Maximal Subgroups of the Semigroup $B_X(D)$
Defined by Semilattices of the Class $\Sigma_3^X(8)$

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Abstract

By the symbol $\Sigma_3^X(8)$ we denote the class of all $X$-semilattices of unions whose every element is isomorphic to an $X$-semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\}$, where

\[
\begin{align*}
Z_7 &\subseteq Z_5 \subseteq Z_3 \subseteq Z_1 \subseteq D, \\
Z_2 &\subseteq Z_6 \subseteq Z_4 \subseteq Z_2 \subseteq D, \\
Z_7 &\subseteq Z_5 \subseteq Z_4 \subseteq Z_3 \subseteq D, \\
Z_7 &\subseteq Z_5 \subseteq Z_4 \subseteq Z_2 \subseteq D, \\
Z_i \setminus Z_j &\neq \emptyset, \quad (i, j) \in \{(5,6),(6,5),(3,6),(6,3),(4,3),(3,4),(2,3),(3,2),(2,1),(1,2)\}.
\end{align*}
\]

In the given paper we give a full description maximal subgroups of the complete semigroups of binary elations defined by semilattices of the class $\Sigma_3^X(8)$.

Keywords: Semilattice, Semigroup, Binary Relation, Idempotent Element.
1 Introduction

Let $X$ be an arbitrary nonempty set, $D$ be a $X$-semilattice of unions, i.e. a nonempty set of subsets of the set $X$ that is closed with respect to the set-theoretic operations of unification of elements from $D$, $f$ be an arbitrary mapping from $X$ into $D$. To each such a mapping $f$ there corresponds a binary relation $\alpha_f$ on the set $X$ that satisfies the condition $\alpha_f = \bigcup \left\{ \{x\} \times f(x) \right\}$. The set of all such $\alpha_f$ ($f : X \rightarrow D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X$-semilattice of unions $D$ (see [1, Item 2.1, p. 34]).

By $\emptyset$ we denote an empty binary relation or empty subset of the set $X$. The condition $(x,y) \in \alpha$ will be written in the form $\alpha xy$. Further let $x,y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $\tau \in D = \bigcup_{y \in D} Y$. Then by symbols we denote the following sets:

\begin{align}
\gamma \alpha &= \{x \in X \mid \gamma \alpha x\}, \ Y \alpha = \bigcup_{\gamma \in D} \gamma \alpha, \ V(D,\alpha) = \{Y \alpha \mid Y \in D\}, \\
X' &= \{T \mid \emptyset \neq T \subseteq X\}, \ D'_T = \{Z' \in D' \mid \exists T \in Z'\}, \ Y''_T = \{x \in X \mid x \alpha = T\}, \\
D'_\tau &= \{Z' \in D' \mid T \subseteq Z'\}, \ D'_\tau = \{Z' \in D' \mid Z' \subseteq T\}, \ l(D',\tau) = \cup(D' \setminus D'_\tau).
\end{align}

Under symbol $\land (D, D_i)$ we mean an exact lower bound of the set $D_i$ in the semilattice $D$.

**Definition 1.1:** Let $\varepsilon \in B_X(D)$. If $\varepsilon \circ \varepsilon = \varepsilon$ or $\alpha \circ \varepsilon = \alpha$ for any $\alpha \in B_X(D)$, then $\varepsilon$ is called an idempotent element or called right unit of the semigroup $B_X(D)$ respectively (see [1], [2], [3]).

**Definition 1.2:** We say that a complete $X$-semilattice of unions $D$ is an $XI$-semilattice of unions if it satisfies the following two conditions:

\begin{enumerate}
\item $\land (D, D_i) \in D$ for any $\iota \in D$;
\item $Z = \bigcup_{\alpha \in Z} \land (D, D_i)$ for any nonempty element $Z$ of $D$ (see [1, Definition 1.14.2], [2, Definition 1.14.2] or [6]).
\end{enumerate}

**Definition 1.3:** The one-to-one mapping $\varphi$ between the complete $X$-semilattices of unions $D'$ and $D''$ is called a complete isomorphism if the condition $\varphi(\cup D_i) = \bigcup_{T \neq 0} \varphi(T')$ is fulfilled for each nonempty subset $D_i$ of the semilattice $D'$ (see [1, Definition 6.2.3], [2, Definition 6.2.3] or [5]).
**Definition 1.4:** We say that a nonempty element \( T \) is a nonlimiting element of the set \( D' \) if \( T \setminus (D',T) \neq \emptyset \) and a nonempty element \( T \) is a limiting element of the set \( D' \) if \( T \setminus (D',T) = \emptyset \) (see [1, Definition 1.13.1 and 1.13.2] or [2, Definition 1.13.1 and 1.13.2]).

**Theorem 1.1:** Let \( X \) be a finite set and \( D(\alpha) \) be the set of all those elements \( T \) of the semilattice \( Q = V(D,\alpha) \setminus \{\emptyset\} \) which are nonlimiting elements of the set \( Q_T \). A binary relation \( \alpha \) having a quasinormal representation \( \alpha = \bigcup_{T \in V(D,\alpha)} (Y_T \times T) \) is an idempotent element of this semigroup iff

a) \( V(D,\alpha) \) is complete \( XI \)–semilattice of unions;

b) \( \bigcup_{T \in D(\alpha)_t} Y_T \supset T \) for any \( T \in D(\alpha) \);

c) \( Y_T \cap T \neq \emptyset \) for any nonlimiting element of the set \( D(\alpha)_t \) (see [1, Theorem 6.3.9], [2, Theorem 6.3.9] or [5]).

