On Some Ideals of Fuzzy Points Semigroups

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Abstract

Kim [Int. J. Math. & Math. Sc. 26:11 (2001), 707-712.] Considered the semigroup $S$ of the fuzzy points of a semigroup $S$. In this paper, we discuss the relation between some ideals $A$ of $S$ and the subset $C_A$ of $S$.

Keywords: Fuzzy set; Semigroup; Fuzzy point; Minimal ideal.

1 Introduction

Zadeh [9] introduced the concept of a fuzzy set for the first time and this concept was applied by Rosenfeld [8] to define fuzzy subgroups and fuzzy ideals. Based on this crucial work, Kuroki [3, 4, 5, 6] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. Authors in [1] investigated the existence of a fuzzy kernel and minimal fuzzy ideals in semigroups. They showed that a subset $A$ of a semigroup $S$ is minimal ideal if and only if the characteristic function of $A$, $C_A$, is minimal fuzzy ideal of $S$. In [2], Kim considered the semigroup $S$ of the fuzzy points of a semigroup $S$, and discussed the relation between the fuzzy interior ideals and the subsets of $S$. In this paper, we discuss the relation between some ideals $A$ of $S$ and the subset $C_A$ of $S$. 
2 Basic Definitions and Results

Let $S$ be a semigroup. A nonempty subset $A$ of $S$ is called a left (resp., right) ideal of $S$ if $SA \subseteq A$ (resp., $AS \subseteq A$), and a two-sided ideal (or simply ideal) of $S$ if $SAS \subseteq A$. An ideal $A$ of $S$ is called minimal ideal of $S$ if $A$ does not properly contains any other ideal of $S$. If the intersection $K$ of all the ideals of a semigroup $S$ is nonempty then we shall call $K$ the kernel of $S$. A subsemigroup $A$ of $S$ is called a bi-ideal of $S$ if $SA \subseteq A$ [7]. A function $f$ from $S$ to the closed interval $[0, 1]$ is called a fuzzy set in $S$. The semigroup $S$ itself is a fuzzy set in $S$ such that $f(x) = 1$ for all $x \in S$, denoted also by $S$. Let $A$ and $B$ be two fuzzy sets in $S$. Then the inclusion relation $A \subseteq B$ is defined by $f(x) \leq g(x)$ for all $x \in S$. $A \cap B$ and $A \cup B$ are fuzzy sets in $S$ defined by $(A \cap B)(x) = \min \{A(x), B(x)\}$, $(A \cup B)(x) = \max \{A(x), B(x)\}$, for all $x \in S$. For any $\alpha \in (0, 1]$ and $x \in S$, a fuzzy set $x_\alpha$ in $S$ is called a fuzzy point in $S$ if

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y, \\ 0 & \text{otherwise}, \end{cases}$$

for all $x \in S$. The fuzzy point $x_\alpha$ is said to be contained in a fuzzy set $A$, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$. The characteristic mapping of a subset $A$ of a semigroup $S$ is

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases}$$

for all $x \in S$.

**Lemma 2.1** (see [1, Lemma 3.]): For any nonempty subsets $A$ and $B$ of a semigroup $S$, we have $A \subseteq B$ if and only if $C_A \subseteq C_B$.

**Lemma 2.2** (see [1, Lemma 4.]): Let $A$ be a nonempty subset of a semigroup $S$, then $A$ is an ideal of $S$ if and only if $C_A$ is a fuzzy ideal of $S$.

Let $\mathcal{F}(S)$ be the set of all fuzzy sets in a semigroup $S$. For each $A, B \in \mathcal{F}(S)$, the product of $A$ and $B$ is a fuzzy set $A \circ B$ defined as follows:

$$(A \circ B)(x) = \{ \sup_{x=ab} A(a) \wedge B(b) \} \quad \text{if } ab = x$$

otherwise.

