On Some New Almost Double Lacunary $\Delta^m$-Sequence Spaces Defined by Orlicz Functions

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(Received: 12-9-11/ Accepted: 24-10-11)

Abstract

In this paper we introduce a new concept for almost double lacunary $\Delta^m$-sequence spaces defined by Orlicz function and give inclusion relations. The results here in proved are analogous to those by Ayhan Esi [General Mathematics (2009), 2(17) 53-66].

Keywords: Lacunary Sequence, Different Double sequence, Orlicz Function, Strongly almost convergence.

1 Introduction

Let $l_\infty$, c and $c_0$ be the spaces of bounded, convergent and null sequences $x = (x_k)$, with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$.

A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent[15] if all Banach limits of $x = (x_k)$ coincide. In [15], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \right\}.$$
In [16,17], Maddox defined a sequence \( x = (x_k) \) to be strongly convergent to a number \( L \) if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ uniformly in } n
\]
By a lacunary sequence \( \theta = (k_r) \), \( r=0,1,2,... \) where \( k_0 = 0 \), we mean an increasing sequence of non negative integers \( h_r = (k_r - k_{r-1}) \to \infty (r \to \infty) \). The intervals determined by \( \theta \) are denoted by \( I_r = (k_{r-1}, k_r] \) and ratio \( \frac{k_r}{k_{r-1}} \) will be denoted by \( q_r \). The space of lacunary strongly convergent sequence \( N_{\theta} \) was defined by Freedman et al.[3] as follows
\[
N_{\theta} = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.
\]
The double lacunary sequence was defined by E.Savas and R.F.Patterson[20] as follows:
The double sequence \( \theta_{r,s} = \{(k_r, l_s)\} \) is called double lacunary if there exist two increasing sequence of integers such that
\[
k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty
\]
and
\[
l_0 = 0, h_s^- = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.
\]
The following intervals are determined by \( \theta \).
\[
I_r = \{(k_r) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\}
\]
\[
I_{r,s} = \{(k, l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},
\]
\[
q_r = \frac{k_r}{k_{r-1}}, q_s^- = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r q_s^- . \text{ We will denote the set of all lacunary sequences by } N_{\theta_{r,s}}.
\]
Let \( x = (x_{kl}) \) be a double sequence that is a double infinite array of elements \( x_{kl} \). The space of double lacunary strongly convergent sequence is defined as follows:
\[
N_{\theta_{r,s}} = \left\{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{kl} - L| = 0 \text{ for some } L \right\} (\text{see[20]}).
\]
Double sequences have been studied by V.A.Khan[8,9,10,11], Moricz and Rhoades[19] and many others.

In [12], Kizmaz defined the sequence spaces $Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$ for $Z = l_\infty, c, c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. After Et. and Colak [1] generalized the difference sequence spaces to the sequence spaces $Z(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in Z\}$ for $Z = l_\infty, c, c_0$, where $m \in \mathbb{N}, \Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v}.$$ 

An Orlicz Function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function $M$ satisfies the $\Delta_2 - condition$ ($M \in \Delta_2$ for short ) if there exist constant $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

An Orlicz function $M$ can always be represented (see[13])in the integral form

$$M(x) = \int_0^x q(t) dt,$$

where $q$ known as the kernel of $M$, is right differentiable for $t \geq 0, q(t) > 0$ for $t > 0, q$ is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x)$$

for all $\lambda$ with $0 < \lambda < 1$,

since $M$ is convex and $M(0) = 0$.

Lindesstrauss and Tzafiri [14] used the idea of Orlicz sequence space;

$$l_M := \{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}$$

which is Banach space with the norm the norm

$$\|x\|_M = \inf \left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$
The space $l_M$ is closely related to the space $l_p$, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Orlicz function has been studied by V.A.Khan[4,5,6,7] and many others.

The purpose of this paper is to introduce and study a concept of lacunary almost generalized $\Delta^m$-convergence function and to examine these new sequence spaces which also generalize the well known Orlicz sequence space $l_M$ and strongly summable sequence $[C, 1, p], [C, 1, P]_0$ and $[C, 1, p]_\infty$(see[18]).

