Common Fixed Theorem on Intuitionistic Fuzzy 2-Metric Spaces

Mona S. Bakry

Department of Mathematics, Faculty of Science
Shaqra University, El-Dawadmi, K. S. A.
E-mail: monabak_1000@yahoo.com; mbakery@su.edu.sa

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Abstract

The aim of this paper is to prove the existence and uniqueness of common fixed point theorem for four mappings in complete intuitionistic fuzzy 2-metric spaces.

Keywords: Fuzzy metric spaces, fuzzy 2-metric spaces, intuitionistic fuzzy metric spaces, common fixed point, intuitionistic fuzzy 2-metric spaces.

1 Introduction

The concept of fuzzy sets was introduced by L. A. Zadeh [24] in 1965, which became active field of research for many researchers. In 1975, Karmosil and Michalek [16] introduced the concept of a fuzzy metric space based on fuzzy sets, this notion was further modified by George and Veermani [11] with the help of t-norms. Many authors made use of the definition of a fuzzy metric space in proving fixed point theorems. In 1976, Jungck [14] established common fixed point theorems for commuting maps generalizing the Banach’s fixed point theorem. Sessa [23] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [15] introduced more generalized commutativity, so called compatibility. Mishra et. al. [21] introduced the concept of compatibility in fuzzy metric spaces. Atanassov [1-8] introduced the notion of intuitionistic fuzzy sets and developed its theory. Park [22] using the idea of intuitionistic fuzzy sets to define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t co-norm as a
generalization of fuzzy metric space. Muralisankar and Kalpana [20] proved a common fixed point theorem in an intuitionistic fuzzy metric space for point-wise R-weakly commuting mappings using contractive condition of integral type and established a situation in which a collection of maps has a fixed point which is a point of discontinuity. Gahler [10] introduced and studied the concept of 2-metric spaces in a series of his papers. Iseki et. al. [13] investigated, for the first time, contraction type mappings in 2-metric spaces. In 2002 Sharma [18] introduced the concept of fuzzy 2- metric spaces. Mursaleen et. al. [19] introduced the concept of intuitionistic fuzzy 2-metric space. In this paper, we prove the existence and uniqueness of common fixed point theorem for four mappings in complete intuitionistic fuzzy 2-metric spaces.

2 Preliminaries

**Definition 2.1 (17)** A binary operation \( \ast : [0, 1] \times [0, 1] \longrightarrow [0, 1] \) is called continuous \( t \)-norm if \( \ast \) is satisfying the following conditions:

- (TN1) \( \ast \) is commutative and associative;
- (TN2) \( \ast \) is continuous;
- (TN3) \( a \ast 1 = a \) for all \( a \in [0, 1] \);
- (TN4) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) and \( a, b, c, d \in [0, 1] \).

Examples of \( t \)-norms are \( a \ast b = ab \) and \( a \ast b = \min\{a, b\} \).

**Definition 2.2 (16)** A binary operation \( \triangle : [0, 1] \times [0, 1] \longrightarrow [0, 1] \) is called continuous \( t \)-conorm if \( \triangle \) is satisfying the following conditions:

- (TCN1) \( \triangle \) is commutative and associative;
- (TCN2) \( \triangle \) is continuous;
- (TCN3) \( a \triangle 0 = a \) for all \( a \in [0, 1] \);
- (TCN4) \( a \triangle b \leq c \triangle d \) whenever \( a \leq c \) and \( b \leq d \) and \( a, b, c, d \in [0, 1] \).

**Definition 2.3 (16)** A fuzzy metric space (shortly, \( FM \)-space) is a triple \((X, M, \ast)\), where \( X \) is a nonempty set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^2 \times [0, \infty) \) satisfying the following conditions: for all \( x, y, z \in X \) and \( s, t > 0 \),

- (FM1) \( M(x, y, 0) = 0 \)
(FM2) \(M(x, y, t) = 1\), for all \(t > 0\) if and only if \(x = y\),

(FM3) \(M(x, y, t) = M(y, x, t)\),

(FM4) \(M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)\),

(FM5) \(M(x, y, ) : [0, 1) \rightarrow [0, 1]\) is left continuous.

Note that \(M(x, y, t)\) can be thought of as the degree of nearness between \(x\) and \(y\) with respect to \(t\). We identify \(x = y\) with \(M(x, y, t) = 1\) for all \(t > 0\) and \(M(x, y, t) = 0\) with \(\infty\).

