence is \( \rho \)-convergent to an element of \( S \).

The modular \( \rho \) has the Fatou property if and only if \( \rho(x) \leq \lim_{n \to \infty} \inf \rho(x_n) \) whenever the sequence \( \{x_n\} \) is \( \rho \)-convergent to \( x \). For further details and proofs, we refer the reader to [17].

Remark 2.3. If \( x \in X_\rho \) then \( \rho(ax) \) is a nondecreasing function of \( a \geq 0 \). Suppose that \( 0 < a < b \), then property (iii) of definition 2.1 with \( y = 0 \) shows that

\[
\rho(ax) = \rho(\frac{a}{b}bx) \leq \rho(bx).
\]

Moreover, if \( \rho \) is convex modular on \( X \) and \( |\alpha| \leq 1 \) then, \( \rho(ax) \leq |\alpha|\rho(x) \) and also \( \rho(x) \leq \frac{1}{2}\rho(2x) \leq \frac{k}{2}\rho(x) \) if \( \rho \) satisfy the \( \Delta_2 \)-condition for all \( x \in X \).

Throughout this paper, \( \mathbb{N} \) and \( \mathbb{R} \) denote the sets of all positive integers and all real numbers, respectively. By the notation \( E_p \) we mean \( E \setminus \{0\} \) provided that \( p < 0 \) and \( E \) otherwise. In order to avoid some definitional problems we also assume for the sake of this paper that \( 0^0 := 1 \).

3 Orthogonal Stability of Eq (1) in Modular Spaces

In this section we assume that the convex modular \( \rho \) has the Fatou property such that satisfies the \( \Delta_2 \)-condition with \( 0 < k \leq 2 \). In addition, we assume that \( (E_p, \perp) \) denotes an orthogonality space, on the other hand, we give the Hyers-Ulam-Rassias stability of orthogonal quadratic functional equation in modular spaces.

Theorem 3.1. Let \( (E_p, \|\cdot\|) \) with \( \dim E_p \geq 2 \) be a real normed linear space with Birkhoff-James orthogonality and \( X_\rho \) is \( \rho \)-complete modular space. If a function \( f : E_p \to X_\rho \) satisfies

\[
\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \leq \epsilon(\|x\|^p + \|y\|^p),
\]

for all \( x,y \in E_p \) with \( x \perp y \), \( \epsilon \geq 0 \) and \( p < 2 \), then there exist unique quadratic mapping \( Q : E_p \to X_\rho \) such that

\[
\rho(f(x) - Q(x)) \leq \begin{cases} 
\frac{\beta^+}{4-2p} \|x\|^p & \text{if } 0 \leq p < 2, \\
\frac{\beta^-}{4-2p} \|x\|^p & \text{if } p < 0,
\end{cases}
\]
for all \( x \in E_p \), where \( \beta^+ = \frac{k^{\alpha^+}}{8}(2 + k + k.3^p) \), \( \beta^- = \frac{k^{\alpha^-}}{8}(2 + k + k.2^{-p}) \), 
\( \alpha^+ = \frac{k}{2}(2p + 22^p + k + k.3^p) \) and \( \alpha^- = \frac{k}{2}(2 + k + k.2^{-p}) \).

**Proof.** Fix \( x \in E_p \) and choose \( y_0, z_0 \in E_p \) such that \( x \perp y_0 \), \( x \perp z_0 \) and \( y_0 \perp z_0 \). Then as well whence \( x + y_0 \perp x - y_0 \) and by (2) we get

\[
\rho(f(2x) + f(2y_0) - 2f(x + y_0) - 2f(x - y_0)) \leq \epsilon(\|x + y_0\|^p + \|x - y_0\|^p). \tag{4}
\]