Let $M$ be an Orlicz function and $p = (p_k)$ be any bounded sequence of strictly positive real numbers. Ayhan Esi[2] defined the following sequence spaces:

$$\hat{c}, M, p((\Delta^m)) = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} M\left(\frac{|\Delta^m x_{k+m} - L|}{\rho}\right) p_k = 0, \right\}$$

uniformly in $m$ for some $\rho > 0$ and $L > 0$.

$$\hat{c}, M, p_0((\Delta^m)) = \left\{ x = (x_k) : \lim_{n \to \infty} \sum_{k=1}^{n} M\left(\frac{|\Delta^m x_{k+m}|}{\rho}\right) p_k = 0, \right\}$$

uniformly in $m$, for some $\rho > 0$.

$$\hat{c}, M, p_\infty((\Delta^m)) = \left\{ x = (x_k) : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta^m x_{k+m}|}{\rho}\right) p_k < \infty, \right\}$$

If $x = (x_k) \in \hat{c}, M, p((\Delta^m))$, we say that $x = (x_k)$ is lacunary almost $\Delta^m$-convergent to $L$ with respect an Orlicz function $M$.

The following inequality will be used throughout the paper

$$|x_{kl} + y_{kl}|^{p_k} \leq K(|x_{kl}|^{p_k} + |y_{kl}|^{p_k}) \quad [1.1]$$

where $x_{kl}$ and $y_{kl}$ are complex numbers, $K = \max(1, 2H^{-1})$ and $H = \sup_{k,l} p_{kl} < \infty$. 
2 Main Results

In the following paper we introduce and examine the following spaces defined by Orlicz function.

**Definition 2.1.** Let \( M \) be an Orlicz function and \( p = (p_{kl}) \) be any bounded sequence of strictly positive real numbers. We have

\[
[\hat{c}_2, M, p]^\theta(\Delta^m) = \{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\left| \Delta^m x_{k+m,l+n} - L \right|}{\rho} \right) \right]^{p_{kl}} = 0, \]

uniformly in \( m \) and \( n \), for some \( \rho > 0 \) and \( L > 0 \}.

\[
[\hat{c}_2, M, p]_0^\theta(\Delta^m) = \{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\left| \Delta^m x_{k+m,l+n} \right|}{\rho} \right) \right]^{p_{kl}} = 0, \]

uniformly in \( m \) and \( n \), for some \( \rho > 0 \}.

\[
[\hat{c}_2, M, p]_\infty^\theta(\Delta^m) = \{ x = (x_{kl}) : \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\left| \Delta^m x_{k+m,l+n} \right|}{\rho} \right) \right]^{p_{kl}} < \infty, \text{ for some } \rho > 0 \}.
\]

where

\[
\Delta^m x = (\Delta^m x_{kl}) = (\Delta^{m-1} x_{kl} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}),
\]

\[
(\Delta^1 x_{kl}) = (\Delta x_{kl}) = (x_{k+l} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}),
\]

\[
\Delta^0 x = (x_{k,l}), \quad \text{for all } k, l \in \mathbb{N},
\]

and also this generalized difference double notion has the following binomial representation:

\[
\Delta^m x_{kl} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{k+i,l+j}.
\]

If \( x = (x_{kl}) \in [\hat{c}_2, M, p]^\theta(\Delta^m) \), we say that \( x = (x_{kl}) \) is double lacunary almost \( \Delta^m \)-convergent to \( L \) with respect an Orlicz function \( M \).
In this section we prove some results involving the double sequence spaces \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\), \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\) and \([\hat{c}_2, M, p]_\infty^\theta(\Delta^m)\).

**Theorem 2.1.** Let \(M\) be an Orlicz function and \(p = (p_{kl})\) be a bounded sequence of strictly real numbers. Then \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\), \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\) and \([\hat{c}_2, M, p]_\infty^\theta(\Delta^m)\) are linear spaces over the set of complex numbers \(\mathbb{C}\).

