On the Nullity of Expanded Graphs

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Abstract

The nullity (degree of singularity) η(G) of a graph G is the multiplicity of zero as an eigenvalue in its spectrum. It is proved that, the nullity of a graph is the number of non-zero independent variables in any of its high zero-sum weightings. Let u and v be nonadjacent conneighbor vertices of a connected graph G, then η(G) = η(G−u) + 1 = η(G−v) + 1. If G is a graph with a pendant vertex (a vertex with degree one), and if H is the subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then η(G) = η(H). Let H be a graph of order n and G₁, G₂, …, Gₙ be given vertex disjoint graphs, then the expanded graph H ² G₁G₂…Gₙ is a graph obtained from the graph H by replacing each vertex vi of H by a graph Gi with extra sets of edges Si,j for each edge vᵢvⱼ of H in which Sᵢ,j = {uw: u ∈ V(Gᵢ), w ∈ V(Gⱼ)}. In this research, we evaluate the nullity of expanded graphs, for some special ones, such as null graphs, complete bipartite graphs, star graphs, complete graphs, nut graphs, paths, and cycles.

Keywords: Graph Theory, Graph Spectra, Nullity of a Graph.

I Introduction

A graph G is said to be a singular graph provided that its adjacency matrix A(G) is a singular matrix. The eigenvalues λ₁, λ₂, …, λₚ of A(G) are said to be the
eigenvalues of the graph $G$, which form the spectrum of $G$. The occurrence of zero as an eigenvalue in the spectrum of the graph $G$ is called its **nullity (degree of singularity)** and is denoted by $\eta(G)$. See [1] and [3].

**Definition 1.1**[2, 5, 7] A graph $G$ is said to be **$\eta$-singular** or the nullity of $G$ is $\eta$, abbreviated $\eta(G)$ or $\eta$ if, the multiplicity of zero (as an eigenvalue) in $S_p(G)$ is $\eta$.

**Definition 1.2**[2] A **vertex weighting** of a graph $G$ is a function $f: V(G) \to \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers, which assigns a real number (weight) to each vertex.

A weighting of $G$ is said to be **non-trivial** if there is at least one vertex $v \in V(G)$ for which $f(v) \neq 0$.

**Definition 1.3**[2] A non-trivial vertex weighting of a graph $G$ is called a **zero-sum weighting** provided that for each $v \in V(G)$, $\sum f(u) = 0$, where the summation is taken over all $u \in N_G(v)$.

Clearly, the following weighting for $G$ is a non-trivial zero-sum weighting, where $x$ and $y$ are weights and $(x, y) \neq (0, 0)$, as indicated in Fig.1.1.

![Figure 1.1: A non-trivial zero-sum weighting of a graph](image)

**Definition 1.4**[5] Out of all zero-sum weightings of a graph $G$, a **high zero-sum weighting** of $G$ is one that uses maximum number of non-zero independent variables, $M_v(G)$.

An important relation between the singularity of a graph, and existence of a zero-sum weighting is, that a graph is singular iff it possesses a non trivial zero-sum weighting.[2]

**Proposition 1.5**[5] In any graph $G$, the maximum number of non-zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of $G$.

In Fig. 1.1, the weighting for the graph $G$ is a high zero-sum weighting that uses 2 independent variables, hence, $\eta(G) = 2$.

Let $r(A(G))$ be the rank of $A(G)$. Clearly, $\eta(G) = p - r(A(G))$. The rank $r(G)$ of a graph $G$ is the rank of its adjacency matrix $A(G)$. Then, each of $\eta(G)$ and $r(G)$ determines the other.
The nullity of some known graphs such as cycle $C_n$, path $P_n$, complete $K_p$ and complete bipartite $K_{r,s}$ graphs are given in the next lemma.

**Lemma 1.6[5, 6, 7]**

i) The eigenvalues of the cycle $C_n$ are of the form:

$$2 \cos \frac{2 \pi r}{n}, r = 0, 1, ..., n-1.$$ According to this,

$$\eta(C_n) = \begin{cases} 2, & \text{if } n \equiv 0 \mod 4, \\ 0, & \text{otherwise}. \end{cases}$$

ii) The eigenvalues of the path $P_n$ are of the form:

$$2 \cos \frac{\pi r}{n+1}, r = 1, 2, ..., n.$$ And thus,

$$\eta(P_n) = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$$

iii) The nullity of the complete graph $K_p$ is:

$$\eta(K_p) = \begin{cases} 1, & \text{if } p = 1, \\ 0, & \text{if } p > 1. \end{cases}$$

iv) The nullity of the complete bipartite graph $K_{r,s}$ is:

$$\eta(K_{r,s}) = \begin{cases} 0, & \text{if } r = s = 1, \\ r + s - 2, & \text{otherwise}. \end{cases}$$

**Definition 1.7:** Two vertices of a graph $G$ are said to be of the same type (coneighbors) if they are not adjacent and have the same set of neighbors. Thus, the two vertices $v_i, v_j$ of the same type have the same row vectors $R_i = R_j$ describing them, where $R_i$ and $R_j$ are the $i^{th}$ and $j^{th}$ row vector of $A(G)$, corresponding to the vertices $v_i$ and $v_j$, $i, j = 1, 2, ..., p$. Each pair of such (same type) vertices results in two dependent (coincide) rows which yield a zero in spectra of the graph $G$. It is clear that the occurrence of $m$ equal rows contributes $(m-1)$ to the nullity.