**Definition 2.4 (9)** The 5-tuple \((X, M, N, *, \Diamond)\) is said to be an intuitionistic fuzzy metric space (shortly, IFM-space) if \(X\) is an arbitrary set, \(*\) is a continuous \(t\)-norm, \(\Diamond\) is a continuous \(t\)-conorm, and \(M, N\) are fuzzy sets on \(X^2 \times [0, \infty)\) satisfying the following conditions:

(IFM1) \(M(x, y, t) + N(x, y, t) \leq 1\);

(IFM2) \(M(x, y, 0) = 0\);

(IFM3) \(M(x, y, t) = 1\), for all \(t > 0\) if and only if \(x = y\);

(IFM4) \(M(x, y, t) = M(y, x, t)\);

(IFM5) \(M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)\) for all \(x, y, z \in X\) and \(s, t > 0\);

(IFM6) \(M(x, y, ) : [0, \infty) \rightarrow [0, 1]\) is left continuous.

(IFM7) \(\lim_{t \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\);

(IFM8) \(N(x, y, 0) = 1\);

(IFM9) \(N(x, y, t) = 0\), for all \(t > 0\) if and only if \(x = y\);

(IFM10) \(N(x, y, t) = N(y, x, t)\);

(IFM11) \(N(x, z, t + s) \leq N(x, y, t) \Diamond N(y, z, s)\) for all \(x, y, z \in X\) and \(s, t > 0\);

(IFM12) \(N(x, y, ) : [0, \infty) \rightarrow [0, 1]\) is right continuous.

(IFM13) \(\lim_{t \to \infty} N(x, y, t) = 0\) for all \(x, y \in X\);

Then \((M, N)\) is called an intuitionistic fuzzy metric on \(X\).

The function \(M(x, y, t)\) and \(N(x, y, t)\) denote the degree of nearness and the degree of non-nearness between \(x\) and \(y\) with respect to \(t\) respectively.
Remark 2.5 Every fuzzy metric \((X, M, \ast)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, \ast, \Diamond)\) such that \(t\)-norm \(\ast\) and \(t\)-conorm \(\Diamond\) are associated [12] i.e., \(x \Diamond y = 1 - ((1 - x) \ast (1 - y))\) for any \(x, y \in X\).

Remark 2.6 In intuitionistic fuzzy metric space \(X, M(x, y, .)\) is non-decreasing and \(N(x, y, .)\) is non-increasing for any \(x, y \in X\).

Definition 2.7 (10) A 2-metric space is a set \(X\) with a real-valued function \(d\) on \(X^3\) satisfying the following conditions:

(2M1) For distinct elements \(x, y \in X\), there exists \(z \in X\) such that \(d(x, y, z) \neq 0\).

(2M2) \(d(x, y, z) = 0\) if at least two of \(x, y\) and \(z\) are equal.

(2M3) \(d(x, y, z) = d(x, z, y) = d(y, z, x)\) for all \(x, y, z \in X\).

(2M4) \(d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)\) \(\forall x, y, z, w \in X\).

The function \(d\) is called a 2-metric for the space \(X\) and the pair \((X, d)\) denotes a 2-metric space. It has shown by Gähler [10] that a 2-metric \(d\) is non-negative and although \(d\) is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric \(d\) which is continuous in all of its arguments is said to be continuous.

Geometrically a 2-metric \(d(x, y, z)\) represents the area of a triangle with vertices \(x, y\) and \(z\).

Example 2.8 Let \(X = \mathbb{R}^3\) and let \(d(x, y, z)\) is the area of the triangle spanned by \(x, y\) and \(z\) which may be given explicitly by the formula, \(d(x, y, z) = [x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)],\) where \(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3).\) Then \((X, d)\) is a 2-metric space.

Definition 2.9 (18) The 3-tuple \((X, M, N, \ast)\) is said to be a fuzzy 2-metric space (shortly, F2M-space) if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm, and \(M\) is fuzzy sets on \(X^3 \times [0, \infty)\) satisfying the following conditions: for all \(x, y, z, u \in X\) and \(r, s, t > 0\).