**Theorem 1.2:** Let \( D = \{D,Z_1,Z_2,...,Z_{m-1}\} \) be some finite \( X \)-semilattice of unions and \( C(D) = \{P_0,P_1,P_2,...,P_{m-1}\} \) be the family of sets of pairwise nonintersecting subsets of the set \( X \). If \( \varphi \) is a mapping of the semilattice \( D \) on the family of sets \( C(D) \) which satisfies the condition \( \varphi(D) = P_0 \) and \( \varphi(Z_i) = P_i \) for any \( i = 1,2,...,m-1 \) and \( \hat{D} = D \setminus \{T \in D \mid Z \subseteq T\} \), then the following equalities are valid:

\[
\hat{D} = P_0 \cup P_1 \cup P_2 \cup ... \cup P_{m-1}, \quad Z_i = P_0 \cup \bigcup_{T \in D_0} \varphi(T). \tag{1.2}
\]

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice \( D \) are represented in the form (1.2), then among the parameters \( P_i \) (\( i = 0,1,2,...,m-1 \)) there exist such parameters that cannot be empty sets. Such sets \( P_i \) (\( 0 < i \leq m-1 \)) are called basis sources, whereas sets \( P_i \) (\( 0 \leq j \leq m-1 \)) which can be empty sets too are called completeness sources.

The number the basis sources we denote by symbol \( \delta \).

It is proved that under the mapping \( \varphi \) the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping \( \varphi \) the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1, Item 11.4], [2, Item 11.4] or [3]).
Denote by the symbol $G_X(D,e)$ a maximal subgroup of the semigroup $B_X(D)$ whose unit is an idempotent binary relation $e$ of the semigroup $B_X(D)$.

**Theorem 1.3:** For any idempotent element $e \in B_X(D)$, the group $G_X(D,e)$ is antiisomorphic to the group of all complete automorphism of the semilattice $V(D,e)$ (see [1, Theorem 7.4.2], [2, Theorem 7.4.2] or [4]).

## 2 Results

Let $X$ and $\Sigma(X,8)$ be respectively an any nonempty set and a class intreisomorphic $X$-semilattices of unions where every element is isomorphic to some $X$-semilattice of unions $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$, that satisfying the conditions.

\[
\begin{align*}
Z_7 & \subseteq Z_6 \subseteq Z_5 \subseteq Z_4 \subseteq Z_3 \subseteq Z_2 \subseteq Z_1 \subseteq D, \\
Z_7 & \subseteq Z_6 \subseteq Z_5 \subseteq Z_4 \subseteq Z_3 \subseteq Z_2 \subseteq D, \\
Z_7 & \subseteq Z_5 \subseteq Z_4 \subseteq Z_3 \subseteq Z_2 \subseteq D, \\
Z_7 & \subseteq Z_5 \subseteq Z_3 \subseteq Z_2 \subseteq D, \\
Z_1 \setminus Z_2 & \neq \emptyset, \; Z_5 \setminus Z_3 \neq \emptyset, \; Z_3 \setminus Z_5 \neq \emptyset, \\
Z_3 \setminus Z_4 & \neq \emptyset, \; Z_5 \setminus Z_3 \neq \emptyset, \; Z_3 \setminus Z_5 \neq \emptyset, \; Z_5 \setminus Z_6 \neq \emptyset, \; Z_6 \setminus Z_5 \neq \emptyset; \\
\end{align*}
\]

(2.1)

The semilattice satisfying the conditions (2.1) is shown in Figure 1.

![Fig. 1](image)

**Lemma 2.1:** Let $D \in \Sigma(X,8)$. Then the following sets exhaust all subsemilattices of the semilattice $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$.

1)\{\{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}. (see diagram 1 of the figure 2);

2)\{\{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_1\}, \{\bar{D}\}\}, \{\{\bar{D}\}\}, \\
\{\{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \\
\{\{Z_2\}, \{Z_1\}, \{\bar{D}\}\}, \{\{Z_2\}, \{Z_1\}, \{\bar{D}\}\}.

(see diagram 2 of the figure 2);
3) \(\{Z_x, Z_y, D\}, \{Z_x, Z_y, Z_1\}, \{Z_x, Z_y, Z_2\}, \{Z_x, Z_y, Z_3\}, \{Z_x, Z_y, Z_4\}\).

(see diagram 3 of the figure 2);

4) \(\{Z_x, Z_y, Z_1, Z_2\}, \{Z_x, Z_y, Z_3, Z_4\}, \{Z_x, Z_y, Z_5, Z_6\}, \{Z_x, Z_y, Z_7, Z_8\}\).

(see diagram 4 of the figure 2);

5) \(\{Z_x, Z_y, Z_1, Z_2, D\}, \{Z_x, Z_y, Z_3, Z_4, D\}, \{Z_x, Z_y, Z_5, Z_6, Z_7, D\}, \{Z_x, Z_y, Z_8, Z_9, D\}\).

(see diagram 5 of the figure 2);

6) \(\{Z_x, Z_y, Z_1, D\}, \{Z_x, Z_y, Z_2, D\}, \{Z_x, Z_y, Z_3, D\}, \{Z_x, Z_y, Z_4, D\}\).

(see diagram 6 of the figure 2);

7) \(\{Z_x, Z_y, Z_1, Z_2, D\}, \{Z_x, Z_y, Z_3, Z_4, Z_5\}, \{Z_x, Z_y, Z_6, Z_7, D\}, \{Z_x, Z_y, Z_8, Z_9, D\}\).

(see diagram 7 of the figure 2);

8) \(\{Z_x, Z_y, Z_1, Z_2, Z_3, D\}, \{Z_x, Z_y, Z_4, Z_5, Z_6, D\}\).

(see diagram 8 of the figure 2);

9) \(\{Z_x, Z_y, Z_1, Z_2, Z_3, D\}\).