**Proof.** Let \(x = (x_{kl}), y = (y_{kl}) \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)\) and \(\alpha, \beta \in \mathbb{C}\). Then there exists positive \(\rho_1\) and \(\rho_2\) such that

\[
\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho_1} \right)^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n
\]

and

\[
\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m y_{k+m,l+n}}{\rho_2} \right)^{p_{kl}} = 0, \text{ uniformly in } m \text{ and } n
\]

Let \(\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)\). Since \(M\) is non decreasing convex function, by using equation [1.1], we have

\[
\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m \alpha x_{k+m,l+n} + \beta y_{k+m,l+n}}{\rho_3} \right)^{p_{kl}}
\]

\[
= \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m \alpha x_{k+m,l+n}}{\rho_3} + \frac{\beta \Delta^m(y_{k+m,l+n})}{\rho_3} \right)^{p_{kl}}
\]

\[
\leq K \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \frac{1}{2^{p_{kl}}} M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho_1} \right)^{p_{kl}} + K \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \frac{1}{2^{p_{kl}}} M \left( \frac{\Delta^m y_{k+m,l+n}}{\rho_2} \right)^{p_{kl}}
\]

\[
\leq K \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho_1} \right)^{p_{kl}} + K \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m y_{k+m,l+n}}{\rho_2} \right)^{p_{kl}}
\]

\[
\to 0 \text{ as } r, s \to \infty \text{ uniformly in } m \text{ and } n.
\]

So \(\alpha x + \beta y \in [\hat{c}_2, M, p]_0^\theta(\Delta^m)\). Hence \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\) is a linear space.

The proof for the cases \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\) and \([\hat{c}_2, M, p]_\infty^\theta(\Delta^m)\) are similar to the above proof.

**Theorem 2.2.** For any Orlicz function \(M\) on a bounded double sequence \(p = (p_{kl})\) of strictly positive real numbers, \([\hat{c}_2, M, p]_0^\theta(\Delta^m)\) is a topological linear space paranormed by

\[
h(x_{kl}) = \sup_k |x_{k1}| + \sup_l |x_{1l}| + \inf \left\{ \rho^{\frac{p_{kl}}{p}} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right)^{p_{kl}} \right)^{\frac{1}{p}} \leq 1 \right\}
\]
where $H = \max(1, \sup_{k,l} p_{kl}) \leq \infty$.

**Proof.** Clearly $h(x_{kl}) \geq 0$, for all $x = (x_{kl}) \in [\hat{c}, M, p]_0^\theta(\Delta^m)$

Since $M(0) = 0$, we get $h(0) = 0$

$h(-x_{kl}) = h(x_{kl})$.

Let $(x_{kl}), (y_{kl}) \in [\hat{c}, M, p]_0^\theta(\Delta^m)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

\[
\left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \right]^{p_{kl}} \right)^{1/H} \leq 1
\]

and

\[
\left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m y_{k+m,l+n}}{\rho} \right) \right]^{p_{kl}} \right)^{1/H} \leq 1
\]

for each $r, s, m$ and $n$.

Let $\rho = \rho_1 + \rho_2$. Then we have,

\[
\left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m x_{k+m,l+n} + y_{k+m,l+n}}{\rho} \right) \right]^{p_{kl}} \right)^{1/H} \leq \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho_1} + \frac{\Delta^m y_{k+m,l+n}}{\rho_2} \right) \right]^{p_{kl}} \right)^{1/H}
\]

By Minkowski’s Inequality

\[
\leq \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho_1} \right) \right]^{p_{kl}} \right)^{1/H} + \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m y_{k+m,l+n}}{\rho_2} \right) \right]^{p_{kl}} \right)^{1/H} \leq 1
\]