(IFM2) \(M(x, y, z, 0) = 0,\)

(IFM3) \(M(x, y, z, t) = 1,\) if and only if at least two of the three points are equal,

(IFM4) \(M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t).\)

(Symmetry about first three variables)
(IFM5) $M(x, y, z, r + s + t) \geq M(x, y, u, r) \ast M(x, u, z, s) \ast M(u, y, z, t)$.
(This corresponds to tetrahedron inequality in 2-metric space, the function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than $t$.)

(IFM6) $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

**Definition 2.10** (19) The 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy 2-metric space (shortly, IF2M-space) if $X$ is an arbitrary set, $*$ is a continuous $t$-norm, $\Diamond$ is a continuous $t$-conorm, and $M, N$ are fuzzy sets on $X^3 \times [0, \infty)$ satisfying the following conditions:

for all $x, y, z, w \in X$ and $r, s, t > 0$.

(IF2M1) $M(x, y, z, t) + N(x, y, z, t) \leq 1$,

(IF2M2) given distinct elements $x, y, z$ of $X$ there exists an element $z$ of $X$ such that $M(x, y, z, 0) = 0$,

(IF2M3) $M(x, y, z, t) = 1$, if at least two of $x, y, z$ of $X$ are equal,

(IF2M4) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$,

(IF2M5) $M(x, y, z, r + s + t) \geq M(x, y, w, r) \ast M(x, w, z, s) \ast M(w, y, z, t)$;

(IF2M6) $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous,

(IF2M7) $N(x, y, z, 0) = 1$,

(IF2M8) $N(x, y, z, t) = 0$, if at least two of $x, y, z$ of $X$ are equal,

(IF2M9) $N(x, y, z, t) = N(x, z, y, t) = N(y, z, x, t)$,

(IF2M10) $N(x, y, z, r + s + t) \leq N(x, y, w, r) \Diamond N(x, w, z, s) \Diamond N(w, y, z, t)$;

(IF2M11) $N(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous,

In this case $(M, N)$ is called an intuitionistic fuzzy 2-metric on $X$. The function $M(x, y, z, t)$ and $N(x, y, z, t)$ denote the degree of nearness and the degree of non-nearness between $x, y$ and $z$ with respect to $t$, respectively.

**Example 2.11** Let $(X, d)$ be a 2-metric space. Denote $a \ast b = ab$ and $a \Diamond b = min\{1, a + b\}$ for all $a, b \in [0, 1]$ and $M_d$ and $N_d$ be fuzzy sets on $X^3 \times [0, \infty)$ defined by

$$M_d(x, y, z, t) = \frac{ht^n}{ht^n + md(x, y, z)}, N_d(x, y, z, t) = \frac{d(x, y, z)}{kt^n + md(x, y, z)}$$

for all $h, k, m, n \in R^+$. Then $(X, M_d, N_d, *, \Diamond)$ is IF2M-space.
Definition 2.12 Let \((X, M, N, *, \triangleleft)\) be an IF2M-space.

(a) A sequence \(\{x_n\}\) in IF2M-space \(X\) is said to be convergent to a point \(x \in X\) (denoted by \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\)) if for any \(\lambda \in (0, 1)\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) and \(a \in X\),
\[
M(x_n, x, a, t) > 1 - \lambda \quad \text{and} \quad N(x_n, x, a, t) < \lambda.
\]
That is \(\lim_{n \to \infty} M(x_n, x, a, t) = 1\) and \(\lim_{n \to \infty} N(x_n, x, a, t) = 0\), for \(a \in X\) and \(t > 0\).

(b) A sequence \(\{x_n\}\) in IF2M-space \(X\) is called a Cauchy sequence, if for any \(\lambda \in (0, 1)\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(m, n \geq n_0\) and \(a \in X\),
\[
M(x_m, x_n, a, t) > 1 - \lambda \quad \text{and} \quad N(x_m, x_n, a, t) < \lambda.
\]
That is \(\lim_{m, n \to \infty} M(x_m, x_n, a, t) = 1\) and \(\lim_{m, n \to \infty} N(x_m, x_n, a, t) = 0\), for \(a \in X\) and \(t > 0\).

(c) The IF2M-space \(X\) is said to be complete if and only if every Cauchy sequence is convergent.

Definition 2.13 Self mappings \(A\) and \(B\) of an IF2M-space \((X, M, N, *, \triangleleft)\) is said be compatible, if
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, a, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ABx_n, BAx_n, a, t) = 0
\]
for all \(a \in X\) and \(t > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z\) for some \(z \in X\).