Then, from (2) and (4) we have

\[
\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) = \rho(f(2x) + f(2y_0) - 2f(x + y_0) - 2f(x - y_0))
\]

\[
-2f(x - y_0) + 2f(x + y_0) + 2f(x - y_0) - 4f(x) - 4f(y_0))
\]

\[
\leq \frac{k}{2} \rho(f(2x) + f(2y_0) - 2f(x + y_0) - 2f(x - y_0))
\]

\[
+ \frac{k^2}{2} \rho(f(x + y_0) + f(x - y_0) - 2f(x) - 2f(y_0))
\]

\[
\leq \frac{k\epsilon}{2} \{\|x + y_0\|^p + \|x - y_0\|^p + k(\|x\|^p + \|y_0\|^p)\}. \tag{5}
\]

From the definition of the orthogonality, since \( x \perp y_0 \), we derive \( \|x\| \leq \|x + y_0\| \) and \( \|x\| \leq \|x - y_0\| \) (for \( \lambda = 1 \) and \( \lambda = -1 \), respectively), and analogously, from \( x + y_0 \perp x - y_0 \) we derive \( \|x + y_0\| \leq 2\|x\| \) and \( \|x + y_0\| \leq 2\|y_0\| \). From this relation and the triangle inequality we have additionally \( \|y_0\| = \|y_0 + x - x\| \leq \|x + y_0\| + \|x\| \leq 3\|x\| \), \( \|x - y_0\| \leq \|y_0\| + \|x\| \leq 4\|x\| \) and \( \|x\| \leq \|x + y_0\| \leq 2\|y_0\| \)

In case \( p \) is a non-negative real number, we have the approximation

\[
\|x + y_0\|^p \leq 2^p \|x\|^p, \|x - y_0\|^p \leq 4^p \|x\|^p \text{ and } \|y_0\|^p \leq 3^p \|x\|^p
\]

otherwise

\[
\|y_0\|^p \leq 2^{-p} \|x\|^p, \|x - y_0\|^p \leq \|x\|^p \text{ and } \|x + y_0\|^p \leq \|x\|^p
\]

**Case 1:** if \( p < 0 \) then (5) become

\[
\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) \leq \alpha^- \|x\|^p \tag{6}
\]

where \( \alpha^- = \frac{k}{2}(2 + k + k.2^{-p}) \).
In the same way, from the conditions \( x + z_0 \perp x - z_0 \) and \( y_0 + z_0 \perp y_0 - z_0 \) we obtain