Since $\rho_1$ and $\rho_2$ are non-negative, so we have

\[
h(x_{kl} + y_{kl}) = \sup_k |x_{k1} + y_{k1}| + \sup_l |x_{1l} + y_{1l}| + \inf \left\{ \rho^{p_{kl}} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\Delta^m x_{kl} + y_{kl}}{\rho} \right) \right]^{p_{kl}} \right)^{1/H} \leq 1 \right\}
\]
\[ \leq \sup_k |x_{k1}| + \sup_l |x_{l1}| + \inf \left\{ \rho_1 \frac{p_{kl}}{\rho_1} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m x_{k+l}+n}{\rho_1} \right) \right)^{\frac{1}{\frac{p_{kl}}{\rho_1}}} \leq 1 \right\} \]
\[ + \sup_k |y_{k1}| + \sup_l |y_{l1}| + \inf \left\{ \rho_2 \frac{p_{kl}}{\rho_2} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m y_{k+l}+n}{\rho_2} \right) \right)^{\frac{1}{\frac{p_{kl}}{\rho_2}}} \leq 1 \right\} \]
\[ = h(x_{kl}) + y(\ell) \]

This implies that
\[ h(x_{kl} + y_{kl}) \leq h(x_{kl}) + y(\ell). \]

Finally, we prove that the scalar multiplication is continuous. Let \( \lambda \) be any complex number. By definition
\[ h(\lambda (x_{kl})) = \sup_k |\lambda(x_{k1})| + \sup_l |\lambda(x_{l1})| + \inf \left\{ \frac{p_{kl}}{\rho} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m \lambda x_{k+l}+n}{\rho} \right) \right)^{\frac{1}{\frac{p_{kl}}{\rho}}} \leq 1 \right\} \]
\[ = |\lambda| \sup_k |x_{k1}| + |\lambda| \sup_l |x_{l1}| + \inf \left\{ \frac{p_{kl}}{\rho} \rho^{1/p} : \left( \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m \lambda x_{k+l}+n - L + L}{\rho} \right) \right)^{\frac{1}{\frac{p_{kl}}{\rho}}} \leq 1 \right\} \]
where \( t = \rho^{1/p} \)

This completes the proof of the theorem.

**Theorem 2.3.** Let \( M \) be an Orlicz function. If \( \sup_{k,l} [M(x)] p_{kl} < \infty \) for all fixed \( x > 0 \) then
\[ [\hat{c}_2, M, p_0^\theta(\Delta^m)] \subset [\hat{c}_2, M, p_0^\theta(\Delta^m)]. \]

**Proof.** Let \( x = (x_{kl}) \in [\hat{c}_2, M, p_0^\theta(\Delta^m)]. \) Then there exists some positive \( \rho_1 \) such that
\[ \lim_{k,l} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m x_{k+l}+n}{\rho_1} \right)^{p_{kl}} = 0, \] uniformly in \( m \) and \( n \)

Define \( \rho = 2\rho_1. \) Since \( M \) is non decreasing and convex, by using (1) we have
\[ \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m x_{k+l}+n}{\rho} \right)^{p_{kl}} \]
\[ = \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} M\left( \frac{\Delta^m x_{k+l}+n - L + L}{\rho} \right)^{p_{kl}} \]
\[
\leq K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ \frac{1}{2^{p_{k,l}}} M\left( \frac{\Delta^m x_{k+m,l+n} - L}{\rho_1} \right) \right]^{p_{k,l}}
+ K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ \frac{1}{2^{p_{k,l}}} M\left( \frac{|L|}{\rho_1} \right) \right]^{p_{k,l}}
\leq K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\Delta^m x_{k+m,l+n} - L}{\rho_1} \right) \right]^{p_{k,l}}
+ K \sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{|L|}{\rho_1} \right) \right]^{p_{k,l}} < 1
\]

Hence \( x = (x_{k,l}) \in \left[ \hat{c}_2, M, p \right]_\infty^\theta (\Delta^m) \).

This completes the proof.