3 Main Results

Lemma 3.1 Let \((X, M, N, *, \triangleleft)\) be an IF2M-space. Then \(M(x, y, z, t)\) is non-decreasing and \(N(x, y, z, t)\) is non-increasing for all \(x, y, z \in X\).

Proof: Let \(s, t > 0\) be any points such that \(t > s\). \(t = s + \frac{t-s}{2} + \frac{t-s}{2}\). Hence we have
\[
N(x, y, z, t) = N(x, y, z, s + \frac{t-s}{2} + \frac{t-s}{2})
\leq N(x, y, z, s)\triangleleft N(x, z, z, \frac{t-s}{2})\triangleleft N(z, y, z, \frac{t-s}{2})
= N(x, y, z, s)
\]
Thus \(N(x, y, z, t) < N(x, y, z, s)\). Similarly, \(M(x, y, z, t) > M(x, y, z, s)\). Therefore, \(M(x, y, z, t)\) is non-decreasing and \(N(x, y, z, t)\) is non-increasing.

From Lemma 3.1, let \((X, M, N, *, \triangleleft)\) be an IF2M-space with the following conditions:
\[
\lim_{t \to \infty} M(x, y, z, t) = 1, \quad \lim_{t \to \infty} N(x, y, z, t) = 0
\]
Lemma 3.2 Let \((X, M, N, *, \Diamond)\) be an IF2M-space. If there exists \(q \in (0, 1)\) such that \(M(x, y, z, q t + 0) \geq M(x, y, z, t)\) and \(N(x, y, z, q t + 0) \leq N(x, y, z, t)\) for all \(x, y, z \in X\) with \(z \neq x, z \neq y\) and \(t > 0\). Then \(x = y\).

**Proof:** Since
\[
M(x, y, z, t) \geq M(x, y, z, q t + 0) \geq M(x, y, z, t), \quad \text{and}
\]
\[
N(x, y, z, t) \leq N(x, y, z, q t + 0) \leq N(x, y, z, t)
\]
for all \(t > 0\), \(M(x, y, z, t)\) and \(N(x, y, z, t)\) are constant. Since \(\lim_{t \to \infty} M(x, y, z, t) = 1\), \(\lim_{t \to \infty} N(x, y, z, t) = 0\). Then \(M(x, y, z, t) = 1\) and \(N(x, y, z, t) = 0\). Consequently, for all \(t > 0\). Hence \(x = y\) because \(z \neq x, z \neq y\).

Lemma 3.3 Let \((X, M, N, *, \Diamond)\) be an IF2M-space and let \(\lim_{t \to \infty} x_n = x, \lim_{t \to \infty} y_n = y\). Then the following are satisfied for all \(a \in X\) and \(t \geq 0\)

(1) \(\lim_{n \to \infty} \inf M(x_n, y_n, a, t) \geq M(x, y, a, t)\) and
\[
\lim_{n \to \infty} \sup N(x_n, y_n, a, t) \leq N(x, y, a, t)
\]

(2) \(M(x, y, a, t + 0) \geq \lim_{n \to \infty} \sup M(x_n, y_n, a, t)\) and
\[
N(x, y, a, t + 0) \leq \lim_{n \to \infty} \inf N(x_n, y_n, a, t)
\]

**Proof:** (1) For all \(a \in X\) and \(t \geq 0\) we have
\[
M(x_n, y_n, a, t) \geq M(x_n, y_n, x, t_1) * M(x_n, x, a, t_2) * M(x_n, y_n, a, t), t_1 + t_2 = 0
\]
\[
\geq M(x, y, a, t_4) * M(y, y, a, t), t_3 + t_4 = 0
\]
which implies \(\lim_{n \to \infty} \inf M(x_n, y_n, a, t) \geq M(x, y, a, t)\)

Also,
\[
N(x_n, y_n, a, t) \leq N(x_n, y_n, x, t_1) \Diamond N(x_n, x, a, t_2) \Diamond N(x_n, y_n, a, t), t_1 + t_2 = 0
\]
\[
\leq N(x, y, a, t_4) \Diamond N(y, y, a, t), t_3 + t_4 = 0
\]
which implies \(\lim_{n \to \infty} \sup N(x_n, y_n, a, t) \leq 0 \Diamond 0 \Diamond N(x, y, a, t) \Diamond 0 = N(x, y, a, t)\)