**Corollary 1.8[4] (End Vertex Corollary):** If $G$ is a graph with a pendant vertex, and $H$ is the subgraph of $G$ obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$. 
Applying Corollary 1.8, several times, deleting $v_1$ and $v_3$ respectively is illustrated in the next figure.

\[
\eta(\text{Figure 1.2: Illustration of Corollary 1.8})
\]

So (End Vertex Corollary) is a strong tool to determine the nullity of trees.

**Operations on Graphs**

Many interesting graphs are obtained from combining pairs (or more) of graphs or operating on a single graph in some way. We now discuss a number of operations which are used to combine graphs to produce new ones.

**Lemma 1.9**[1, 3] Let $G = G_1 \cup G_2 \cup \ldots \cup G_t$, where $G_1$, $G_2$, ..., $G_t$ are connected components of $G$, then

\[
\eta(G) = \sum_{i=1}^{t} \eta(G_i)
\]

**Definition 1.10**[1, 3] The **join** $G_1 + G_2$ of two graphs $G_1$ and $G_2$ is a graph whose vertex set, $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv: \text{for all } u \in G_1 \text{ and all } v \in G_2\}$.

**Definition 1.11**[1] The **sequential join** $G_1 + G_2 + \ldots + G_n$ of $n$ disjoint graphs $G_1, G_2, \ldots, G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{n-1} + G_n)$ and denoted by

\[
\sum_{i=1}^{n} G_i, \text{ for } i=1,2,\ldots,n \text{ and defined by}
\]

\[
V(\sum_{i=1}^{n} G_i) = \bigcup_{i=1}^{n} V(G_i), \quad E(\sum_{i=1}^{n} G_i) = \bigcup_{i=1}^{n} E(G_i) \cup \{uv: \text{for all } u \in G_i \text{ and all } v \in G_{i+1}, i=1,2,\ldots,n-1\}.
\]
As depicted in Fig.1.3.

\[ G_1 \rightarrow G_2 \rightarrow G_{n-1} \rightarrow G_n \]

**Figure 1.3:** The sequential join graph \( \sum_{i=1}^{n} G_i \)

It is clear that, \( p(\sum_{i=1}^{n} G_i) = \sum_{i=1}^{n} p_i \), \( q(\sum_{i=1}^{n} G_i) = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1} \),

In which \( p_i = p(G_i) \) and \( q_i = q(G_i) \).

**Definition 1.12**[4] Let \( P_n \) be a path with vertex \( \{v_1, v_2, \ldots, v_p\} \). Replacing each vertex \( v_i \) by an empty graph \( N_{p_i} \) of order \( p_i \), for \( i=1,2,\ldots,p \) and joining edges between each vertex of \( N_{p_i} \) and each vertex of \( N_{p_{i+1}} \) for \( i=1,2,\ldots,n-1 \), we get a graph order \( p_1+p_2+\ldots+p_n \) denoted by \( \sum_{i=1}^{n} N_{p_i} \). Such graph is called a **sequential join**.

**Definition 1.13**[1] The **strong product graph** \( G_1 \boxtimes G_2 \) of \( G_1 \) and \( G_2 \), is the union of the Cartesian product graph \( G_1 \times G_2 \) and the Kronecker product graph \( G_1 \otimes G_2 \).

Clearly, \( p(G_1 \boxtimes G_2) = p(G_1)p(G_2) \) and \( q(G_1 \times G_2) = p(G_1)q(G_2) + p(G_2)q(G_1) + 2q(G_1)q(G_2) \).

It is apparent that \( K_m \boxtimes K_n = K_{mn} \).

Results, relating the nullity of the graphs \( G_1 \) and \( G_2 \) and their strong product \( G_1 \boxtimes G_2 \), are not studied widely.

We conclude that, if both \( G_1 \) and \( G_2 \) are singular graphs, then so is \( G_1 \boxtimes G_2 \).

**Definition 1.14**[1] The **corona** \( G = G_1 \circ G_2 \) of two disjoint graphs \( G_1 \) and \( G_2 \) is defined as the graph obtained from taking one copy of \( G_1 \) and \( p_i \) copies of \( G_2 \), and then joining the \( i^{th} \) vertex of \( G_1 \) to every vertex in the \( i^{th} \) copy of \( G_2 \).
As illustrated in Figure 1.8, where the copies of $G_2$ are denoted by $G'_1$, $G'_2$, ..., $G'_{p_1}$, $V_1 = V(G_1) = \{v_1, v_2, ..., v_{p_1}\}$, $U^{(i)} = V(G'_i) = \{u_1^{(i)}, u_2^{(i)}, ..., u_{p_1}^{(i)}\}$, for $i = 1, 2, ..., p_1$, and $V(G) = V_1 \cup \bigcup_{i=1}^{p_1} U^{(i)}$

![Figure 1.4: The corona $G_1 \ast G_2$](image)

From the definition of the corona, it is clear that $G_1 \ast G_2$ is connected iff $G_1$ is connected. Also if $G_2$ contains at least one edge, then $G_1 \ast G_2$ is not bipartite graph.

And $p(\ G_1 \ast G_2\ ) = p_1(1 + p_2)$.

$q(\ G_1 \ast G_2\ ) = q_1 + p_1q_2 + p_1p_2$, with a diameter $diam(\ G_1 \ast G_2\ )=diam(G_1)+2$.

Note that $G_1 \ast G_2 \neq G_2 \ast G_1$ unless $G_1 \cong G_2$.

