Minimality Conditions on $\Pi^c_k$-Connectedness in Graphs

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Abstract

Let $k$ be a positive integer. A graph $G = (V,E)$ is said to be $\Pi^c_k$ - connected if for any given edge subset $F$ of $E(G)$ with $|F| = k$, the subgraph induced by $F$ is connected. In this paper, we explore the minimality conditions on $\Pi^c_k$-connectedness of a graph and also its properties of prism and corona graphs are obtained.

Keywords: Graph, subgraph, prism graph, corona graph, $\Pi^c_k$ - connected graph.

1 Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V,E)$ with vertex set $V$ and edge set $E$. As such $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G$, respectively. An edge-induced subgraph is a subset of the edges of a graph $G$ together with any vertices that are their endpoints. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of edges $X \subseteq E$. A graph $G$ is connected if it has a $u - v$ path whenever $u,v \in V(G)$ (otherwise, $G$ is disconnected). A graph with no cycle is acyclic. A tree $T$ is a connected acyclic graph. An edge-independent
set in a graph is a set of pairwise nonadjacent edges. A cut-edge or cut-vertex of a graph \( G \) is an edge or a vertex whose deletion increases the number of components. Difference between two sets \( A \) and \( B \) consists of the elements of \( A \) that are not in \( B \) and is denoted by \( A \setminus B \). Unless mentioned otherwise, for terminology and notation the reader may refer Harary [8] and Bondy et.al. [3].

The concept of \( \Pi_k \) - connectedness was suggested by Sampathkumar [10] and [11], and studied by Chaluvaraju et al. [4] in the following manner. For any positive integer \( k \). A graph \( G \) is said to be \( \Pi_k \) - connected if for any given subset \( S \) of \( V(G) \) with \( |S| = k \), the subgraph induced by \( S \) is connected.

Here, we shall introduce an edge analogue of this concept as follows: A graph \( G \) is said to be \( \Pi_e \) - connected if for any given edge subset \( F \) of \( E(G) \) with \( |F| = k \), the subgraph induced by \( F \) is connected. A \( \Pi_e \) - connected graph \( G \) is said to be edge minimal \( \Pi_e \) - connected if the graph \( G \) is not \( \Pi_{e-1} \) - connected. Let \( G \) be a nontrivial graph. Then a generalized vertex (edge)induced connected subsets of a graph is denoted as \( \Pi_k(G) \) (\( \Pi_e^k(G) \)). For more details on related concepts, we refer [1], [2], [5], [9] and [12].

2 \( \Pi_e^k \) - Connectedness

**Theorem 2.1** Let \( s \) and \( t \) be a positive integer. If the graph \( G \) is not \( \Pi_e^t \) - connected graph, then it is also not \( \Pi_e^s \) - connected graph, where \( 2 \leq s \leq t \).

**Proof.** Let the graph \( G \) be not a \( \Pi_e^t \) - connected graph, then there exists \( t \)-edges whose edge induced subgraph, say \( F_1 \) is disconnected. Let \( F_2, 2 \leq |F_2| \leq t \) be set of edges formed by taking at least one edge from each component of \( F_1 \). Clearly the subgraph induced by \( F_2 \) is also disconnected. Hence \( G \) is not \( \Pi_e^s \) - connected graph, where \( 2 \leq s \leq t \).

**Theorem 2.2** For any connected graph \( G \) is an edge minimal \( \Pi_{e}^q \) - connected graph if and only if it has a cut edge.

**Proof.** Let \( G \) be an edge minimal \( \Pi_{e}^q \) - connected graph then there exists an edge \( e \) such that the subgraph induced by \( E(G) \setminus \{e\} \) is disconnected. Hence \( e \) is a cut edge of a graph \( G \). Conversely, if the graph \( G \) has a cut edge \( e \), then the subgraph induced by \( E(G) \setminus \{e\} \) is disconnected and the graph on \( q \)-edges is connected. Hence the graph \( G \) is an edge minimal \( \Pi_{e}^q \) - connected graph.