(2) Let \(\epsilon > 0\) be given. For all \(a \in X\) and \(t > 0\) we have
\[
M(x, y, a, t + 2\epsilon) \geq M(x, y, x_n, \frac{\epsilon}{2}) * M(x, x_n, a, \frac{\epsilon}{2}) * M(x_n, y, a, t + \epsilon)
\]
\[
\geq M(x, y, x_n, \frac{\epsilon}{2}) * M(x, x_n, a, \frac{\epsilon}{2}) * M(x_n, y_n, \frac{\epsilon}{2})
\]
\[
* M(x_n, y_n, a, t) * M(y_n, y_n, a, \frac{\epsilon}{2}).
\]
Consequently,

\[ M(x, y, a, t + 2\epsilon) \geq \lim_{n \to \infty} \sup M(x_n, y_n, a, t). \]

Letting \( \epsilon \to 0 \), we have

\[ M(x, y, a, t + 0) \geq \lim_{n \to \infty} \sup M(x_n, y_n, a, t). \]

Also, we have

\[
N(x, y, a, t + 2\epsilon) \leq N(x, y, x_n, \frac{\epsilon}{2}) \Diamond N(x, y, a, \frac{\epsilon}{2}) \Diamond N(x_n, y_n, \frac{\epsilon}{2}) \Diamond N(y, y, a, \frac{\epsilon}{2}).
\]

Consequently,

\[ N(x, y, a, t + 2\epsilon) \leq \lim_{n \to \infty} \inf N(x_n, y_n, a, t). \]

Letting \( \epsilon \to 0 \), we have

\[ N(x, y, a, t + 0) \leq \lim_{n \to \infty} \inf N(x_n, y_n, a, t). \]

**Lemma 3.4** Let \((X, M, N, *, \Diamond)\) be an IF2M-space and let \(A\) and \(B\) be continuous self mappings of \(X\) and \([A, B]\) are compatible. Let \(x_n\) be a sequence in \(X\) such that \(Ax_n \to z\) and \(Bx_n \to z\). Then \(ABx_n \to Bz\).

**Proof:** Since \(A, B\) are continuous maps, \(ABx_n \to Az, BAx_n \to Bz\) and so, \(M(ABx_n, Az, a, \frac{t}{3}) \to 1\) and \(M(BAx_n, Bz, a, \frac{t}{3}) \to 1\) for all \(a \in X\) and \(t > 0\).

Since the pair \([A, B]\) is compatible, \(M(BAx_n, ABx_n, a, \frac{t}{3}) \to 1\) for all or all \(a \in X\) and \(t > 0\). Thus

\[
M(ABx_n, Bz, a, t) \geq M(ABx_n, Bz, BAx_n, \frac{t}{3}) \ast M(ABx_n, BAx_n, a, \frac{t}{3})
\]
\[
\ast M(BAx_n, Bz, a, \frac{t}{3})
\]
\[
\geq M(BAx_n, Bz, ABx_n, \frac{t}{3}) \ast M(BAx_n, ABx_n, a, \frac{t}{3})
\]
\[
\ast M(BAx_n, Bz, a, \frac{t}{3})
\]
\[
\to 1
\]
Also we have
\[ N(ABx_n, Bz, a, t) \leq N(ABx_n, Bz, BAx_n, \frac{t}{3}) \triangle N(ABx_n, Bz, A, \frac{t}{3}) \]
\[ \leq N(BAx_n, Bz, ABx_n, \frac{t}{3}) \triangle N(BAx_n, Bz, a, \frac{t}{3}) \]
\[ \rightarrow 0 \]
for all \( a \in X \) and \( t > 0 \).
Hence \( ABx_n \rightarrow Bz \).

**Theorem 3.5** Let \((X, M, N, *, \triangle)\) be a complete IF2M-space with continuous \( t \)-norm \(*\) and continuous \( t \)-conorm \(\triangle\). Let \( S \) and \( T \) be continuous self mappings of \( X \). Then \( S \) and \( T \) have a unique common fixed point in \( X \) if and only if there exists two self mappings \( A, B \) of \( X \) satisfying

1. \( AX \subset TX, BX \subset SX \),
2. the pair \( \{A, S\} \) and \( \{B, T\} \) are compatible,
3. there exists \( q \in (0, 1) \) such that for every \( x, y, a \in X \) and \( t > 0 \)
   \[ M(Ax, By, a, qt) \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(By, Ty, a, t), M(Ax, Ty, a, t)\} \]
   \[ N(Ax, By, a, qt) \leq \max\{N(Sx, Ty, a, t), N(Ax, Sx, a, t), N(By, Ty, a, t), N(Ax, Ty, a, t)\} \]
   Then \( A, B \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** Suppose that \( S \) and \( T \) have a (unique) common fixed point say \( z \in X \). Define \( A : X \rightarrow X \) be \( Ax = z \) for all \( x \in X \), and \( B : X \rightarrow X \) be \( Bx = z \) for all \( x \in X \).