for each $x \in S$. If $S$ is a semigroup, then $\mathcal{F}(S)$ is a semigroup with the product "$\circ$" [2]. Let $\mathcal{F}_S$ be the set of all fuzzy points in a semigroup $S$. Then $x_\alpha \circ y_\beta = (xy)_{\alpha\beta} \in \mathcal{F}_S$ for $x_\alpha, y_\beta \in \mathcal{F}_S$ [2]. For any $A \in \mathcal{F}(S)$, $A$ denotes the set of all fuzzy points contained in $A$, that is, $A = \{ x_\alpha \in \mathcal{F}_S : A(x) \geq \alpha \}$. For any $A, B \in \mathcal{F}_S$, we define the product of $A$ and $B$ as $A \circ B = \{ x_\alpha \circ y_\beta : x_\alpha \in A, y_\beta \in B \}$. 
Lemma 2.3 (see [2, Lemma 3.2]): Let $A$ and $B$ be two fuzzy subsets of a semigroup $S$, then

1) $A \cup B = A \cup B$.
2) $A \cap B = A \cap B$.
3) $A \circ B \subseteq A \circ B$.

Lemma 2.4: Let $A$ be nonempty subset of a semigroup $S$, we have $x_\alpha \in C_A$ if and only if $x \in A$.

Proof: Suppose that $x_\alpha \in C_A$ for any $x \in S$, then $C_A(x) \geq \alpha$. Hence $C_A(x) = 1$ for any $\alpha > 0$, which implies that $x \in A$. Conversely, Let $x \in A$, then $C_A(x) = 1 \geq \alpha$ for any $\alpha > 0$. This means that $x_\alpha \in C_A$. $\blacksquare$

Lemma 2.5: For any nonempty subsets $A$ and $B$ of a semigroup $S$, we have

1) $A \subseteq B$ if and only if $C_A \subseteq C_B$.
2) $C_A \subseteq C_B$ if and only if $C_A \subseteq C_B$.

Proof: (1) Assume that $A \subseteq B$, and let $x_\alpha \in C_A$. By lemma 2.4, $x \in A \subseteq B$ and $x_\alpha \in C_B$, this implies that $C_A \subseteq C_B$. Conversely, suppose that $C_A \subseteq C_B$. Let $x \in A$, then by lemma 2.4, $x_\alpha \in C_A$ for any $\alpha > 0$, $x_\alpha \in C_B$ and hence $x \in B$. (2) Let $x_\alpha \in C_A \subseteq C_B$, then lemma 2.5 implies that $A \subseteq B$ and from lemma 2.1, we have $C_A \subseteq C_B$. This completes the proof. $\blacksquare$

3 Main Results

Lemma 3.1: Let $A$ be a nonempty subset of a semigroup $S$. Then $A$ is an ideal of $S$ if and only if $C_A$ is an ideal of $S$.

Proof: By lemma 2.2, $A$ is an ideal of $S$ if and only if $C_A$ is a fuzzy ideal of $S$, and from lemma 3.1[2], $C_A$ is a fuzzy ideal of $S$ if and only if $C_A$ is an ideal of $S$. $\blacksquare$

Theorem 3.2: Let $A$ be a nonempty subset of a semigroup $S$. Then $A$ is a minimal ideal of $S$ if and only if $C_A$ is a minimal ideal of $S$.

Proof: By theorem 7[1], $A$ is a minimal ideal of $S$ if and only if $C_A$ is a fuzzy minimal ideal of $S$. We only need to prove that $C_A$ is a minimal fuzzy ideal of $S$ if and only if $C_A$ is a minimal ideal of $S$. Let $C_A$ be a minimal fuzzy ideal of $S$, then by lemma 3.1[2], $C_A$ is an ideal of $S$. Suppose that $C_A$ is not minimal, then there exists some ideals $C_B$ of $S$ such that $C_B \subseteq C_A$. Hence by lemma 2.5,
\[ C_B \subseteq C_A, \text{ where } C_B \text{ is a fuzzy ideal of } S. \] This is a contradiction to \( C_A \) is a minimal fuzzy ideal of \( S \). Conversely, assume \( C_A \) is a minimal ideal of \( S \) and that \( C_A \) is not a minimal fuzzy ideal of \( S \). Then there exists a fuzzy ideal \( C_B \) of \( S \) such that \( C_B \subseteq C_A \). Now, lemma 2.5 implies that \( C_B \subseteq C_A \), where \( C_B \) is an ideal of \( S \). This contradicts that \( C_A \) is a minimal ideal of \( S \). This completes the proof of the theorem. \( \blacksquare \)

**Theorem 3.3:** Let \( A \) be a nonempty subset of a semigroup \( S \). Then \( A \) is the kernel of \( S \) if and only if \( C_A \) is the kernel of \( S \).