\[
\rho(f(2x) + f(2z_0) - 4f(x) - 4f(z_0)) \leq \alpha^- \|x\|^p \tag{7}
\]
and
\[
\rho(f(2y) + f(2z) - 4f(y) - 4f(z)) \leq \alpha^- \|y_0\|^p \leq 2^{-p} \alpha^- \|x\|^p. \tag{8}
\]
From (6), (7) and (8) we get
\[
\rho(2f(2x) - 8f(x)) = \rho(f(2x) + f(2y) - 4f(x) - 4f(y) + f(2x) + f(2z) - 4f(x) - 4f(z) + f(y_0) + f(z) - f(2y_0) - f(2z_0)) \\
\leq \frac{k}{2} \rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0) \\
+ \frac{k}{2} \rho(f(2x) + f(2z_0) - 4f(x) - 4f(z_0) + f(y_0) + f(z_0) - f(2y_0) - f(2z_0)) \\
\leq \frac{k\alpha^-}{2} \|x\|^p + \frac{k^2}{4} (\alpha^- \|x\|^p + 2^{-p} \alpha^- \|x\|^p) \leq \frac{k\alpha^-}{4} (2 + k + 2^{-p} k) \|x\|^p.
\]
Hence
\[
\rho(f(2x) - 4f(x)) = \rho\left(\frac{1}{2} (2f(2x) - 8f(x))\right) \leq \frac{1}{2} \rho(2f(2x) - 8f(x)) \\
\leq \frac{k}{8} \alpha^- (2 + k + 2^{-p} k) \|x\|^p \\
\leq \beta^- \|x\|^p, \tag{9}
\]
for all $x \in E_p$, where $\beta^- = \frac{k\alpha^-}{8} (2 + k + 2^{-p}).$ Thus
\[
\rho\left(\frac{f(2x)}{4} - f(x)\right) = \rho\left(\frac{1}{4} (f(2x) - 4f(x))\right) \leq \frac{1}{4} \beta^- \|x\|^p, \tag{10}
\]
Replacing $x$ by $2x$ in (9) we get
\[
\rho(f(4x) - 4f(2x)) \leq \beta^- \|2x\|^p, \tag{11}
\]
for all $x \in E_p$. By (11) and (9) we have
\[
\rho\left(\frac{f(2^2x)}{4} - 4f(x)\right) = \rho\left(\frac{f(2^2x)}{4} - f(2x) + f(2x) - 4f(x)\right) \\
\leq \frac{1}{2} \rho\left(\frac{f(2^2x)}{2} - 2f(2x)\right) + \frac{k}{2} \rho(f(2x) - 4f(x)) \\
\leq \frac{1}{4} \rho(f(2^2x) - 4f(2x)) + \frac{k^2}{4} \rho(f(2x) - 4f(x)) \\
\leq \frac{\beta^-}{4} \|2x\|^p + \frac{k^2 \beta^-}{4} \|x\|^p \leq \beta^- \left(\frac{1}{4} \|2x\|^p + \frac{k^2}{4} \|x\|^p\right).
\]
Thus
\[
\rho\left(\frac{f(2^2x)}{4^2} - f(x)\right) = \rho\left(\frac{1}{4} \left(\frac{f(2^2x)}{4} - 4f(x)\right)\right) \\
\leq \beta^- \left(\frac{1}{4^2} \|2x\|^p + \frac{k^2}{4^2} \|x\|^p\right). \tag{12}
\]
By mathematical induction, we can easily see that
\[
\rho \left( \frac{f(2^n x)}{4^n} - f(x) \right) \leq \frac{\beta^2}{4^n} \sum_{i=1}^{n} k^{2(n-i)} \|2^{i-1} x\|^p \tag{13}
\]
for all \( x \in E_p \). Indeed, for \( n = 1 \) the relation (13) is true. Assume that the relation (13) is true for \( n \), and we show this relation rest true for \( n + 1 \), thus we have
\[
\rho \left( \frac{f(2^{n+1} x)}{4^{n+1}} - f(x) \right) \leq \frac{1}{4} \rho \left( \frac{f(2^{n+1} x)}{4^n} - f(x) \right)
\]
\[= \frac{1}{4} \rho \left( \frac{f(2^{n+1} x)}{4^n} - f(2x) + f(2x) - f(x) \right) \]
\[\leq \frac{k}{8} \left[ \rho \left( \frac{f(2^{n+1} x)}{4^n} - f(2x) \right) + \rho \left( f(2x) - f(x) \right) \right] \]
\[\leq \frac{k \beta}{8} \left[ \frac{1}{4} \sum_{i=1}^{n} k^{2(n-i)} \|2^{i-1} x\|^p + \|x\|^p \right] \]
\[= \frac{k \beta}{8} \sum_{i=0}^{n} \frac{1}{4} k^{2(n-i)} \|2^{i} x\|^p \]
\[\leq \frac{k \beta}{2} \sum_{i=0}^{n} k^{2(n-i)} \|2^{i} x\|^p \]
\[\leq \frac{\beta}{4^{n+1}} \sum_{i=1}^{n+1} k^{2(n+1-i)} \|2^{i-1} x\|^p ,
\]
hence the relation (13) is true for all \( x \in E_p \) and \( n \in \mathbb{N}^* \). Thus
\[
\rho \left( \frac{f(2^n x)}{4^n} - f(x) \right) \leq \frac{\beta}{4^n} \sum_{i=1}^{n} k^{2(n-i)} \|2^{i-1} x\|^p \]
\[\leq \beta \sum_{i=1}^{n} 2^{-2i} \|2^{i-1} x\|^p \]
\[= \beta \frac{1 - 2^n(p-2)}{4 - 2^p} \|x\|^p \tag{14}
\]
for all \( x \in E_p \). Replacing \( x \) by \( 2^m x \) (with \( m \in \mathbb{N}^* \)) in (14) we obtain
\[
\rho \left( \frac{f(2^{m+n} x)}{4^n} - f(2^m x) \right) \leq \frac{\beta}{4^n} \frac{2^{mp}}{4 - 2^p} \|2^{n(p-2)} x\|^p \tag{15}
\]
for all \( x \in E_p \). Whence
\[
\rho \left( \frac{f(2^{m+n} x)}{4^{n+m}} - f(2^m x) \right) = \rho \left( \frac{1}{4^m} \left( \frac{f(2^{m+n} x)}{4^{n+m}} - f(2^m x) \right) \right) \]
\[\leq \frac{\beta}{4^n} \frac{2^{mp}}{4 - 2^p} \|2^{n(p-2)} x\|^p \tag{16}
\]
for all $x \in E_p$. If $m, n \to \infty$ we get, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is $\rho-$Cauchy sequence in the $\rho-$complete modular space $X_{\rho}$. Hence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is $\rho-$convergent in $X_{\rho}$, and we well define the mapping $Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}$ from $E_p$ into $X_{\rho}$ satisfying