Then one can see that (1)-(3) are satisfied.

Conversely, assume that there exist two self mappings \( A, B \) of \( X \) satisfying condition (1)-(3). From condition (1) we can construct two sequences \( x_n \) and \( y_n \) of \( X \) such that \( y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \) and \( y_{2n} = Sx_{2n} = Bx_{2n-1} \) for \( n = 1, 2, 3, \ldots \). Putting \( x = x_{2n} \) and \( x = x_{2n+1} \) in condition (3), we have that for all \( a \in X \) and \( t > 0 \)
\[ M(y_{2n+1}, y_{2n+2}, a, qt) = M(Ax_{2n}, Bx_{2n+1}, a, qt) \geq \min\{M(Sx_{2n}, Tx_{2n+1}, a, t), M(Ax_{2n}, Sx_{2n}, a, t)\} \]
Let 

\[ \left\{ \right. \]

We now show that

\[ \geq \min \{ M(yx_{2n}, yx_{2n+1}, a, qt), M(yx_{2n+1}, yx_{2n+1}, a, qt) \} \]

and

\[ N(yx_{2n+1}, yx_{2n+2}, a, qt) = N(Ax_{2n}, Bx_{2n+1}, a, qt) \]
\[ \leq \max \{ N(Sx_{2n}, Tx_{2n+1}, a, t), N(Ax_{2n}, Sx_{2n}, a, t) \} \]
\[ N(Bx_{2n+1}, Tx_{2n+1}, a, t), N(Ax_{2n}, Tx_{2n+1}, a, t) \} \]
\[ \leq \max \{ N(yx_{2n}, yx_{2n+1}, a, qt), N(yx_{2n+1}, yx_{2n+1}, a, qt) \} \]

which implies

\[ M(yx_{2n+1}, yx_{2n+1}, a, qt) \geq M(yx_{2n+1}, yx_{2n+1}, a, qt) \]

by Lemma 3.1. Also, letting \( x = x_{2n+2} \) and \( y = x_{2n+1} \) in condition (3), we have that

\[ M(y_{2n+2}, y_{2n+3}, a, qt) \geq M(y_{2n+1}, y_{2n+2}, a, t) \]
\n
In general we obtain that for all \( a \in X \) and \( t > 0 \) and \( n = 1, 2, ... \)

\[ M(y_{n}, y_{n+1}, a, qt) \geq M(y_{n-1}, y_{n}, a, t) \]
\n
Thus, for all \( a \in X \) and \( t > 0 \) and \( n = 1, 2, ... \)

\[ M(y_{n}, y_{n+1}, a, t) \geq M(0, y_{1}, a, \frac{t}{q^n}) \quad (3.1) \]

and

\[ N(y_{n}, y_{n+1}, a, t) \leq N(y_{0}, y_{1}, a, \frac{t}{q^n}) \quad (3.2) \]

We now show that \( \{ y_{n} \} \) is a Cauchy sequence in \( X \).

Let \( m > n \). Then for all \( a \in X \) and \( t > 0 \) we have

\[ M(y_{m}, y_{n}, a, t) \]
\[ \geq M(y_{m}, y_{n}, y_{n+1}, \frac{t}{3}) * M(y_{n+1}, y_{n}, a, \frac{t}{3}) * M(y_{m}, y_{n+1}, y_{n+2}, \frac{t}{32}) * M(y_{n+2}, y_{n+1}, a, \frac{t}{32}) \]
\[ M(y_{m}, y_{n+2}, a, \frac{t}{32}) \]
\[
M(y_m, y_{m-n}, a, \frac{t}{3m-n})
\]