**Theorem 2.3** For any tree \( T \) with \( p \geq 3 \) vertices is an edge minimal \( \Pi_{e}^{p-1} \) - connected graph.

**Proof.** Let \( T \) be any tree with \( p \geq 3 \) vertices. Since the total number of edges in \( T \) is \( p - 1 \), the graph induced by \( p - 1 \) edges is isomorphic to \( T \) and hence
connected. Let $e$ be an edge whose end vertices have degree greater than one, then the subgraph induced by $E(T)\setminus\{e\}$ is disconnected. Hence the tree $T$ with $p \geq 3$ vertices is an edge minimal $\Pi^e_{p-1}$ - connected graph.

**Theorem 2.4** For any Complete graph $K_p$ with $p \geq 4$ vertices is an edge minimal $\Pi^e_k$ - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$.

**Proof.** Let $K_p$ be a complete graph with $p \geq 4$ vertices. The maximum number of independent edges of a complete graph $K_p$ is $\left\lfloor \frac{p}{2} \right\rfloor$. Therefore only disconnected edge induced subgraph of $K_p$ with maximum number of vertices is $K_2 \cup K_{p-2}$. Hence $K_p$ is not $\Pi^e_{k-1}$ - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$. Addition of any edge makes the subgraph $K_2 \cup K_{p-2}$ is connected. Hence the complete graph $K_p$ with $p \geq 4$ vertices is an edge minimal $\Pi^e_k$ - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$.

By the above two results, we have the following Theorem.

**Theorem 2.5** Let $\Pi^e_k(G)$ be an edge minimal $\Pi^e_k$ - connected graph of a $(p,q)$-graph. Then

$$q \leq \Pi^e_k(G) \leq \frac{(p-2)(p-3)}{2} + 2.$$

**Theorem 2.6** For any Cycle $C_p$ with $p \geq 4$ vertices is an edge minimal $\Pi^e_{p-1}$ - connected graph.

**Proof.** Let $C_p$ be any cycle on $p \geq 4$ vertices and $e$ be any edge in $C_p$. The subgraph induced by $E(C_p)\setminus\{e\}$ is connected. Since an edge $e$ is arbitrarily chosen, the cycle $C_p$ with $p \geq 4$ vertices is $\Pi^e_{p-1}$ - connected graph. Now we prove $C_p$ is an edge minimal $\Pi^e_{p-1}$ - connected graph, i.e., $C_p$ is not $\Pi^e_{p-2}$ - connected graph. Let $e_1$ and $e_2$ be two independent edges in $C_p$. The subgraph induced by $E(C_p)\setminus\{e_1, e_2\}$ is disconnected. Hence $C_p$ with $p \geq 4$ vertices is an edge minimal $\Pi^e_{p-1}$ - connected graph.

3 $\Pi^e_k$ - Connectedness in Prism Graphs

The prism of a graph $G$ is defined as the cartesian product $G \times K_2$. For more details, we refer [7].

**Theorem 3.1** For any Prism $C^*_p$ of a cycle $C_p$ with $p \geq 5$ vertices is $\Pi^e_{5p-3}$ - connected graph.
Proof. Let $C_p$ be a cycle with $p \geq 5$ vertices and $C_p^*$ be its prism. Let $C_p^1$ and $C_p^2$ be two copies of $C_p$ in the prism. We have to prove the subgraph induced by any set of $(3p - 3)$-edges is connected. The total number of edges in $C_p^*$ is $3p$. Let $E(C_p^*)$ be the set of edges in $C_p^*$ and $e_1$, $e_2$, $e_3$ be any three edges in $E(C_p^*)$. Instead of proving the subgraph induced by any set of $3p - 3$ edges is connected, we prove the equivalent statement the subgraph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$ is connected for every set of three edges. Hence the following cases arise depending on the selection of edges in $E(C_p^*)$.