Studying the nullity of the corona graph $G_1 \ast G_2$ is one of the main subjects discussed in the present study.

**II On the Nullity of the Sequential Join of Some Special Graphs**

The nullities of the sequential join of some special graphs, $N_p$, $K_{r,s}$, $S_p$, $K_p$, $P_m$ and $C_p$ are determined in this section.
**Definition 2.1:** Let the graphs $G_1$, $G_2$, ..., $G_n$ be given. An **expanded graph** (expanded join graph) $H_n^{G_i}$ of a labeled graph $H$ of vertex set $\{v_1, v_2, \ldots, v_n\}$, is a graph obtained from $H$ by replacing each vertex $v_i$ by the graph $G_i$, $i = 1, 2, \ldots, n$, with extra sets of edges $S_{i,j}$ for each edge $v_i v_j$ of $H$ in which $S_{i,j} = \{uw : u \in G_i, w \in G_j\}$. We call $G_i$’s inserting graphs and $H$ the base graph. Thus, the order $p(H_n^{G_i}) = \sum_{i=1}^{n} p_i$ and the size $q(H_n^{G_i}) = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1}$, where the last summations is taken over all $i$ for which the vertex $v_i$ is adjacent with $v_{i+1}$ in $H$. That is $G_j + G_k$ is an induced subgraph of $H_n^{G_i}$ for each pair of adjacent vertices $v_j, v_k$ of $H$. Moreover, if $v_j, v_k, v_l$ forms a path $P_3$ in $H$, then $(G_j \cup G_l) + G_k$ is an induced subgraph of $H_n^{G_i}$. While if $v_j, v_k$ and $v_l$ forms a triangle in $H$, then $(G_j + G_k) + G_l$ is a subgraph of $H_n^{G_i}$.

An illustration for Definition 2.1 is given in the next example.

**Example 2.2:** Let the graphs $G_1$, $G_2$, $G_3$, $G_4$, $G_5$ and $H$ be given as follows:

$G_1 = P_3$, $G_2 = K_2$, $G_3 = K_1$, $G_4 = C_3$, $G_5 = K_1$ and,

![Graph H](image)

Then the expanded graph of $H$ by inserting the above $G_i$ graphs is indicated in Figure 2.1, in which $p(G) = \sum_{i=1}^{5} p_i = 10$ and

$q(G) = \sum_{i=1}^{5} q_i + \sum_{i=1}^{4} p_i p_{i+1} = 18$.

![Expanded Graph](image)

**Figure 2.1:** The expanded graph $G = H_5^{G_i}$

Moreover, if the base graph $H$ is a path $P_n$, then the expanded graph $P_n^{G_i}$ is the sequential join of the graphs $G_1$, $G_2$, ..., $G_n$. If $H = P_5$ and $G_i$s, $i = 1, 2, 3, 4, 5$ are
given as in Example 2.2, then, the sequential join graph \( \sum_{i=1}^{5} G_i \) is depicted in Fig. 2.2, with \( p(G) = 10 \) and \( q(G) = 6 + 14 = 20 \).

Figure 2.2: The sequential join graph \( \sum_{i=1}^{5} G_i \)

Next, we determine the nullity of the sequential join of some special graphs such as \( G_i \cong N_p, K_{r,s}, S_p, K_p, P_m, \) or \( C_p \), for \( i = 1, 2, \ldots, n \).

If \( G = \sum_{i=1}^{n} N_{p_i} \), where \( N_{p_i} = N_1 \) (the trivial graph) for all \( i \), then the graph \( G \) is simply a path of order \( n \).

**Proposition 2.3:** For an expanded graph \( \sum_{i=1}^{n} N_2 \), we have:

i) If \( n = 2k \), for \( k = 1, 2, \ldots \), then \( \eta(\sum_{i=1}^{2k} N_2) \) is
\[
2 + \eta(\sum_{i=1}^{2k-1} N_2) = 2k = n.
\]

ii) If \( n = 2k+1 \), for \( k = 1, 2, \ldots \), then \( \eta(\sum_{i=1}^{2k+1} N_2) \) is
\[
2 + \eta(\sum_{i=1}^{2k} N_2) = 2k + 2 = n + 1.
\]

**Proof:** i) Let \( w_{(i,j)} \), \( i=1, 2 \) and \( j=1, 2, \ldots, 2k \), be a zero-sum weighting for the vertex \( v_{i,j} \) in the graph \( \sum_{i=1}^{2k} N_2 \) (or \( \sum_{i=1}^{2k+1} N_2 \)), as indicated in Fig. 2.3.
Figure 2.3: A weighting of the graph $\sum_{i=1}^{2k} N_2$

If $k = 1$, then using the weights technique it is easy to evaluate that $\eta\left(\sum_{i=1}^{2} N_2\right) = 2$ and $\eta\left(\sum_{i=1}^{3} N_2\right) = 4$ by Lemma 1.6 ii. Since the vertices $v_1, 2k$ and $v_2, 2k$ are coneighbors as well as $v_1, 2k-1$ and $v_2, 2k-1$ hence removing the vertices $v_1, 2k$ and $v_1, 2k-1$ from the graph $\sum_{i=1}^{2k} N_2$, a graph with end vertex (namely $v_2, 2k$) is obtained.