and
\[
N(y_m, y_n, a, t) \leq N(y_m, y_{n+1}, \frac{t}{3}) \triangleq N(y_{n+1}, y_n, a, \frac{t}{3})
\]
\[
N(y_m, y_{n+1}, a, \frac{t}{3})
\]
\[
N(y_m, y_{n+1}, y_{n+2}, \frac{t}{3^2}) \triangleq N(y_{n+2}, y_{n+1}, a, \frac{t}{3^2})
\]
\[
N(y_m, y_{n+2}, a, \frac{t}{3^2})
\]
\[
N(y_m, y_{m-n}, a, \frac{t}{3m-n})
\]

letting \(m, n \to \infty\) we have
\[
\lim_{n \to \infty} M(y_m, y_n, a, t) = 1, \lim_{n \to \infty} N(y_m, y_n, a, t) = 0. \text{ Thus } \{y_n\} \text{ is a Cauchy sequence in } X.
\]

It follows from completeness of \(X\) that there exists \(z \in X\) such that \(\lim_{n \to \infty} y_n = z\). Hence \(\lim y_{2n-1} = \lim_{n \to \infty} Tx_{2n-1} = \lim_{n \to \infty} Ax_{2n-2} = z\) and \(\lim y_{2n} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Bx_{2n-1} = z\). From Lemma 3.4, \(ASx_{2n+1} = Sz\) and \(BTx_{2n+1} = Tz\) (3.3)

Meanwhile, for all \(a \in X\) with \(a \neq Sz\) and \(a \neq Tz\) and \(t > 0\),
\[
M(ASx_{2n+1}, BTx_{2n+1}, a, qt) \geq \min\{M(SSx_{2n+1}, TTx_{2n+1}, a, t), \newline M(ASx_{2n+1}, Sx_{2n+1}, a, t), \newline M(BTx_{2n+1}, TTx_{2n+1}, a, t), \newline M(ASx_{2n+1}, TTx_{2n+1}, a, t)\}
\]

and
\[
N(ASx_{2n+1}, BTx_{2n+1}, a, qt) \leq \max\{N(SSx_{2n+1}, TTx_{2n+1}, a, t), \newline N(ASx_{2n+1}, Sx_{2n+1}, a, t), \newline N(BTx_{2n+1}, TTx_{2n+1}, a, t), \newline N(ASx_{2n+1}, TTx_{2n+1}, a, t)\}.
\]
Taking limit as \( n \to \infty \) and using (3.3), we have for all \( a \in X \) with \( a \neq Sz \) and \( a \neq Tz \) and \( t > 0 \).

\[
M(Sz, Tz, a, qt + 0) \geq \min \{ M(Sz, Tz, a, t), M(Sz, Sz, a, t), \\
M(Tz, Tz, a, t), M(Sz, Tz, a, t) \}
\]

and

\[
N(Sz, Tz, a, qt + 0) \leq \max \{ N(Sz, Tz, a, t), N(Sz, Sz, a, t), \\
N(Tz, Tz, a, t), N(Sz, Tz, a, t) \}
\]

By Lemma 3.2, we have \( Sz = Tz \) \hspace{1cm} (3.4)

From condition (3), we get for all \( a \in X \) with \( a \neq Az, a \neq Tz \) and \( t > 0 \)

\[
M(Az, BTx_{2n+1}, a, qt) \geq \min \{ M(Sz, TTx_{2n+1}, a, t), M(Az, Sz, a, t), \\
M(BTx_{2n+1}, TTx_{2n+1}, a, t), M(Az, TTx_{2n+1}, a, t) \}
\]

and

\[
N(Az, BTx_{2n+1}, a, qt) \leq \max \{ N(Sz, TTx_{2n+1}, a, t), N(Az, Sz, a, t), \\
N(BTx_{2n+1}, TTx_{2n+1}, a, t), N(Az, TTx_{2n+1}, a, t) \}
\]

Taking limit as \( n \to \infty \) and using condition (3), and Lemma 3.3, we have for all \( a \in X \)

\[
M(Az, Tz, a, qt + 0) \geq \min \{ M(Sz, Tz, a, t), M(Az, Sz, a, t), \\
M(Tz, Tz, a, t), M(Az, Tz, a, t) \}
\]

and

\[
N(Az, Tz, a, qt + 0) \leq \max \{ N(Sz, Tz, a, t), N(Az, Sz, a, t), \\
N(Tz, Tz, a, t), N(Az, Tz, a, t) \}
\]