(see diagram 9 of the figure 2);

10) \(\{Z_x, Z_y, Z_1, Z_2, D\}, \{Z_x, Z_y, Z_3, Z_4, Z_5, D\}, \{Z_x, Z_y, Z_6, Z_7, Z_8, D\}\).

(see diagram 10 of the figure 2);

11) \(\{Z_x, Z_y, Z_1, Z_2, D\}, \{Z_x, Z_y, Z_3, Z_4, Z_5, D\}\).

(see diagram 11 of the figure 2);
12) \( \{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_3, Z_2, Z_1, \bar{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1, \bar{D}\}, \{Z_3, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \).
(see diagram 12 of the figure 2);

13) \( \{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 13 of the figure 2);

14) \( \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 14 of the figure 2);

15) \( \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 15 of the figure 2);

16) \( \{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 16 of the figure 2);

17) \( \{Z_1, Z_2, \bar{D}\}, \{Z_1, \bar{D}\}, \{Z_2, Z_1, \bar{D}\}, \{Z_3, Z_1, \bar{D}\}, \{Z_4, Z_1, \bar{D}\} \).
(see diagram 17 of the figure 2);

18) \( \{Z_6, Z_2, Z_4, \bar{D}\}, \{Z_6, Z_2, Z_3, \bar{D}\}, \{Z_6, Z_2, Z_4, \bar{D}\}, \{Z_6, Z_2, Z_5, \bar{D}\}, \{Z_6, Z_2, Z_6, \bar{D}\} \).
(see diagram 18 of the figure 2);

19) \( \{Z_6, Z_2, Z_1, \bar{D}\}, \{Z_6, Z_1, \bar{D}\} \). (see diagram 19 of the figure 2);

20) \( \{Z_6, Z_4, Z_1, \bar{D}\}, \{Z_6, Z_4, Z_1, \bar{D}\} \). (see diagram 20 of the figure 2);

21) \( \{Z_6, Z_4, Z_1, \bar{D}\} \). (see diagram 21 of the figure 2);

22) \( \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}, \{Z_7, Z_3, Z_2, Z_1, \bar{D}\} \).
(see diagram 22 of the figure 2);

23) \( \{Z_7, Z_3, Z_2, Z_1, \bar{D}\}, \{Z_7, Z_1, Z_2, Z_1, \bar{D}\} \).
(see diagram 23 of the figure 2);

24) \( \{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 24 of the figure 2);

25) \( \{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 25 of the figure 2);

26) \( \{Z_6, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 26 of the figure 2);

27) \( \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 27 of the figure 2);

28) \( \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \). (see diagram 28 of the figure 2);
29) \( \{Z_6, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D\} \), (see diagram 29 of the figure 2);

30) \( \{Z_7, Z_6, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, D\} \), (see diagram 30 of the figure 2);

**Proof:** It is easy to see that, the sets \( \{Z_1\}, \{Z_2\}, \{Z_3\}, \{Z_4\}, \{Z_5\}, \{Z_6\}, \{Z_7\}, \{D\} \) are subsemilattices of the semilattice \( D \).

The number all subsets of the semilattice \( D \), every set of which contains two elements, is equal to \( C_2^7 = 28 \). In this case \( X \)-subsemilattices of the semilattice \( D \) are the following sets:

\[
\{Z_1, Z_2\}, \{Z_1, Z_3\}, \{Z_1, Z_4\}, \{Z_1, Z_5\}, \{Z_1, Z_6\}, \{Z_1, Z_7\}, \{Z_2, Z_3\}, \{Z_2, Z_4\}, \{Z_2, Z_5\}, \{Z_2, Z_6\}, \{Z_2, Z_7\}, \{Z_3, Z_4\}, \{Z_3, Z_5\}, \{Z_3, Z_6\}, \{Z_3, Z_7\}, \{Z_4, Z_5\}, \{Z_4, Z_6\}, \{Z_4, Z_7\}, \{Z_5, Z_6\}, \{Z_5, Z_7\}, \{Z_6, Z_7\}, \{D\}
\]

Remainder 5 subsets of the semilattice \( D \), whose every element contains two elements is not an \( X \)-subsemilattice.

The number all subsets of the semilattice \( D \), every set of which contains three elements, is equal to \( C_3^7 = 56 \). In this case \( X \)-subsemilattices of the semilattice \( D \) are the following sets:

\[
\{Z_1, Z_2, Z_3\}, \{Z_1, Z_2, Z_4\}, \{Z_1, Z_2, Z_5\}, \{Z_1, Z_2, Z_6\}, \{Z_1, Z_2, Z_7\}, \{Z_1, Z_3, Z_4\}, \{Z_1, Z_3, Z_5\}, \{Z_1, Z_3, Z_6\}, \{Z_1, Z_3, Z_7\}, \{Z_1, Z_4, Z_5\}, \{Z_1, Z_4, Z_6\}, \{Z_1, Z_4, Z_7\}, \{Z_1, Z_5, Z_6\}, \{Z_1, Z_5, Z_7\}, \{Z_1, Z_6, Z_7\}, \{Z_2, Z_3, Z_4\}, \{Z_2, Z_3, Z_5\}, \{Z_2, Z_3, Z_6\}, \{Z_2, Z_3, Z_7\}, \{Z_2, Z_4, Z_5\}, \{Z_2, Z_4, Z_6\}, \{Z_2, Z_4, Z_7\}, \{Z_2, Z_5, Z_6\}, \{Z_2, Z_5, Z_7\}, \{Z_2, Z_6, Z_7\}, \{Z_3, Z_4, Z_5\}, \{Z_3, Z_4, Z_6\}, \{Z_3, Z_4, Z_7\}, \{Z_3, Z_5, Z_6\}, \{Z_3, Z_5, Z_7\}, \{Z_3, Z_6, Z_7\}, \{Z_4, Z_5, Z_6\}, \{Z_4, Z_5, Z_7\}, \{Z_4, Z_6, Z_7\}, \{Z_5, Z_6, Z_7\}, \{D\}
\]

Remainder 20 subsets of the semilattice \( D \), whose every element contains three elements is not an \( X \)-subsemilattice.