Apply (End Vertex Corollary) to it, we get $\eta\left(\sum_{i=1}^{2k} N_2\right) = 2 + \eta\left(\sum_{i=1}^{2(k-1)} N_2\right)$.

ii) Similar argument holds for the odd case also. ■

**Theorem 2.4:** For $n \geq 2$, if $G = \sum_{i=1}^{n} N p_i$, then

$$\eta(G) = \begin{cases} 
\sum_{i=1}^{n} p_i - n & \text{if } n \text{ is even,} \\
\sum_{i=1}^{n} p_i - n + 1 & \text{if } n \text{ is odd.}
\end{cases}$$

**Proof:** The proof is just an extension to that of Proposition 2.3, and hence it is omitted. ■

**Corollary 2.5:** In Theorem 2.4 if $p_i = p$ $\forall i$, then the nullity of $G = \sum_{i=1}^{n} N p_i$ is

$$\eta(G) = \begin{cases} 
n(p-1) & \text{if } n \text{ is even,} \\
n(p-1) + 1 & \text{if } n \text{ is odd.}
\end{cases}$$

**The Sequential Join of Complete Bipartite Graph**

A graph $G$ is said to be a bipartite graph if it contains no odd cycles. Thus, the sequential join of complete bipartite graphs is also a bipartite graph. Moreover,
the sequential join of complete bipartite graphs \( \sum_{i=1}^{n} K_{r_i,s_i} \) has \( p = \sum_{i=1}^{n} (r_i + s_i) \) and \( q = \sum_{i=1}^{n} r_i s_i + \sum_{i=1}^{n-1} (r_i + s_i)(r_{i+1} + s_{i+1}) \)

while, the diameter of \( \sum_{i=1}^{n} K_{r_i,s_i} \) is n-1, for \( n \geq 3 \).

We define the next term:

**Definition 2.6:** A singular graph \( G \) is said to be a **completely non stable** if \( \sum_{u \in G} w(u) = 0 \) for a high zero-sum weighting of \( G \). It is clear that if the above condition holds for only a high zero-sum weighting then it holds for any other one.

Thus, complete bipartite graphs with order greater than 2 are completely non stable graphs, while \( P_{4n+1} \) is not.

**Lemma 2.7 (Coneighbor Lemma):** Let \( u \) and \( v \) be coneighbor vertices of a connected graph \( G \), then

\[ \eta(G) = \eta(G-u) + 1 = \eta(G-v) + 1. \]

**Proof:** Label the vertices of \( G \) by \( u \equiv v_1, v_2, v_3, \ldots, v_p \). Let \( A(G), A(G-u), A(G-v) \) be the adjacency matrices of \( G, G-u, G-v \), respectively. Applying row elimination \( R_1 \rightarrow R_1 - R_2 \) and column elimination \( C_1 \rightarrow C_1 - C_2 \) to the matrix \( A \), we get zero in each entry of row one and zero in each entry of column one.

And obtain a new matrix \( A^* \), where \( A^* = A(K_1 \cup (G-u)) \).

Hence, \( r(G) = r(G-u) \Rightarrow p - \eta(G) = (p-1) - \eta(G-u) \),

\[ \therefore \eta(G) = \eta(G-u) + 1, \text{ similarly } \eta(G) = \eta(G-v) + 1. \]

**Definition 2.8:** Two adjacent vertices \( v_1 \) and \( v_2 \) in a graph \( G \) are said to be **semi-coneighbors** if \( N(v_1) = N(v_2) \) in the graph \( G-e \) where \( e = v_1v_2 \).

**Remark 2.9:** Let \( w \) be any zero-sum weighting of a graph \( G \). If \( v_1 \) and \( v_2 \) are semi-coneighbors, then they must be weighted by the same variable (weight), say \( x \), because in any zero-sum weighting for \( G \) we have:

\[ \sum_{u \in N_G(v_1)} w(u) = w(v_2) + \sum_{u \in N_{G-e}(v_1)} w(u) = 0, \] \hspace{1cm} (1)
\[ \sum_{u \in N_G(v_1)} w(u) = w(v_1) + \sum_{u \in N_G(v_2)} w(u) = 0. \] (2)

Therefore, from (1) and (2), we get \( w(v_1) = w(v_2) \)

**Proposition 2.10:** The strong product graph \( K_2 \boxtimes P_n = \sum K_2 \) is non-singular.

**Proof:** For \( n = 2 \) or \( 3 \), it is easy to prove that any zero-sum weighting for \( K_2 \boxtimes P_2 \) and \( K_2 \boxtimes P_3 \) is trivial. ■

For any zero-sum weighting of the graph \( K_2 \boxtimes P_n \) as indicated in Fig.2.4, we can put \( w(v_{i,j}) = x_j \), for \( i = 1,2 \) and \( j = 1,2,...,n \). This is possible by Remark 2.9

![Figure 2.4: The strong product graph \( K_2 \boxtimes P_n \)](image)

Apply \( \sum_{u \in N(v_{i,j})} w(u) = 0 \), for all \( i,j \), we get:

\[
\begin{align*}
x_1 + 2x_2 &= 0 \\
2x_1 + x_2 + 2x_3 &= 0 \\
2x_2 + x_3 + 2x_4 &= 0 \\
\vdots \\
2x_{n-2} + x_{n-1} + 2x_n &= 0 \\
2x_{n-1} + x_n &= 0
\end{align*}
\] (1, 2, 3, ..., n-1, n)

Then, from Equation (1'), we get:

\[ x_2 = -\frac{1}{2} x_1 \] (1')

From Equations (1') and (2), we get:

\[ x_3 = -\frac{1}{2} [2x_1 - \frac{1}{2} x_1] = -\frac{3}{4} x_1 \] (2')

From Equations (2') and (3), we get:

\[ x_4 = \frac{1}{2} [x_1 + \frac{3}{4} x_1] = \frac{7}{8} x_1 \] (3')
And so on, all the values of \(x_2, x_3, \ldots, x_n\) are defined in term of \(x_1\).