**Proof:** Suppose that \( A \) is the kernel of \( S \), then \( A = \bigcap_1 I_i \) where \( I_i \) is an ideal of \( S \). Let \( C_B \) be an ideal of \( S \), then by lemma 3.1, \( B \) is an ideal of \( S \). Now we need to show that, \( C_A \subseteq C_B \). Let \( x_a \in C_A \), by lemma 2.4, \( x \in A \) and also \( x \in B \) since \( A \) is the kernel of \( S \). This implies that \( x_a \in C_B \) and hence, \( C_A \) is the kernel of \( S \). Conversely, Let \( C_A \) be the kernel of \( S \), then \( C_A \subseteq C_B \), for every ideal \( C_B \) of \( S \). Thus \( A \subseteq B \), that is, \( A \) is the kernel of \( S \). \( \blacksquare \)

The following lemma weakens the condition of theorem 3.3.

**Lemma 3.4:** Let \( A \) be a minimal ideal of a semigroup \( S \), then \( C_A \) is the kernel of \( S \).

**Proof:** Since \( A \) be a minimal ideal of \( S \), then \( C_A \) is a minimal fuzzy ideal of \( S \) [1, theorem 7]. Also theorem 8 in [1] implies that \( C_A \) is the fuzzy kernel of \( S \). Now, let \( C_B \) be a fuzzy ideal of \( S \), then we have \( C_A \subseteq C_B \). By lemma 2.5, \( C_A \subseteq C_B \), so \( C_A \) is a minimal ideal contained in every ideal of \( S \). Thus \( C_A \) is the kernel of \( S \). \( \blacksquare \)

**Lemma 3.5:** Let \( A \) be a nonempty subset of a semigroup \( S \). Then \( A \) is an interior ideal of \( S \) if and only if \( C_A \) is an interior ideal of \( S \).

**Proof:** Let \( A \) be an interior ideal of \( S \), and let \( y \beta, z \gamma \in S \) and \( x_a \in C_A \). Since \( x \in A \), hence \( y \beta \circ x_a \circ z \gamma = (yxz) \beta \alpha \gamma \in C_A \). This implies that \( S \circ C_A \circ S \subseteq C_A \), thus \( C_A \) is an interior ideal of \( S \). Conversely, suppose that \( C_A \) is an interior ideal of \( S \). Let \( y, z \in S \) and \( x \in A \), then \( x_a \in C_A \). Assume that, \( y \beta \circ x_a \circ z \gamma = (yxz) \alpha \in S \circ C_A \circ S \subseteq C_A \), then \( yxz \in A \). This implies that \( SAS \subseteq A \), and hence \( A \) is an interior ideal of \( S \). \( \blacksquare \)

**Lemma 3.6:** Let \( A \) be a nonempty subset of a semigroup \( S \). Then \( A \) is a bi-ideal of \( S \) if and only if \( C_A \) is a bi-ideal of \( S \).
Proof: Let $A$ be a bi-ideal of $S$, and let $y_\beta, z_\gamma \in \mathcal{C}_A$ and $x_\alpha \in S$. Since $y, z \in A$ and $yxz \in A$ then $y_\beta \circ x_\alpha \circ z_\gamma = (yxz)_\beta \alpha \gamma \in \mathcal{C}_A$. This implies that $\mathcal{C}_A \circ \mathcal{S} \circ \mathcal{C}_A \subseteq \mathcal{C}_A$, thus $\mathcal{C}_A$ is a bi-ideal of $S$. Conversely, suppose that $\mathcal{C}_A$ is a bi-ideal of $S$. Let $y, z \in A$ and $x \in S$, then by lemma 2.4, $y_\alpha, z_\alpha \in \mathcal{C}_A$. Assume that, $y_\alpha \circ x_\alpha \circ z_\alpha = (yxz)_\alpha \in \mathcal{C}_A \circ \mathcal{S} \circ \mathcal{C}_A \subseteq \mathcal{C}_A$, then $yxz \in A$. This implies that $ASA \subseteq A$, and hence $A$ is a bi-ideal of $S$. 

References