The number all subsets of the semilattice \( D \), every set of which contains four elements, is equal to \( C_4^7 = 70 \). In this case \( X \)-subsemilattices of the semilattice \( D \) are the following sets:

\[
\{Z_1, Z_2, Z_3, Z_4\}, \{Z_1, Z_2, Z_3, Z_5\}, \{Z_1, Z_2, Z_3, Z_6\}, \{Z_1, Z_2, Z_3, Z_7\}, \{Z_1, Z_2, Z_4, Z_5\}, \{Z_1, Z_2, Z_4, Z_6\}, \{Z_1, Z_2, Z_4, Z_7\}, \{Z_1, Z_2, Z_5, Z_6\}, \{Z_1, Z_2, Z_5, Z_7\}, \{Z_1, Z_2, Z_6, Z_7\}, \{Z_1, Z_3, Z_4, Z_5\}, \{Z_1, Z_3, Z_4, Z_6\}, \{Z_1, Z_3, Z_4, Z_7\}, \{Z_1, Z_3, Z_5, Z_6\}, \{Z_1, Z_3, Z_5, Z_7\}, \{Z_1, Z_3, Z_6, Z_7\}, \{Z_1, Z_4, Z_5, Z_6\}, \{Z_1, Z_4, Z_5, Z_7\}, \{Z_1, Z_4, Z_6, Z_7\}, \{Z_2, Z_3, Z_4, Z_5\}, \{Z_2, Z_3, Z_4, Z_6\}, \{Z_2, Z_3, Z_4, Z_7\}, \{Z_2, Z_3, Z_5, Z_6\}, \{Z_2, Z_3, Z_5, Z_7\}, \{Z_2, Z_3, Z_6, Z_7\}, \{Z_2, Z_4, Z_5, Z_6\}, \{Z_2, Z_4, Z_5, Z_7\}, \{Z_2, Z_4, Z_6, Z_7\}, \{Z_3, Z_4, Z_5, Z_6\}, \{Z_3, Z_4, Z_5, Z_7\}, \{Z_3, Z_4, Z_6, Z_7\}, \{Z_4, Z_5, Z_6, Z_7\}, \{D\}
\]

Remainder 33 subsets of the semilattice \( D \), whose every element contains four elements is not an \( X \)-subsemilattice.
The number all subsets of the semilattice $D$, every set of which contains five elements, is equal to $C^5_8 = 56$. In this case $X$-subsemilattices of the semilattice $D$ are the following sets:

$$
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{16}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{16}, Z_{17}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{16}, Z_{17}, Z_{18}\}.
$$

Remainder 29 subsets of the semilattice $D$, whose every element contains five elements is not an $X$-subsemilattice.

The number all subsets of the semilattice $D$, every set of which contains six elements, is equal to $C^6_8 = 28$. In this case $X$-subsemilattices of the semilattice $D$ are the following sets:

$$
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}\}.
$$

Remainder 13 subsets of the semilattice $D$, whose every element contains six elements is not an $X$-subsemilattice.

The number all subsets of the semilattice $D$, every set of which contains seven elements, is equal to $C^7_8 = 8$. In this case $X$-subsemilattices of the semilattice $D$ are the following sets:

$$
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}\}, \{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}\}, \\
\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}\}.
$$

Remainder 3 subsets of the semilattice $D$, whose every element contains seven elements is not an $X$-subsemilattice.

The number all subsets of the semilattice $D$, every set of which contains eight elements, is equal to $C^8_8 = 1$. This set is $\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z, D\}$.

The Lemma is proved.

From the proven lemma it follows that diagrams shown in fig. 2, exhaust all diagrams of subsemilattices of the semilattice $D$. 

\[\text{Giuli Tavadgiridze}\]
Lemma 2.2: Let $D \in \Sigma_3(x, 8)$ and $Z_7 \neq \emptyset$. Then any subsemilattices of the semilattice $D$ having diagram $17 - 30$ of the figure 2 are never XI–semilattice.

Proof: Remark, that the all subsemilattices of semilattice $D$ which has diagrams of form $17 - 30$ are never XI–semilattices. For example we consider the semilattice which has the diagram of the form 30 of the figure 3 (see diagram figure 30).

Let $D' = \{Z_1, Z_6, Z_4, Z_3, Z_2, Z_1, D\}$ and $C(D') = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6\}$ is a family sets, where

$P_0, P_1, P_2, P_3, P_4, P_5, P_6$ are pairwise disjoint subsets of the set $X$ and

$\varphi = \left( \begin{array}{cccccc} D & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \right)$ is a mapping of the semilattice $D'$ onto the family sets $C(D')$. Then for the formal equalities of the semilattice $D'$ we have a form:

\[
\begin{align*}
\bar{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\
Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\
Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_6 \cup P_7, \\
Z_4 &= P_0 \cup P_3 \cup P_6 \cup P_7, \\
Z_5 &= P_0 \cup P_3 \cup P_7 \\
Z_6 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\
Z_7 &= P_0
\end{align*}
\]

Here the elements $P_1, P_2, P_3, P_6$ are basis sources, the element $P_0, P_2, P_7$ is sources of completenes of the semilattice $D'$. Therefore $|X| \geq 3$ and, that
We have $D' = \{Z_i, Z_6, Z_7, Z_1\}$ and $\Lambda(D', D') \in D'$ for all $i \in \tilde{D}$. But element $Z_4$ is not union of some elements of the set $D'$. So, from the Definition 1.2 follows that semilattice $D'$ which has diagram 41 of the figure 3 never is $XI-$semilattice.