By Lemma 3.2, we have, \( Az = Tz \) \hspace{1cm} (3.5)

And for all \( a \in X \) with \( a \neq Az \) and \( a \neq Bz \), and \( t > 0 \).

\[
M(Az, Bz, a, qt) \geq \min \{ M(Sz, Tz, a, t), M(Az, Sz, a, t), \\
M(Bz, Tz, a, t), M(Az, Tz, a, t) \} \\
\geq \min \{ M(Tz, Tz, a, t), M(Tz, Tz, a, t), \\
M(Bz, Az, a, t), M(Tz, Tz, a, t) \} \\
M(Az, Bz, a, t)
\]
and

\[ N(Az, Bz, a, qt) \leq \min\{N(Sz, Tz, a, t), N(Az, Sz, a, t), \\
N(Bz, Tz, a, t), N(Az, Tz, a, t)\} \]
\[ \leq \max\{N(Tz, Tz, a, t), N(Tz, Tz, a, t), \\
N(Bz, Az, a, t), N(Tz, Tz, a, t)\} \]
\[ N(Az, Bz, a, t) \]

By Lemma 3.2, \( Az = Bz \) 

(3.6)

It follows that \( Az = Bz = Sz = Tz \). For all \( a \in X \) with \( a \neq Bz \) and \( a \neq z \), and \( t > 0 \).

\[ M(Ax_{2n}, Bz, a, qt) \geq \min\{M(Sx_{2n}, Tz, a, t), M(Ax_{2n}, Sx_{2n}, a, t), \\
M(Bz, Tz, a, t), M(Ax_{2n}, Tz, a, t)\} \]

and

\[ N(Ax_{2n}, Bz, a, qt) \leq \max\{N(Sx_{2n}, Tz, a, t), N(Ax_{2n}, Sx_{2n}, a, t), \\
N(Bz, Tz, a, t), N(Ax_{2n}, Tz, a, t)\} \]

Taking limit as \( n \to \infty \) and using (3.3) and Lemma 3.3, we have for all \( a \in X \) with \( a \neq Bz, a \neq z \) and \( t > 0 \).

\[ M(z, Bz, a, qt + 0) \geq \min\{M(z, Tz, a, t), M(z, z, a, t), \\
M(Bz, Bz, a, t), M(z, Tz, a, t)\} \]
\[ \geq M(z, Tz, a, t) \geq M(z, Bz, a, t) \]

and

\[ N(z, Bz, a, qt + 0) \leq \max\{N(z, Tz, a, t), N(z, z, a, t), \\
N(Bz, Bz, a, t), N(z, Tz, a, t)\} \]
\[ \leq N(z, Tz, a, t) \leq N(z, Bz, a, t), \]

and so we have, \( M(z, Bz, a, qt) \geq M(z, Bz, a, t) \) and \( N(z, Bz, a, qt) \leq N(z, Bz, a, t) \), and hence \( Bz = z \). Thus, \( z = Az = Bz = Sz = Tz \), and so \( z \) is a common fixed point of \( A, B, C \) and \( T \).

For uniqueness, let \( w \) be another common fixed point of \( A, B, S, T \). Then, for all \( a \in X \) with \( a \neq z \), \( a \neq w \) and \( t > 0 \).

\[ M(z, w, a, qt) = M(Az, Bw, a, qt) \]
\[ \geq \min\{M(Sz, Tw, a, t), M(Az, Sz, a, t), \\
M(Bw, Tw, a, t), M(Az, Tw, a, t)\} \]
\[ \geq \min\{M(z, w, a, t), M(z, z, a, t), \\
M(w, w, a, t), M(z, w, a, t)\} \]
\[ \geq M(z, w, a, t). \]

and
\[ N(z, w, a, qt) = N(Az, Bw, a, qt) \]
\[ \leq \max\{N(Sz, Tw, a, t), N(Az, Sz, a, t), \\
N(Bw, Tw, a, t), N(Az, Tw, a, t)\} \]
\[ \leq \max\{N(z, w, a, t), N(z, z, a, t), \\
N(w, w, a, t), N(z, w, a, t)\} \]
\[ \leq N(z, w, a, t). \]

which implies that \(M(z, w, a, qt) \geq M(z, w, a, t)\) and \(N(z, w, a, qt) \geq N(z, w, a, t)\), hence \(z = w\). This complete the proof of.

References