**Theorem 2.4.** Let \( 0 < \inf p_{k,l} = h \leq p_{k,l} = H \leq \infty \) and \( M, M_1 \) be Orlicz functions satisfying \( \Delta_2 \)-condition, then we have \( [\hat{c}_2, M_1, p]_0^\theta (\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]_0^\theta (\Delta^m) \), \( [\hat{c}_2, M_1, p]_\infty^\theta (\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]_\infty^\theta (\Delta^m) \).

**Proof.** Let \( x = (x_{k,l}) \in [\hat{c}_2, M_0 M_1, p]_0^\theta (\Delta^m) \). Then we have

\[
\lim_{r,s \to \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ uniformly in } m \text{ and } n.
\]

Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M(t) < \epsilon \) for \( 0 \leq t \leq \delta \).

Let \( y_{k,l} = M_1\left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \) for \( k, l, m, n \in \mathbb{N} \).

We can write

\[
\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} [M(y_{k,l})]^{p_{k,l}} = \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}, y_{k,l} \leq \delta} [M(y_{k,l})]^{p_{k,l}} + \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}, y_{k,l} > \delta} [M(y_{k,l})]^{p_{k,l}}
\]

since \( M \) is continuous and \( M(t) < \epsilon \) for \( t \leq \delta \).

For \( y_{k,l} > \delta \) we use the fact that

\[ y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta} \]

Since \( M \) is non decreasing and convex, it follows that

\[ M(y_{k,l}) < M(1 + \delta^{-1} y_{k,l}) = M\left( \frac{2}{2} + \frac{2}{2} \delta^{-1} y_{k,l} \right) \]
Since $M$ satisfies $\Delta_2$-condition, there is a constant $K > 2$ such that

$$M(2\delta^{-1}y_{k,l}) \leq \frac{1}{2} K \delta^{-1}y_{k,l} M(2)$$

Hence

$$\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}, y_{k,l}>\delta} [M(y_{k,l})]^{p_{kl}} \leq \max \left( 1, \left( \frac{KM(2)}{\delta} \right) \right) \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}, y_{k,l}>\delta} [(y_{k,l})]^{p_{kl}} \rightarrow 0 \text{ as } r, s \rightarrow \infty \quad [2.2]$$

By [2.1] and [2.2] we have $x = (x_{k,l} \in [\hat{c}_2, M_0 M_1, p]_0^\theta(\Delta^m)$.

Similarly we can prove that

$[\hat{c}_2, M_1, p]^\theta(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]^\theta(\Delta^m)$ and $[\hat{c}_2, M_1, p]^\theta_\infty(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]^\theta_\infty(\Delta^m)$.

This completes the proof.

Taking $M_1(x)$ in above theorem we have the following result.

**Corollary 2.5.** Let $0 < \inf p_{k,l} = h \leq p_{k,l} = H \leq \infty$ and $M$ be Orlicz function satisfying $\Delta_2$-condition, then we have $[\hat{c}_2, M, p]^\theta_\infty(\Delta^m)$ and $[\hat{c}_2, M, p]^\theta_0(\Delta^m) \subset [\hat{c}_2, M_0 M_1, p]^\theta(\Delta^m)$.

**Theorem 2.6.** Let $M$ be an Orlicz function. Then the following statements are equivalent:

(i) $[\hat{c}_2, p]^\theta_\infty(\Delta^m) \subset [\hat{c}_2, M, p]^\theta_\infty(\Delta^m)$.

(ii) $[\hat{c}_2, p]^\theta_0(\Delta^m) \subset [\hat{c}_2, M, p]^\theta_0(\Delta^m)$.

(iii) $\sup \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{t}{\rho} \right) \right]^{p_{kl}} < \infty \quad (t, \rho > 0)$.

**Proof.** (i) $\Rightarrow$ (ii): It is obvious, since $[\hat{c}_2, p]^\theta_0(\Delta^m) \subset [\hat{c}_2, M, p]^\theta_\infty(\Delta^m)$.

(ii) $\Rightarrow$ (iii): Let $[\hat{c}_2, p]^\theta_0(\Delta^m) \subset [\hat{c}_2, M, p]^\theta_\infty(\Delta^m)$.