In the same manner it can be proved that any subsemilattice of the semilattice $D$ having diagrams 17-30 are never an $XI-$semilattice.

Lemma is proved.

**Lemma 2.3:** Let $D \in \Sigma_{(X,8)}$ and $Z_7 \neq \emptyset$. Then the following sets are all $XI$-subsemilattices of the given semilattice $D$:

1) $\{Z_1\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\tilde{D}\}$; (see diagram 1 of the figure 2);

2) $\{Z_7, Z_6\}, \{Z_7, Z_5\}, \{Z_7, Z_4\}, \{Z_7, Z_3\}, \{Z_7, Z_2\}, \{Z_7, Z_1\}, \{Z_7, \tilde{D}\}, \{Z_6, Z_4\}, \{Z_6, Z_2\}, \{Z_6, Z_1\}, \{Z_5, Z_4\}, \{Z_5, Z_3\}, \{Z_5, Z_2\}, \{Z_5, Z_1\}, \{Z_4, Z_2\}, \{Z_4, Z_1\}, \{Z_4, \tilde{D}\}, \{Z_3, Z_1\}, \{Z_3, \tilde{D}\}, \{Z_2, \tilde{D}\}, \{Z_1, \tilde{D}\}$; (see diagram 2 of the figure 2);

3) $\{Z_7, Z_6, Z_5\}, \{Z_7, Z_6, Z_4\}, \{Z_7, Z_6, Z_3\}, \{Z_7, Z_6, Z_2\}, \{Z_7, Z_6, Z_1\}, \{Z_7, Z_6, \tilde{D}\}, \{Z_7, Z_5, Z_4\}, \{Z_7, Z_5, Z_3\}, \{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, Z_1\}, \{Z_7, Z_5, \tilde{D}\}, \{Z_7, Z_4, Z_3\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \tilde{D}\}, \{Z_7, Z_3, Z_2\}, \{Z_7, Z_3, Z_1\}, \{Z_7, Z_3, \tilde{D}\}, \{Z_7, Z_2, Z_1\}, \{Z_7, Z_2, \tilde{D}\}, \{Z_7, Z_1, \tilde{D}\}, \{Z_7, \tilde{D}\}, \{Z_6, Z_4\}, \{Z_6, Z_3\}, \{Z_6, Z_2\}, \{Z_6, Z_1\}, \{Z_6, \tilde{D}\}, \{Z_5, Z_4\}, \{Z_5, Z_3\}, \{Z_5, Z_2\}, \{Z_5, Z_1\}, \{Z_5, \tilde{D}\}, \{Z_4, Z_3\}, \{Z_4, Z_2\}, \{Z_4, Z_1\}, \{Z_4, \tilde{D}\}, \{Z_3, Z_2\}, \{Z_3, Z_1\}, \{Z_3, \tilde{D}\}, \{Z_2, \tilde{D}\}, \{Z_1, \tilde{D}\}$; (see diagram 3 of the figure 2);

4) $\{Z_7, Z_6, Z_5, Z_4\}, \{Z_7, Z_6, Z_5, Z_3\}, \{Z_7, Z_6, Z_5, Z_2\}, \{Z_7, Z_6, Z_5, Z_1\}, \{Z_7, Z_6, Z_5, \tilde{D}\}, \{Z_7, Z_6, Z_4, Z_3\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, \tilde{D}\}, \{Z_7, Z_6, Z_3, Z_2\}, \{Z_7, Z_6, Z_3, Z_1\}, \{Z_7, Z_6, Z_3, \tilde{D}\}, \{Z_7, Z_6, Z_2, Z_1\}, \{Z_7, Z_6, Z_2, \tilde{D}\}, \{Z_7, Z_6, Z_1, \tilde{D}\}, \{Z_7, Z_6, \tilde{D}\}, \{Z_7, Z_5, Z_4\}, \{Z_7, Z_5, Z_3\}, \{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, Z_1\}, \{Z_7, Z_5, \tilde{D}\}, \{Z_7, Z_4, Z_3\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \tilde{D}\}, \{Z_7, Z_3, Z_2\}, \{Z_7, Z_3, Z_1\}, \{Z_7, Z_3, \tilde{D}\}, \{Z_7, Z_2, Z_1\}, \{Z_7, Z_2, \tilde{D}\}, \{Z_7, Z_1, \tilde{D}\}, \{Z_7, \tilde{D}\}$; (see diagram 4 of the figure 2);

5) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, \tilde{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, \tilde{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_1, \tilde{D}\}, \{Z_7, Z_6, Z_5, Z_4, \tilde{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1\}, \{Z_7, Z_5, Z_4, Z_3, Z_2, \tilde{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_1, \tilde{D}\}, \{Z_7, Z_5, Z_4, Z_3, \tilde{D}\}, \{Z_7, Z_5, Z_4, Z_2, Z_1\}, \{Z_7, Z_5, Z_4, Z_2, \tilde{D}\}, \{Z_7, Z_5, Z_4, Z_1, \tilde{D}\}, \{Z_7, Z_5, Z_4, \tilde{D}\}, \{Z_7, Z_5, Z_3, Z_2, Z_1\}, \{Z_7, Z_5, Z_3, Z_2, \tilde{D}\}, \{Z_7, Z_5, Z_3, Z_1, \tilde{D}\}, \{Z_7, Z_5, Z_3, \tilde{D}\}, \{Z_7, Z_5, Z_2, Z_1\}, \{Z_7, Z_5, Z_2, \tilde{D}\}, \{Z_7, Z_5, Z_1, \tilde{D}\}, \{Z_7, Z_5, \tilde{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1\}, \{Z_7, Z_4, Z_3, Z_2, \tilde{D}\}, \{Z_7, Z_4, Z_3, Z_1, \tilde{D}\}, \{Z_7, Z_4, Z_3, \tilde{D}\}, \{Z_7, Z_4, Z_2, Z_1\}, \{Z_7, Z_4, Z_2, \tilde{D}\}, \{Z_7, Z_4, Z_1, \tilde{D}\}, \{Z_7, Z_4, \tilde{D}\}, \{Z_7, Z_3, Z_2, Z_1\}, \{Z_7, Z_3, Z_2, \tilde{D}\}, \{Z_7, Z_3, Z_1, \tilde{D}\}, \{Z_7, Z_3, \tilde{D}\}, \{Z_7, \tilde{D}\}$; (see diagram 5 of the figure 2).
6) \( \{Z_1, Z_6, Z_2, Z_4, Z_5, Z_4\}, \{Z_7, Z_6, Z_3, Z_1\}, \{Z_7, Z_4, Z_2, Z_1\}, \{Z_7, Z_3, Z_2, Z_1\}, \{Z_7, Z_3, Z_1, D\}, \{Z_6, Z_5, Z_3, Z_1, D\} \)
(see diagram 6 of the figure 2);