Finally, put the values of \(x_{n-1}\) and \(x_n\) in Equation (n) to get:

\[ax_1 = 0,\] for some number \(a\), this implies that \(x_1 = 0\) and hence all the remaining variables are zeros.

Therefore, there exist no non trivial zero-sum weighting for the graph \(K_2 \boxtimes P_n\).

Hence, by Proposition 1.5, \(\eta(K_2 \boxtimes P_n) = 0\). ■

Since, \(K_2 \boxtimes P_n\) is singular graph, then \(\lambda \mu + \lambda + \mu\) must equal to zero for some eigenvalue \(\lambda\) of \(K_2\) and \(\mu\) of \(P_n\). This follows from the relations between the eigenvalues of the strong product graph and the eigenvalues of its product components. See [5].

But \(\lambda = 1\) or \(\lambda = -1\), hence \(\mu + 1 + \mu = 0 \Rightarrow \mu = -\frac{1}{2}\) or \(-\mu -1 + \mu = 0 \Rightarrow -1 = 0\) which is impossible.

Thus, we conclude that \(-\frac{1}{2}\) is not an eigenvalue for \(P_n\) for any \(n\). This leads that

\[2 \cos\left(\frac{i \Pi}{n + 1}\right) \neq -\frac{1}{2}\] for any \(i, i = 1, 2, \ldots, n\), and any \(n\). That is \(\exists\) no such integers \(i\) and \(n\) that satisfies \(\cos\left(\frac{i \Pi}{n + 1}\right) = -\frac{1}{4}\).

The nullity of the expanded graph \(G\), \(G = \sum_{i=1}^{n} K_{r_i,s_i}\) is determined in the next theorem.

**Theorem 2.11:** For \(n \geq 2\), \(\eta(\sum_{i=1}^{n} K_{r_i,s_i}) = \sum_{i=1}^{n} (r_i + s_i - 2)\).

**Proof:** Apply (Coneighbor Lemma) for each pair of coneighbor vertices \(u\) and \(v\) in \(\sum_{i=1}^{n} K_{r_i,s_i}\), that is removing a vertex out of each such a pair which are exactly \(\sum_{i=1}^{n} (r_i + s_i - 2)\), we obtain the graph \(K_2 \boxtimes P_n\).

Then, \(\eta(G) = \sum_{i=1}^{n} (r_i + s_i - 2) + \eta(K_2 \boxtimes P_n)\). ■

The Star graph \(S_p\) (or \(S_{1,p-1}\)) is a complete bipartite graph, with one of its partite sets consisting of exactly one vertex. It is a special type of trees, trees with diameter 2, with only three distinct eigenvalues, namely, \(\sqrt{p-1}, 0, -\sqrt{p-1}\).
Moreover, the expanded graph of \( n \) stars \( S_{p_i}, i=1,2,\ldots,n \), \( G = \sum_{i=1}^{n} S_{p_i} \) has order,

\[
p = \sum_{i=1}^{n} p_i\quad \text{and size } q,
\]

\[
q = \sum_{i=1}^{n} p_i - n + \sum_{i=1}^{n-1} p_i p_{i+1},\quad \text{with diameter } n-1, \, n \geq 3.
\]

**Corollary 2.12:** For \( n, p_i \geq 2 \), the nullity of \( G \) is \( \sum_{i=1}^{n} p_i - 2n \).

**Proof:** \( S_{p_i} \cong K_{1, p_i-1} \), hence it is a special case of Theorem 2.3.7. \( \square \)

**Corollary 2.13:** For \( n \geq 2 \), if \( G = \sum_{i=1}^{n} S_{p_i}, \, p \geq 2 \).

Then, \( \eta(G) = n(p-2) \).

**Proof:** Put \( r_i+s_i = p \) in Theorem 2.3.7, then, the prove is immediate. \( \square \)

### The Sequential Join of Complete Graphs

The complete graph \( K_p, p > 2 \), is a simple graph with neither a cut vertex nor a bridge, and has maximum size \( q, \quad q = \frac{p(p-1)}{2} \). While the sequential join \( \sum_{i=1}^{n} K_{p_i} \), \( n \geq 3 \), is not a complete graph. Its order is \( p = \sum_{i=1}^{n} p_i \)

and size \( q = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1} = \sum_{i=1}^{n} p_i (p_i - 1) + \sum_{i=1}^{n-1} p_i p_{i+1} \).

The nullity of \( \sum_{i=1}^{n} K_{p_i} \) is determined in the next theorem.

**Theorem 2.14:** If \( n \geq 2 \), and \( p_i \geq 2 \) for \( i=1,2,\ldots,n \), then \( \sum_{i=1}^{n} K_{p_i} \) is non-singular.

**Proof:** For \( n = 2 \), then, the sequential join of \( K_{p_1} \) and \( K_{p_2} \) is \( K_{p_1+p_2} \), which is a complete graph, hence by Lemma 1.4.11(iii), \( \eta(G) = 0 \).