Suppose that (iii) does not hold. Then for some $t, \rho > 0$

$$\sup \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{t}{\rho} \right) \right]^{p_{kl}} = \infty.$$
and therefore we can find a subinterval $I_{r(i),s(j)}$ of the set of interval $I_{r,s}$ such that

$$\frac{1}{h_{r(i),s(j)}} \sum_{(k,l) \in I_{r(i),s(j)}} \left[ M\left( \frac{(ij)^{-1}}{\rho} \right) \right]^{p_{kl}} > ij$$

(i) $i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots$

Define the double sequence $x = (x_{kl})$ by

$$\Delta^m x_{k+m,l+n} = \begin{cases} 
(ij)^{-1} & (k,l) \in I_{r(i),s(j)} \\
0 & (k,l) \notin I_{r(i),s(j)}
\end{cases}$$

for all $m,n \in \mathbb{N}$

Then $x = (x_{kl}) \in [\hat{c}_2, p]_0^0(\Delta^m)$ but by equation [2.3] $x = (x_{kl}) \notin [\hat{c}_2, M, p]_0^0(\Delta^m)$.

Which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i): Let (iii) hold and $x = (x_{kl}) \in [\hat{c}_2, p]_0^0(\Delta^m)$.

Suppose that $x = (x_{kl}) \notin [\hat{c}_2, M, p]_0^0(\Delta^m)$. Then

$$\sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \right]^{p_{kl}} = \infty$$

Let $t = |\Delta^m x_{k+m,l+n}|$ for each $k,l$ and fixed $m,n$ then by equation [2.4]

$$\sup_{r,s,m,n} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{t}{\rho} \right) \right]^{p_{kl}} = \infty$$

Which contradicts (iii). Hence (i) must hold.

**Theorem 2.7.** Let $1 \leq p_{kl} \leq \sup p_{kl} < \infty$ and $M$ be an Orlicz function. Then the following statements are equivalent:

(i) $[\hat{c}_2, M, p]_0^0(\Delta^m) \subset [\hat{c}_2, p]_0^0(\Delta^m)$.

(ii) $[\hat{c}_2, M, p]_0^0(\Delta^m) \subset [\hat{c}_2, p]_0^0(\Delta^m)$.

(iii) $\inf_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{t}{\rho} \right) \right]^{p_{kl}} > 0$ $(t, \rho > 0)$

**Proof.** (i) $\Rightarrow$ (ii): It is obvious.

(ii) $\Rightarrow$ (iii): Let (ii) hold. Suppose (iii) doesnot hold. Then
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\[ \inf_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{t}{\rho} \right) \right]^{p_{kl}} = 0 \quad (t, \rho > 0) \]

So we can find a subinterval \( I_{r(i),s(j)} \) of the set of interval \( I_{r,s} \) such that

\[ \frac{1}{h_{r(i),s(j)}} \sum_{(k,l) \in I_{r(i),s(j)}} \left[ M\left( \frac{ij}{\rho} \right) \right]^{p_{kl}} < (ij)^{-1} \quad \text{[2.5]} \]

\[ i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots \]

Define the double sequence \( x = (x_{kl}) \) by

\[ \Delta^m x_{k+m,l+n} = \begin{cases} (ij)^{-1} & (k, l) \in I_{r(i),s(j)} \\ 0 & (k, l) \notin I_{r(i),s(j)} \end{cases} \]

for all \( m, n \in \mathbb{N} \)

Thus by equation [2.5], \( x = (x_{kl}) \in [\hat{c}_2, M, p]_0(\Delta^m) \) but by equation [2.3] \( x = (x_{kl}) \notin [\hat{c}_2, p]_0(\Delta^m) \). Which contradicts (ii). Hence (iii) must hold.