7) \( \{Z_7, Z_4, Z_2, Z_1, D\}, \{Z_7, Z_5, Z_3, Z_1, D\}, \{Z_7, Z_5, Z_2, Z_1, D\}, \{Z_7, Z_5, Z_2, Z_1, D\}, \{Z_7, Z_5, Z_1, D\} \)
(see diagram 7 of the figure 2);

8) \( \{Z_7, Z_6, Z_4, Z_2, Z_1, D\}, \{Z_7, Z_5, Z_4, Z_2, Z_1, D\} \)
(see diagram 8 of the figure 2);

9) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 9 of the figure 2);

10) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\}, \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 10 of the figure 2);

11) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\}, \{Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 11 of the figure 2);

12) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\}, \{Z_1, Z_5, Z_4, Z_3, Z_1, D\}, \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 12 of the figure 2);

13) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 13 of the figure 2);

14) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 14 of the figure 2);

15) \( \{Z_1, Z_5, Z_4, Z_3, Z_1, D\} \)
(see diagram 15 of the figure 2);

16) \( \{Z_1, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \)
(see diagram 16 of the figure 2);

Now we will proof the statement 16). Let \( D = \{ Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D} \} \) where

\[
Z_2 \sqsubseteq Z_3 \sqsubseteq Z_4 \sqsubseteq D, \quad Z_7 \sqsubseteq Z_6 \sqsubseteq Z_2 \sqsubseteq \tilde{D}, \\
Z_2 \sqsubseteq Z_3 \sqsubseteq Z_4 \sqsubseteq D, \quad Z_7 \sqsubseteq Z_6 \sqsubseteq Z_4 \sqsubseteq \tilde{D}, \\
Z_2 \sqsubseteq Z_4 \sqsubseteq Z_1 \sqsubseteq D, \quad Z_5 \sqsubseteq Z_6 \# \emptyset, \quad Z_6 \sqsubseteq Z_5 \# \emptyset, \\
Z_4 \sqsubseteq Z_3 \# \emptyset, \quad Z_3 \sqsubseteq Z_4 \# \emptyset, Z_2 \sqsubseteq Z_3 \# \emptyset, \quad Z_1 \sqsubseteq Z_2 \# \emptyset, \\
Z_6 \sqsubseteq Z_5 = Z_4, \quad Z_4 \sqsubseteq Z_3 = Z_7, \quad Z_2 \sqsubseteq Z_1 = D \\
\]

Let \( C(D) = \{ P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7 \} \) is a family sets, where \( P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7 \) are pairwise disjoint subsets of the set \( X \) and \( \varphi = (D, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}) \) is a mapping of the semilattice \( D \) onto the family sets \( C(D) \). Then for the formal equalities of the semilattice \( D \) we have a form:

\[
\begin{align*}
\tilde{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
Z_1 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
Z_2 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
Z_3 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
Z_4 &= P_0 \cup P_1 \cup P_2 \cup P_6 \cup P_7, \\
Z_5 &= P_0 \cup P_1 \cup P_3 \cup P_6 \cup P_7, \\
Z_6 &= P_0 \cup P_1 \cup P_5 \cup P_7, \\
Z_7 &= P_0. \\
\end{align*}
\]

Of the formal equalities (2.5) follows, that

\[
D_1' = \begin{cases} 
D, & \text{if } t \in P_0, \\
\{ Z_2, \tilde{D} \}, & \text{if } t \in P_1, \\
\{ Z_3, Z_1, \tilde{D} \}, & \text{if } t \in P_2, \\
\{ Z_6, Z_4, Z_2, Z_1, \tilde{D} \}, & \text{if } t \in P_3, \\
\{ Z_5, Z_2, Z_1, \tilde{D} \}, & \text{if } t \in P_4, \\
\{ Z_6, Z_4, Z_2, Z_1, \tilde{D} \}, & \text{if } t \in P_5, \\
\{ Z_5, Z_4, Z_2, Z_1, \tilde{D} \}, & \text{if } t \in P_6, \\
\{ Z_6, Z_4, Z_2, Z_1, \tilde{D} \}, & \text{if } t \in P_7, \\
\end{cases} \quad \wedge (D, D_1) = \begin{cases} 
Z_7, & \text{if } t \in P_0, \\
Z_5, & \text{if } t \in P_1, \\
Z_5, & \text{if } t \in P_2, \\
Z_6, & \text{if } t \in P_3, \\
Z_5, & \text{if } t \in P_4, \\
Z_5, & \text{if } t \in P_5, \\
Z_7, & \text{if } t \in P_6, \\
Z_7, & \text{if } t \in P_7, \\
\end{cases}
\]

We have \( D_1 = \{ Z_6, Z_4, Z_2, Z_1, \tilde{D} \} \), \( \wedge (D, D_1) \in D \) for all t and \( Z_4 = Z_6 \cup Z_5, \ Z_5 = Z_6 \cup Z_7, \) \( \tilde{D} = Z_6 \cup Z_7 \). So, from the Definition 1.2 follows that semilattice \( D \) which has diagram of the figure 4 is \( XI \)– semilattice.
In the same manner it can be proved that any subsemilattice of the semilattice $D$ having diagrams 13 and 14 are an $XI$–semilattice.