For \( n > 2 \), no non zero-sum weighting for \( G \) exists. This holds from the fact \( \sum K_2 \) is an induced subgraph of \( \sum K_{p_i} \) and each extra vertex \( u \) in \( K_{p_i} \) is semi-
coneighbor with each vertex \( v \in K_{p_j} \). By Remark 2.9, the weight of \( u \) is the same as the weight of each \( v \), which is zero; hence \( G \) is a non-singular graph by Proposition 1.5. ■

The Sequential Join of Paths

Paths are extreme graphs to determine many invariants of the graph such as the diameter, and the average distance. Moreover, the sequential join of path graphs, \( \sum_{i=1}^{n} P_{m_i} \), has order \( p = \sum_{i=1}^{n} m_i \) and size \( q = \sum_{i=1}^{n} (m_i - 1) + \sum_{i=1}^{n-1} m_i m_{i+1} \), and the diameter is \( n-1 \), for \( n > 2 \), while it is equal to 2, where \( n=2 \), and \( m_1 \) or \( m_2 > 2 \).

Lemma 2.15: If \( G_1 = P_m \) and \( G_2 = P_n \), and \( G = G_1 + G_2 \), then, the nullity of the graph \( G \) is given by:

\[
\eta(G) = \begin{cases} 
2 & \text{if both } m = 4k - 1 \text{ and } n = 4t - 1, \text{ for } k, t \in \mathbb{Z}^+, \\
1 & \text{if either } m = 4k - 1 \text{ or } n = 4t - 1, \text{ but not both,} \\
0 & \text{if neither } m \text{ nor } n = 4t - 1, \text{ for any } k \text{ and any } t.
\end{cases}
\]

Proof: Label the vertices of \( G_1 \) by \( v_1, v_2, \ldots, v_m \), with a high zero-sum weighting \( w_{(1,1)}, w_{(1,2)}, \ldots, w_{(1,m)} \), and the vertices of \( G_2 \) by \( u_1, u_2, \ldots, u_n \), with a high zero-sum weighting \( w_{(2,1)}, w_{(2,2)}, \ldots, w_{(2,n)} \), in \( G \).

By Definition 2.6, if \( m = 4k - 1 \) and \( n = 4t - 1 \), then both \( P_m \) and \( P_n \) are completely non stable graphs, and the same weighting can be used for the join graph, hence the nullity of the join graph is 2 in this case. If either of them is completely non stable, say \( P_m \) but not \( P_n \), then \( w_{(2,1)} = w_{(2,2)} = \ldots = w_{(2,n)} = 0 \) in any high zero-sum weighting of the join. Finally, if both \( n \) and \( m \) cannot be written as \( 4k-1 \) for any \( k \) and any \( t \), then in the join graph \( w_{(i,j)} = 0 \) \( \forall \ i,j \). Thus, \( G \) is non-singular. ■

Theorem 2.16: If \( G = \sum_{i=1}^{n} P_{m_i} \), then nullity of the graph \( G \) is

\[
\eta(G) = \begin{cases} 
n & \text{if } m = 4k - 1, \text{ for } k \in \mathbb{Z}^+, \\
0 & \text{if } m \neq 4k - 1, \text{ for any } k \in \mathbb{Z}^+.
\end{cases}
\]

Proof: If \( m = 3 \), then the proof follows from Lemma 2.15. Moreover, if \( m = 4k-1, k \in \mathbb{Z}^+ \), then the path graph \( P_m \) is a completely non stable graph, and each component of the compound graph \( \sum_{i=1}^{n} P_{m_i} \) uses exactly one variable, hence there exist exactly \( n \) variables in any high zero-sum weighting of \( G \), then, \( \eta(G) = n \).
If $m \neq 4k-1$, $k \in \mathbb{Z}^+$, then each $P_m$ is not a completely non stable, hence there exists no non-trivial zero-sum weighting for $G$, for $n > 1$. Thus, by Proposition 1.5, we conclude that $G$ is non-singular. ■

It is clear that $P_5 + P_5$ has no non trivial zero-sum weighting, hence $\eta(G) = 0$.

**Observation 2.17:** If $G = \sum_{i=1}^{n} P_{m_i}$, then $\eta(G) =$ no. of paths $P_{m_i}$ of order $4k_i-1$, for $k_i \in \mathbb{Z}^+$.

**Example 2.18:** Let $G_1 = P_2$, $G_2 = P_3$ and $G_3 = P_4$, then there is a zero-sum weighting of the graph $\sum_{i=1}^{3} P_{m_i}$, where $m_i = 2, 3, 4$ as indicated in Fig. 2.5.

![Figure 2.5](image)

**Figure 2.5:** A zero-sum weighting of the graph $P_{3}P_{m_i}$.

This zero-sum weighting uses exactly one independent variables, namely $w_{(1,1)}$, hence $\eta(\sum_{i=1}^{3} P_{m_i}) = 1$.

### III The Sequential Join of Cycles

Cycles are 2-regular, critical 2-connected graphs, odd cycles are critical 3-colourable graphs. The nullity of the sequential join of $n$ cycles, $\sum_{i=1}^{n} C_{p_i}$, is our goal in the next.