$$
\rho(f(x) - Q(x)) \leq \frac{\beta^{-} \|x\|^{p}}{4 - 2^p},
$$

(17)

for all $x \in E_p$, since $\rho$ has Fatou property. For all $x, y \in E_p$ with $x \perp y$, by applying (2) and (O3) we get

$$
\rho(4^{-n}(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y))) \leq \epsilon 2^{n(p-2)}(\|x\|^p + \|y\|^p).
$$

(18)

If $n \to \infty$ then, we conclude that

$$
Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y) = 0, \ x \perp y
$$

for all $x, y \in E_p$ and on account of the results by F. Vajzović [28] and M. Fochi [6], $Q$ is quadratic. To prove the uniqueness, assume $Q' : E_p \to X_{\rho}$ to be another quadratic mapping satisfying (17). Then, for each $x \in E_p$ and all $n \in \mathbb{N}$ one has

$$
\rho(Q(x) - Q'(x)) = \rho\left(\frac{1}{n^2}(Q(nx) - Q'(nx))\right) \leq \frac{1}{n^2}\rho(Q(nx) - Q'(nx))
$$

$$
= \frac{1}{n^2}\rho(Q(nx) - f(nx) + f(nx) - Q'(nx))
$$

$$
\leq \frac{k}{n^2}[\rho(Q(nx) - f(nx)) + \rho(f(nx) - Q'(nx))]
$$

$$
\leq \frac{k n^{-2} \|x\|^p}{4 - 2^p}.
$$

If $n \to \infty$ we obtain $Q = Q'$.

**Case 2:** if $0 \leq p < 2$ then (5) become

$$
\rho(f(2x) + f(2y_0) - 4f(x) - 4f(y_0)) \leq \alpha^+ \|x\|^p
$$

(19)

where $\alpha^+ = \frac{k}{2}(2^p + 4^p + k + k.3^p)$, and by the case 1 we have

$$
\rho(f(2x) - 4f(x)) \leq \beta^+ \|x\|^p,
$$

(20)

for all $x \in E$, where $\beta^+ = \frac{k}{8}(2 + k + k.3^p)$. The rest of the proof is similar to the proof of the first case, just the constants $\beta^+$ and $\alpha^+$ serve as $\beta^-$ and $\alpha^-$, respectively. This completes the proof of theorem. 

In the following theorem we take the integers in the set $2^{\mathbb{N}} := \{2^m : m \in \mathbb{N}\}$. 
Theorem 3.2. Let \((E, \|\cdot\|)\) with \(\dim E \geq 2\) be a real normed linear space with Birkhoff-James orthogonality and \(X_\rho\) is \(\rho\)-complete modular space. If a function \(f : E \to X_\rho\) satisfying
\[
\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \leq \epsilon(\|x\|^p + \|y\|^p),
\]
for all \(x, y \in E\) with \(x \perp y\), \(\epsilon \geq 0\) and \(p > 2\), then there exist unique quadratic mapping \(Q : E \to X_\rho\) such that
\[
\rho(f(x) - Q(x)) \leq \frac{\beta^+}{2^p - 4} \|x\|^p,
\]
for all \(x \in E\), where \(\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)\) and \(\alpha^+ = \frac{k}{2}(2^p + 2^{2p} + k + k.3^p)\).