Lemma is proved

**Corollary 2.1:** Let $D \in \Sigma_3(X,8)$ and $Z_7 = \emptyset$. Then the following sets are all $XI$–subsemilattices of the given semilattice $D$:

1) $\{\emptyset\}$ (see diagram 1 of the figure 2);

2) $\{\emptyset, Z_6\}, \{\emptyset, Z_5\}, \{\emptyset, Z_4\}, \{\emptyset, Z_3\}, \{\emptyset, Z_2\}, \{\emptyset, Z_1\}, \{\emptyset, D\}$

(see diagram 2 of the figure 2);

3) $\{\emptyset, Z_6, Z_4\}, \{\emptyset, Z_6, Z_3\}, \{\emptyset, Z_6, Z_1\}, \{\emptyset, Z_6, D\}, \{\emptyset, Z_5, Z_2\}, \{\emptyset, Z_5, Z_1\}, \{\emptyset, Z_5, D\}$

(see diagram 3 of the figure 2);

4) $\{\emptyset, Z_6, Z_4, Z_2\}, \{\emptyset, Z_6, Z_4, Z_1\}, \{\emptyset, Z_6, X, Z_2, D\}, \{\emptyset, Z_6, Z_1, D\}$

(see diagram 4 of the figure 2);

5) $\{\emptyset, Z_6, Z_4, Z_2, D\}, \{\emptyset, Z_6, Z_4, Z_1, D\}, \{\emptyset, Z_5, Z_4, Z_2, D\}, \{\emptyset, Z_5, Z_4, Z_1, D\}$

(see diagram 5 of the figure 2);

6) $\{\emptyset, Z_6, Z_4, Z_3, Z_1\}, \{\emptyset, Z_4, Z_3, Z_1\}, \{\emptyset, Z_3, Z_2, D\}, \{\emptyset, Z_3, Z_1, D\}$

(see diagram 6 of the figure 2);

7) $\{\emptyset, Z_6, Z_2, Z_1, D\}, \{\emptyset, Z_4, Z_2, Z_1, D\}, \{\emptyset, Z_4, Z_2, Z_1, D\}$

(see diagram 7 of the figure 2);

8) $\{\emptyset, Z_6, Z_4, Z_3, Z_1\}, \{\emptyset, Z_4, Z_3, Z_1\}$

(see diagram 8 of the figure 2);

9) $\{\emptyset, Z_5, Z_4, Z_3, Z_1\}$

(see diagram 9 of the figure 2);

10) $\{\emptyset, Z_6, Z_5, Z_4, Z_2\}, \{\emptyset, Z_6, Z_5, Z_4, Z_1\}, \{\emptyset, Z_6, Z_5, Z_4, D\}, \{\emptyset, Z_6, Z_5, Z_1, D\}$

(see diagram 10 of the figure 2);

11) $\{\emptyset, Z_6, Z_5, Z_4, Z_2, D\}, \{\emptyset, Z_6, Z_5, Z_4, Z_1, D\}$

(see diagram 11 of the figure 2);

12) $\{\emptyset, Z_6, Z_5, Z_3, Z_1\}, \{\emptyset, Z_6, Z_5, Z_3, Z_1, D\}, \{\emptyset, Z_4, Z_3, Z_2, Z_1, D\}$
(see diagram 12 of the figure 2);

13) \( \emptyset, Z_2, Z_3, Z_4, Z_5, D \); (see diagram 13 of the figure 2);

14) \( \emptyset, Z_6, Z_4, Z_3, Z_2, D \); (see diagram 14 of the figure 2);

15) \( \emptyset, Z_6, Z_5, Z_3, Z_4, D \); (see diagram 15 of the figure 2);

16) \( \emptyset, Z_6, Z_3, Z_4, Z_5, Z_2, D \); (see diagram 16 of the figure 2);

**Proof:** This corollary immediately follows from the lemmas 2.2.

The corollary is proved.

**Theorem 2.1:** Let \( D \in \Sigma(X, z) \), \( Z \neq \emptyset \) and \( \alpha \in B(X, D) \). Binary relation \( \alpha \) is an idempotent relation of the semigroup \( B(X, D) \) iff binary relation \( \alpha \) satisfies only one conditions of the following conditions:

1) \( \alpha = X \times T \), where \( T \in D \);

2) \( \alpha = (Y_a \times T) \cup (Y_b \times T') \), where \( T, T' \in D \), \( T \subset T' \), \( Y_a, Y_b \in \{ \emptyset \} \), and satisfies the conditions: \( Y_a \supseteq T \), \( Y_b \cap T' \neq \emptyset \);

3) \( \alpha = (Y_a \times T) \cup (Y_b \times T') \cup (Y_a \times T'') \), where \( T, T', T'' \in D \), \( T \subset T' \subset T'' \), \( Y_a, Y_b, Y_c \in \{ \emptyset \} \), and satisfies the conditions: \( Y_a \supseteq T \), \( Y_b \cup Y_c \supseteq T' \), \( Y_a \cup Y_b \cup Y_c \supseteq T'' \);