**Case 1.** Let $e_1 \in C_1$ and $e_2, e_3 \in C_2$. Then again we have the following two subcases,

**Subcase 1.1.** Let $e_2$ and $e_3$ are consecutive edges. Then both the subgraphs $F_1$ and $F_2$ induced by $E(C_1) \setminus \{e_1\}$ and $E(C_2) \setminus \{e_2, e_3\}$, respectively are connected. Now by adding all the edges between $F_1$ and $F_2$ from the $C_p^*$, we get a connected graph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$.

**Subcase 1.2.** Let $e_2$ and $e_3$ are nonconsecutive edges. Then the subgraph induced by $E(C_1) \setminus \{e_1\}$ is connected and the subgraph induced by $E(C_2) \setminus \{e_2, e_3\}$ is disconnected. But the edges between $H_1$ and $H_2$ from the $C_p^*$ makes the subgraph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$ connected.

**Case 2.** Let $e_1, e_2 \in C_1$ and $e_3 \in C_2$. Proof follows on similar lines as in Case 1.

**Case 3.** Let all three edges $e_1$, $e_2$ and $e_3$ are in the first copy of $C_p$ in the prism $C_p^*$. Then the subgraph induced by $E(C_1) \setminus \{e_1, e_2, e_3\}$ may be connected or disconnected. If the subgraph induced by $E(C_1) \setminus \{e_1, e_2, e_3\}$ is connected, then clearly the subgraph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$ is connected. If the subgraph induced by $E(C_1) \setminus \{e_1, e_2, e_3\}$ is disconnected having two or three components then the edges between each of these components and $C_2$ from the prism $C_p^*$ makes the subgraph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$ connected.

**Case 4.** Let all three edges $e_1$, $e_2$ and $e_3$ are in the second copy of $C_p$ in the prism $C_p^*$. Proof of this case follows on the similar lines as in Case 3.

**Case 5.** Let $e_1, e_2 \in C_1$ and $e_3$ belongs to the set of edges between the two copies of $C_p$ in the prism. The subgraph $H_1$ induced by $E(C_1) \setminus \{e_1, e_2\}$ may be connected or disconnected depending on the edges $e_1$ and $e_2$ are consecutive or not. As seen before, the subgraph induced by $E(C_p^*) \setminus \{e_1, e_2, e_3\}$ connected, since $p \geq 5$, there exists edges between the two copies of $C_p$ even after removal of $e_3$, connecting each component of $H_1$ with the second copy of $C_p$ in the
prism. Hence the subgraph induced by $E(C^*_p)\backslash\{e_1, e_2, e_3\}$ is connected.

Similarly we can prove the remaining cases as follows.

**Case 6.** $e_1 \in C_1$ and $e_2, e_3$ belong to the set of edges between the two copies of $C_p$ in the prism.

**Case 7.** All three edges $e_1, e_2$ and $e_3$ are in the set of edges between the two copies of $C_p$ in the prism.

**Case 8.** $e_1, e_2 \in C_2$ and $e_3$ belongs to the set of edges between the two copies of $C_p$ in the prism.

**Case 9.** $e_1 \in C_2$ and $e_2, e_3$ belongs to the set of edges between the two copies of $C_p$ in the prism.

Hence the result follows.