**Proof.** Using Theorem 3.1, the case \(0 \leq p < 2\) we have
\[
\rho(f(2x) - 4f(x)) \leq \beta^+ \|x\|^p,
\]
for all \(x \in E\), where \(\beta^+ = \frac{k\alpha^+}{8}(2 + k + k.3^p)\) and \(\alpha^+ = \frac{k}{2}(2^p + 2^{2p} + k + k.3^p)\). Replacing \(x\) by \(\frac{x}{2}\) in (23) we get
\[
\rho(f(x) - 4f(x/2)) \leq \beta^+ \left\| \frac{x}{2} \right\|^p.
\]
Replacing \(x\) by \(\frac{x}{2}\) in (24) we obtain
\[
\rho(f(x/2) - 4f(x/2^2)) \leq \beta^+ \left\| \frac{x}{2^2} \right\|^p.
\]
From (24) and (25) we get
\[
\rho(f(x) - 4^2 f(x/2^2)) = \rho(f(x) - 4f(x/2) + 4f(x/2) - 4^2 f(x/2^2))
\[
\leq \frac{k}{2} \rho(f(x) - 4f(x/2)) + \frac{k}{2} \rho(4f(x/2) - 4^2 f(x/2^2))
\leq \frac{k^2}{4} \rho(f(x) - 4f(x/2)) + \frac{k^3}{2} \rho(f(x/2) - 4f(x/2^2))
\leq \frac{k^2}{4} \beta^+ \left\| \frac{x}{2} \right\|^p + \frac{k^4}{4} \beta^+ \left\| \frac{x}{2^2} \right\|^p
\leq \frac{\beta^+}{4} \left( k^2 \left\| \frac{x}{2} \right\|^p + k^4 \left\| \frac{x}{2^2} \right\|^p \right)
\]
for all \(x \in E\). By mathematical induction, we can easily see that
\[
\rho(f(x) - 4^n f(x/2^n)) \leq \frac{\beta^+}{4} \sum_{i=1}^{n} k^{2i} \left\| \frac{x}{2^i} \right\|^p.
\]
Whence

\[ \rho(f(x) - 4^n f\left(\frac{x}{2^n}\right)) \leq \frac{\beta^+}{4} \sum_{i=1}^{n} k^{2i} \left\| \frac{x}{2^n} \right\|^p \leq \frac{\beta^+}{4} \sum_{i=1}^{n} 2^{i(2-p)} \|x\|^p \]

\[ = \frac{\beta^+}{2^p - 4} (1 - 2^{n(2-p)}) \|x\|^p \]  \hspace{1cm} (28)

Same as the first case in the theorem 3.1, we find, for each \( x \in E \) the sequence \( \{4^n f\left(\frac{x}{2^n}\right)\} \) is \( \rho \)-Cauchy sequence in \( \rho \)-complete modular space \( X_\rho \). Hence \( \{4^n f\left(\frac{x}{2^n}\right)\} \) is \( \rho \)-convergent in \( X_\rho \) and we well define the mapping \( Q(x) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \) from \( E \) into \( X_\rho \) satisfying

\[ \rho(f(x) - Q(x)) \leq \frac{\beta^+}{2^p - 4} \|x\|^p, \]  \hspace{1cm} (29)

for all \( x \in E \), since \( \rho \) has Fatou property. For all \( x, y \in E \), with \( x \perp y \), we obtain

\[ \rho(4^n f(2^{-n}(x+y)) + f(2^{-n}(x-y)) - 2f(2^{-n}x) - 2f(2^{-n}y)) \leq \epsilon 2^{n(2-p)}(\|x\|^p + \|y\|^p). \]  \hspace{1cm} (30)