\((iii) \Rightarrow (i)\): Let (iii) hold and suppose that \( x = (x_{kl}) \in [\hat{c}_2, M, p]_0(\Delta^m) \), that is

\[ \lim_{r,s \to \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left( \frac{|\Delta^m x_{k+m,l+n}|}{\rho} \right) \right]^{p_{kl}} = 0 \quad \text{uniformly in } m \text{ and, for some } \rho > 0. \quad \text{[2.6]} \]

Again suppose that \( x = (x_{kl}) \in [\hat{c}, p]_0(\Delta^m) \). Then for some \( \epsilon > 0 \) and a subinterval \( I_{r(j),s(j)} \) of the set interval \( I_{r,s} \), we have \( |\Delta^m x_{k+m,l+n}| \geq \epsilon \) for all \( k, l \in \mathbb{N} \) and some \( i \geq i_0, j \geq j_0 \). Then, from the properties of the Orlicz function, we can write

\[ M\left( \frac{|\Delta^m x_{k+m,l+n}|}{\rho} \right) \geq M\left( \frac{\epsilon}{\rho} \right) \]

and consequently by equation [2.6]

\[ \lim_{r,s \to \infty} \frac{1}{h_{r(i),s(j)}} \sum_{(k,l) \in I_{r(i),s(j)}} \left[ M\left( \frac{t}{\rho} \right) \right]^{p_{kl}} = 0 \]

Which contradicts (iii). Hence (i) must hold.

**Theorem 2.8.** Let \( 0 < p_{k,l} \leq q_{k,l} \) for all \( k, l \in \mathbb{N} \) and \( \left( \frac{q_{k,l}}{p_{k,l}} \right) \) be bounded. Then,

\[ [\hat{c}_2, M, q]^\theta(\Delta^m) \subset [\hat{c}_2, p]^\theta(\Delta^m) \].
Proof. Let \( x \in [\hat{c}_2, M, q]^{\theta}(\Delta^m) \)
Write
\[
t_{k,l} = \left[ M \left( \frac{\Delta^m x_{k+m,l+n}}{\rho} \right) \right]^{p_{k,l}}
\]
and \( \lambda_{k,l} = \frac{p_{k,l}}{q_{k,l}} \)
Since \( 0 < p_{k,l} \leq q_{k,l} \), therefore \( 0 < \lambda_{k,l} \leq 1 \).
Take \( 0 < \lambda \leq \lambda_{k,l} \).
Define
\[
u_{k,l} = \begin{cases} t_{k,l} & t_{k,l} \geq 1 \\ 0 & t_{k,l} < 1 \end{cases}
\]
\[
u_{k,l} = \begin{cases} 0 & t_{k,l} \geq 1 \\ t_{k,l} & t_{k,l} < 1 \end{cases}
\]
So \( t_{k,l} = u_{k,l} + v_{k,l} \) and
\[
\lambda_{k,l} = u_{k,l} + v_{k,l}
\]
Now it follows that
\[
u_{k,l}^{\lambda_{k,l}} \leq u_{k,l} \leq t_{k,l} \quad \text{and} \quad v_{k,l}^{\lambda_{k,l}} \leq v_{k,l}
\]
Therefore
\[
\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \lambda_{k,l} \leq \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} t_{k,l} + \left[ \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} v_{k,l} \right]^{\lambda}
\]
Hence \( x \in [\hat{c}_2, M, p]^{\theta}(\Delta^m) \).

By using above theorem it is easy to prove the following result.

Corollary 2.9(i). If \( 0 < \inf p_{k,l} \leq p_{k,l} \leq 1 \) for all \( k, l \in \mathbb{N} \) then,
\[
[\hat{c}_2, M, p]^{\theta}(\Delta^m) \subset [\hat{c}_2, p]^{\theta}(\Delta^m).
\]

(ii). If \( 0 \leq p_{k,l} \leq \sup p_{k,l} \leq \infty \) for all \( k, l \in \mathbb{N} \) then,
\[
[\hat{c}_2, M]^{\theta}(\Delta^m) \subset [\hat{c}_2, M, p]^{\theta}(\Delta^m).
\]

Acknowledgements

The authors would like to record their gratitude to the reviewer for her careful reading and making some useful corrections which improved the presentation of the paper.
References