**Proposition 3.1:** If $G = C_n + C_m$, then the nullity of the graph $G$ is given by:

$$
\eta(G) = \begin{cases} 
4 & \text{if both } n=4k \text{ and } m=4t, \text{ for } k, t \in \mathbb{Z}^+, \\
2 & \text{if } n=4k \text{ or } m=4t \text{ but not both,} \\
0 & \text{if neither the order of } C_n \text{ nor of } C_m \text{ is } 0 \text{ mod } 4.
\end{cases}
$$

**Proof:** The proof is similar to that of Theorem 2.16. ■
Theorem 3.2: For \( n \geq 2 \), if \( G = \sum_{j=1}^{n} C_{p_j} \), \( p_j > 2 \), \( j = 1,2,\ldots,n \), the nullity of \( G \) is

\[
\eta(G) = \begin{cases} 
2n & \text{if } p_j \equiv 4k_j, k_j \geq 1 \text{ for each } j, j = 1,2,\ldots,n, \\
0 & \text{if no order of } C_{p_j} \text{ is equal to zero } \pmod{4}.
\end{cases}
\]

Proof: It is known that \( \eta(C_{4k}) = 2 \), and using weights, it is easy to show that \( \eta(C_{4k_1} + C_{4k_2}) = 2+2 = 4 \).

If \( w_{(1,j)}, w'_{(1,j)}, -w_{(1,j)}, -w'_{(1,j)}, \ldots \), are weights of \( C_{p_j} \)'s, \( p_j = 4k_j \), then from the condition \( \sum_{\forall v \in N(i,j)} w_{(i,j)} = 0 \), there is no relation between \( w_{(1,j)} \) and \( w'_{(1,j)} \), \( \forall j \), \( j = 1,2,\ldots,n \), if \( j = 1 \).

Hence, if \( j = n \) where \( p_j = 4k_j \), then \( \eta(G) = 2n \). For \( n \) subgraphs of \( G \), if no non trivial zero-sum weighting exists (that is order of non of them is zero mod 4), then no non trivial zero-sum weighting for \( G \) exists, hence by Proposition 1.5, \( \eta(G) = 0 \).

Observation 3.3: If \( G = \sum_{i=1}^{n} C_{p_i} \), then the nullity of the graph \( G \) is \( 2j \), if \( j \) orders of the cycles \( C_{p_i} \)'s are of form \( p_i \equiv 4k_j \).

IV Nullity of the Corona of a Path with other Special Graphs

In this section, we study the nullity of the corona of two graphs.

In Definition 1.12, we choose \( G_1 = P_n \) and \( G_2 \) is a known graph, such as \( N_m \), \( K_2 \), \( P_m \), \( C_4 \), \( K_{r,s} \) or \( K_p \).

Proposition 4.1: Let \( G_1 = P_n \) and \( G_2 = N_m \) then the nullity of the corona graph \( \eta(G) \), \( G = P_n \circ N_m \) is \( n(m-1) \).

Proof: Follows from applying (End Vertex Corollary) \( n \)-times namely to the vertices \( u_{1j}, j = 1, 2, \ldots, m \), as illustrated in Fig.4.1.
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Proposition 4.2: The nullity of the graph $G = P_n \square K_2$ is zero.

Proof: Let $w_{(i,j)}$ be a weighting of the corona graph $P_n \square K_2$. From the condition $\sum_{v \in N(i,j)} w_{(i,j)} = 0$ for all $v$ in $P_n \square K_2$, the graph $P_n \square K_2$ can be weighted as indicated in Fig.4.2.

Where, $w_{(1,1)} = y$

![Fig. 4.2: The corona graph $P_n \square K_2$](image)

Now, the sum of the weights over the neighborhood of the vertex weighted $nw_{(1,1)}$ is $(n-1)w_{(1,1)} - nw_{(1,1)} - nw_{(1,1)} = 0 \implies w_{(1,1)} = 0$.

The graph $P_n \square K_2$ has no non trivial zero-sum weighting. Hence it is non-singular.

Proposition 4.3: For the corona graph $G = P_n \square K_m$ is non singular.

Proof: Is similar to that of Lemma 4.2, hence $\eta(G) = 0$.

Proposition 4.4: Let $G_1 = P_n$ and $G_2 = C_4$, then the nullity of $G = P_n \square C_4$ is $\eta(G) = 2n$.

Proof: Follows from the fact that there exists exactly $2n$ pairs of coneighbors vertices in $G$ and after removing a vertex out of each such a pair, we obtain the graph indicated in Figure 4.2.

$\therefore \eta(P_n \square C_4) = 2n + \eta(P_n \square K_2)$

$= 2n$. ■
Proposition 4.5: For any complete bipartite graph $K_{r,s}$, the nullity of $G = P_n \circ K_{r,s}$ is given by

$$\eta(G) = n(r + s - 2).$$

Proof: Follows by applying Coneighbor Lemma, hence it is omitted. ■

Proposition 4.6: Let $G^*$ be the graph $K_1 + P_n$, then,

$$\eta(G^*) = \begin{cases} 1 & \text{if } n \equiv 4k - 1, \text{ for } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise}. \end{cases}$$

Proof: Let $w_{(2,1)}$ and $w_{(1,j)}$, $j=1,2,\ldots,n$ be a zero-sum weighting for the graph $G^*$, as indicated in Fig. 4.3.

Figure 4.3: The cone graph $G^*$ of the path $P_n$.