**Theorem 3.2** Let $T$ be a nontrivial tree. Then Prism $T^*$ of $T$ is an edge minimal $\Pi_{3q}^e$-connected graph.

**Proof.** Let $T$ be any nontrivial tree and $T^*$ be its prism. The number of edges in $T^*$ is equal to the number of edges in the first copy of a tree $T +$ the number of edges in the second copy of $T +$ the number of edges between these two copies of a tree $T,$ i.e., $|E(T^*)| = 2q + p = 3q + 1.$ First, we prove $T^*$ is not $\Pi_{3q-1}^e$-connected graph. Let $e$ be an edge in the first copy of a tree $T$ in its prism $T^*$ and $f(e)$ in the second copy of $T$ in $T^*$ be the mirror image of $e.$ The subgraph induced by $E(T^*)\backslash\{e, f(e)\}$ is disconnected, since $e$ is a bridge in the first copy of a tree $T$ and $f(e)$ is a bridge in the second copy of a tree $T$ in the prism $T^*.$ Hence there exist a set $E(T^*)\backslash\{e, f(e)\}$ of $3q - 1$ edges whose induced subgraph is disconnected. Hence $T^*$ is not $\Pi_{3q-1}^e$-connected graph. The subgraph induced by $E(T^*)\backslash\{e\}$ is connected for all $e$ in the first copy or second copy of a tree $T.$ Now suppose $e \in E(T_1 - T_2),$ where $E(T_1 - T_2)$ is the set of all edges between the two copies of a tree $T$ in the prism $T^*,$ clearly in this case also the subgraph induced by $E(T^*)\backslash\{e\}$ is connected for all $e$ in the $E(T_1 - T_2).$ Hence the prism $T^*$ of any nontrivial tree $T$ is an edge minimal $\Pi_{3q}^e$-connected graph.

4 $\Pi_k^e$-Connectedness in Corona Graphs

The corona $G_1 \circ G_2$ was defined by Frucht and Harary [6] as the graph $G$ obtained by taking one copy of $G_1$ of order $p_1$ and $p_1$ copies of $G_2,$ and then joining the $i^{th}$ node of $G_1$ to every node in the $i^{th}$ copy of $G_2.$
Theorem 4.1 Let $C_p$ be a cycle with $p \geq 3$ vertices and $G(p_1,q_1)$ be a graph. Then the corona $C_p \circ G$ is an edge minimal $\Pi_{p[p_1+q_1+1]-1}^e$-connected graph.

Proof. Let $C_p : u_1, u_2, \ldots, u_p$, $p \geq 3$ be any cycle and $G(p_1,q_1)$ be any graph of order $p_1$ and size $q_1$. Let $G_1, G_2, \ldots, G_p$ be $p$ copies of $G$ in the corona $C_p \circ G$. Let $E(C_p)$ be the set of edges in $C_p$, $E_i$ be the set of edges from $u_i$ to $G_i$ in the corona and $E(G_i)$ be the set of edges in $G_i$. We first prove the corona $C_p \circ G$ is not $\Pi_{p[p_1+q_1+1]-2}^e$-connected. Then there exists a set of $p[p_1+q_1+1]-2$ edges in $C_p \circ G$ whose edge induced subgraph is disconnected or equivalently we show the existence of a pair of edges $e_1, e_2$ in $C_p \circ G$ such that the subgraph induced by $E(C_p \circ G) \setminus \{e_1, e_2\}$ is disconnected, as the total number of edges in $C_p \circ G$ is $p[p_1+q_1+1]$. Suppose $e_1$ and $e_2$ be any two edges on the cycle $C_p$ in the corona, then clearly the subgraph induced by $E(C_p \circ G) \setminus \{e_1, e_2\}$ is disconnected. Hence the corona $C_p \circ G$ is not $\Pi_{p[p_1+q_1+1]-2}^e$-connected. Now we prove $C_p \circ G$ is $\Pi_{p[p_1+q_1+1]-1}^e$-connected or equivalently we prove for every edge $e$ in $C_p \circ G$, the subgraph induced by $E(C_p \circ G) \setminus \{e\}$ is connected. Thus the following cases arise.

Case 1. If $e \in G_i$ then the subgraph induced by $E(C_p \circ G) \setminus \{e\}$ is connected, since every left out edge in $G_i$ after removal of $e$ is connected with $u_i$ of $C_p$.

Case 2. If $e \in E_i$ then the subgraph induced by $E(C_p \circ G) \setminus \{e\}$ is connected, $u_i$ is adjacent to all the vertices of $G_i$.

Case 3. If $e \in C_p$ then the removal $e$ does not disconnect $C_p$. Hence the subgraph induced by $E(C_p \circ G) \setminus \{e\}$ is connected. Hence the proof.

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References