If \( n \to \infty \) then, we conclude that \( Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0, \ x \perp y \) for all \( x, y \in E \) and on account of the results by F. Vajzović [28] and M. Fochi [6], \( Q \) is quadratic. To prove the uniqueness, assume \( Q' : E \to X_\rho \) to be another quadratic mapping satisfying (29). Then, for each \( x \in E \) and for all \( n \in 2^\mathbb{N} \) one has

\[ \rho(Q(x) - Q'(x)) = \rho(n^2(Q\left(\frac{x}{n}\right) - Q'(\left(\frac{1}{n}\right)))) \leq k^{2m} \rho(Q\left(\frac{x}{2^{2m}}\right) - Q'(\left(\frac{x}{2^{2m}}\right))) \]

\[ \leq 2^{m^2} k^2 \left[ \rho(Q\left(\frac{x}{2^{2m}}\right) - f\left(\frac{x}{2^{2m}}\right)) + \rho\left(Q'(\left(\frac{x}{2^{2m}}\right)) - f\left(\frac{x}{2^{2m}}\right)) \right] \]

\[ \leq 2^{m(2-p)} k^2 \frac{k}{2} \|x\|^p. \]

If \( m \to \infty \) we obtain \( Q = Q' \). This completes the proof of theorem. \( \Box \)

**Corollary 3.3.** Let \( E \) is a real linear space with \( \dim E \geq 2 \) and \( X_\rho \) is \( \rho \)-complete modular space. If a function \( f : E \to X_\rho \) satisfying

\[ \rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \leq \epsilon, \]  \hspace{1cm} (31)

for all \( x, y \in E \) with \( x \perp y \) and \( \epsilon \geq 0 \), then there exist unique quadratic mapping \( Q : E \to X_\rho \) such that

\[ \rho(f(x) - Q(x)) \leq \frac{\epsilon[k(k + 1)]^2}{24} \]  \hspace{1cm} (32)

for all \( x \in E \).
Corollary 3.4. Let \((E_p, \|\cdot\|)\) with \(\dim E_p \geq 2\) be a real normed linear space with Birkhoff-James orthogonality and \((X, \|\cdot\|)\) is Banach space. If a function \(f : E_p \to X\) satisfying
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),
\]
for all \(x, y \in E_p\) with \(x \perp y\), \(\epsilon \geq 0\) and \(p \in \mathbb{R} \setminus \{2\}\), then there exist unique quadratic mapping \(Q : E_p \to X\) such that
\[
\|f(x) - Q(x)\| \leq \frac{\beta^+ \text{sgn}(p-2)}{2p-4} \|x\|^p \text{ if } p \in \mathbb{R}^+ \setminus \{2\},
\]
\[
\|f(x) - Q(x)\| \leq \frac{\beta^-}{4-2p} \|x\|^p \text{ if } p < 0,
\]
for all \(x \in E_p\), where \(\beta^+ = \alpha^+ = \frac{\alpha^+}{4}(4 + 2\sqrt{3})\), \(\beta^- = \alpha^- = \frac{\alpha^-}{4}(4 + 2^{1-p})\), \(\alpha^+ = \epsilon(2^p + 2^{2p} + 2 + 2\sqrt{3})\) and \(\alpha^- = \epsilon(4 + 2^{1-p})\).

Proof. It is well known that every normed space is a modular space with the modular \(\rho(x) = \|x\|\) and \(k = 2\). \(\square\)

Corollary 3.5. Let \(E\) is a real linear space with \(\dim E \geq 2\) and \((X, \|\cdot\|)\) is Banach space. If a function \(f : E \to X\) satisfying
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon,
\]
for all \(x, y \in E\) with \(x \perp y\) and \(\epsilon \geq 0\), then there exist unique quadratic mapping \(Q : E \to X\) such that
\[
\|f(x) - Q(x)\| \leq \frac{3}{2} \epsilon
\]
for all \(x \in E\).

Proof. It is well known that every normed space is a modular space with the modular \(\rho(x) = \|x\|\), \(p = 0\) and \(k = 2\). \(\square\)

References