From the condition \( \sum_{v \in N(i,j)} w_{(i,j)} = 0 \), for all $v$ in $G^*$

\[
\begin{align*}
w_{(1,2)} + w_{(2,1)} &= 0 \implies w_{(1,2)} = -w_{(2,1)} \quad (1) \\
w_{(1,j)} + w_{(1,j+2)} + w_{(2,1)} &= 0, \quad j=1,2,\ldots,n-2 \\
w_{(1,n-1)} + w_{(2,1)} &= 0 \quad (2)
\end{align*}
\]

From Equation (2) we get:

\[
\begin{align*}
w_{(1,j+2)} &= -w_{(1,j)} - w_{(2,1)}, \text{ for } j = 1,2,\ldots,n-2 \\
\text{Thus, } w_{(1,3)} &= -w_{(1,1)} - w_{(2,1)} \quad (3) \\
w_{(1,4)} &= -w_{(1,2)} - w_{(2,1)} \quad (4) \\
\therefore \quad w_{(1,4)} &= 0 \quad (5)
\end{align*}
\]

This gives that $w_{(1,4k)} = 0$, for $k \in \mathbb{Z}^+$, and $w_{(1,2k-1)} = -w_{(1,1)} + w_{(1,2)}$. Assume that $w_{(1,2)} \neq 0$, then the sum of the weights over the neighborhood of the vertex weighted $w_{(2,1)}$ is \( \sum_{j=1}^{n} w_{(1,j)} \neq 0 \), which is a contradiction from which we assumed
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w_{(1,2)} = 0. Now if n ≠ 4k–1, then we get w_{(1,1)} = 0 and hence, no non-trivial zero-sum weighting exists. Thus, any high zero-sum weighting of $G^\wedge$ will use only one non-zero variable say $w_{(1,1)}$ where n = 4k–1, and hence the prove is complete. ■

If $G^{\wedge\wedge}$ is a cone over the cone $G^\wedge$ then $\eta(G^{\wedge\wedge}) = \eta(G^\wedge)$, and moreover if this process is continued any t times, then $\eta(G^{\wedge^t}) = \eta(G^\wedge)$, because of the fact that the second vertex of $G^{\wedge\wedge}$ which is added and joined to all vertices of $G^\wedge$ is semi-coneighbor with the first vertex of $G^\wedge$ which is added and joined to all vertices of $G$. Hence, both must be zero weighted and hence, the result follows for any t.

**Proposition 4.7:** Let $G_1 = P_n$ and $G_2 = P_m$, then the nullity of $G = P_n \circ P_m$ is

$$\eta(G) = \begin{cases} n & \text{if } m = 4k - 1, \ k = 1,2..., \\ 0 & \text{otherwise}. \end{cases}$$

**Proof:** If $m = 3$, then the graph G is $P_n \circ P_3$, as indicated in Fig.4.4.

![Figure 4.4: The graph $P_n \circ P_3$](image)

Applying (Coneighbour Lemma), we have n-pairs of coneighbors. After removing a vertex out of each such a pair, we obtain the graph $P_n \circ K_2$ thus, $\eta(P_n \circ P_3) = n + \eta(P_n \circ K_2)$. But, $\eta(P_n \circ K_2) = 0$. Hence, $\eta(P_n \circ P_3) = n$.

If $m>3$, then we have n induced subgraphs of the graph $P_n \circ P_m$ and each of them is a cone of a path $P_m$. But the nullity of a cone of a path $P_m$, $m = 4k - 1$ is one, and we have n such a cones.

$\therefore \eta(G) = \begin{cases} n & \text{if } m = 4k - 1, \ k = 1,2... \\ 0 & \text{otherwise}. \end{cases} $

**Open Problem:** Evaluate $\eta(G_1 \circ G_2)$ in terms of invariants of $G_1$ and $G_2$?

**Nullity of the Semi-Corona of a Path with other Special Graphs**

In the following section we are going to define the semi-corona.

**Definition 4.8:** Let $H_n$ be a graph, whose vertices labeled $h_1,h_2,\ldots,h_n$ and $G_1,G_2,\ldots,G_n$ be distinct graphs with orders $p_1,p_2,\ldots,p_n$ and their vertices labeled by $U_i^{\downarrow} l \leq i \leq n, l \leq j \leq p_i$. 
We define the semi-corona graph $G = H_n \Theta \bigcup_{i=1}^{n} G_i$, to be the graph whose vertex set is $V(G) = V(H_n) \bigcup_{i=1}^{n} V(G_i)$ and edge set $E(G) = E(H_n) \bigcup_{i=1}^{n} E(G_i) \bigcup \{h_iu_j^j\}$ for each $j, 1 \leq j \leq p_i$.

That is the vertex $h_1$ is adjacent with $u_1^1, u_2^1, \ldots, u_{p_1}^1$, $h_2$ is adjacent with $u_1^2, u_2^2, \ldots, u_{p_2}^2$, and $h_n$ is adjacent with $u_1^n, u_2^n, \ldots, u_{p_n}^n$, as illustrated in Fig.4.5, where $G_1 = P_2$, $G_2 = P_3$, $G_3 = C_3$, $G_4 = C_4$ and $H = C_4$.

![Graph Image]

**Figure 4.5:** The semi-corona graph $G = C_4 \Theta \bigcup_{i=1}^{n} G_i$

**Proposition 4.9:**

1) $\eta(P_n \Theta \bigcup_{i=1}^{n} N_{m_i}) = \sum_{i=1}^{n} (m_i - 1)$.

2) $\eta(P_n \Theta \bigcup_{i=1}^{n} K_{p_i}) = 0$.

3) $\eta(P_n \Theta \bigcup_{i=1}^{n} K_{r_i,s_i}) = \sum_{i=1}^{n} (r_i + s_i - 2)$.

**Proof:** The proof is a generalization for that of Propositions 4.1, 4.2 and 4.5, respectively.

**References**