4) \( \alpha = (Y_a \times T) \cup (Y_b \times T') \cup (Y_a \times T'') \), where \( T, T', T'' \in D \), \( T \subset T' \subset T'' \), \( Y_a, Y_b \in \{ \emptyset \} \), and satisfies the conditions: \( Y_a \supseteq T \), \( Y_b \cup Y_c \supseteq T' \), \( Y_a \cup Y_b \cup Y_c \supseteq T'' \);

5) \( \alpha = \sum_{a} Y_a \times z_{a} \cup (Y_a \times T) \cup (Y_a \times T') \cup (Y_a \times T'') \cup D \), where \( Z \subset T \subset T' \subset T'' \subset D \), \( Y_a, Y_b, Y_c \in \{ \emptyset \} \), and satisfies the conditions: \( Y_a \supseteq Z \), \( Y_a \cup Y_b \cup Y_c \supseteq T' \), \( Y_a \cup Y_b \cup Y_c \supseteq T'' \), \( Y_a \supseteq T' \supseteq T'' \), \( Y_a \supseteq T' \supseteq T'' \);
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15) \( \alpha=\{Y^a \times Z\} \cup \{Y^b \times Z\} \cup \{Y^c \times Z\} \cup \{Y^d \times Z\} \cup \{Y^e \times Z\} \cup \{Y^f \times D\} \), where 
\( Y^a, Y^b, Y^c, Y^d, Y^e, Y^f \in \mathcal{O} \) and satisfies the conditions: 
\( Y^a \supseteq Z, Y^b \supseteq Z, Y^c \supseteq Z, 
Y^d \supseteq Z, Y^e \supseteq Z, Y^f \supseteq Z \), 
\( Y^a \cup Y^b \cup Y^c \cup Y^d \cup Y^e \cup Y^f \supseteq Z \), 
\( Y^a \cap Z \neq \emptyset, Y^b \cap Z \neq \emptyset, \)
\( Y^c \cap Z \neq \emptyset, Y^d \cap Z \neq \emptyset, \)
\( Y^e \cap Z \neq \emptyset; \)

16) \( \alpha=\{Y^a \times Z\} \cup \{Y^b \times Z\} \cup \{Y^c \times Z\} \cup \{Y^d \times Z\} \cup \{Y^e \times Z\} \cup \{Y^f \times D\} \), where,
\( Y^a, Y^b, Y^c, Y^d, Y^e, Y^f \in \mathcal{O} \) and satisfies the conditions: 
\( Y^a \supseteq Z, Y^b \supseteq Z, Y^c \supseteq Z, 
Y^d \supseteq Z, Y^e \supseteq Z, Y^f \supseteq Z \), 
\( Y^a \cup Y^b \cup Y^c \cup Y^d \cup Y^e \cup Y^f \supseteq Z \), 
\( Y^a \cap Z \neq \emptyset, Y^b \cap Z \neq \emptyset, \)
\( Y^c \cap Z \neq \emptyset, Y^d \cap Z \neq \emptyset, \)
\( Y^e \cap Z \neq \emptyset; \)


Now we will prove statement 16. \( D=\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \) to begin with, we note that \( D \) is an \( \mathcal{X}_i \) - semilattice of unions(see lemma 2.2) then it is easy to see, that the set \( D(\alpha)=\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6 \} \) is a generating set of the semilattice \( D \).

Then the following equalities are hold:

\( \bar{D}(\alpha)_{Z_7} = \{Z_7\}, \bar{D}(\alpha)_{Z_8} = \{Z_8, Z_9\}, \bar{D}(\alpha)_{Z_9} = Z_9, \bar{D}(\alpha)_{Z_1} = Z_1, \bar{D}(\alpha)_{Z_2} = Z_2, \bar{D}(\alpha)_{Z_3} = Z_3, \bar{D}(\alpha)_{Z_4} = Z_4, \bar{D}(\alpha)_{Z_5} = Z_5, \bar{D}(\alpha)_{Z_6} = Z_6, \bar{D}(\alpha)_{Z_7} = Z_7, \bar{D}(\alpha)_{Z_8} = Z_8, \bar{D}(\alpha)_{Z_9} = Z_9, \bar{D}(\alpha)_{Z_1} = Z_1, \bar{D}(\alpha)_{Z_2} = Z_2, \bar{D}(\alpha)_{Z_3} = Z_3, \bar{D}(\alpha)_{Z_4} = Z_4, \bar{D}(\alpha)_{Z_5} = Z_5, \bar{D}(\alpha)_{Z_6} = Z_6, \bar{D}(\alpha)_{Z_7} = Z_7, \bar{D}(\alpha)_{Z_8} = Z_8, \bar{D}(\alpha)_{Z_9} = Z_9. \)

By statement b) of the Theorem 1.3 follows that the following conditions are true:

\( Y^a \supseteq Z, Y^b \supseteq Z, Y^c \supseteq Z, Y^d \supseteq Z, Y^e \supseteq Z, 
Y^f \supseteq Z, Y^g \supseteq Z, Y^h \supseteq Z, Y^i \supseteq Z, 
Y^j \supseteq Z, 
Y^k \supseteq Z, Y^l \supseteq Z, Y^m \supseteq Z, \)

For last conditions we have:

\( Y^a \supseteq Z, Y^b \supseteq Z, Y^c \supseteq Z, Y^d \supseteq Z, 
Y^e \supseteq Z, Y^f \supseteq Z, Y^g \supseteq Z, Y^h \supseteq Z, 
Y^i \supseteq Z, Y^j \supseteq Z, Y^k \supseteq Z, Y^l \supseteq Z, Y^m \supseteq Z. \)

Since \( \varphi \) is isomorphism. Further, it is to see, that the following equality is true:

\( l(\bar{D}_2, Z_a) = \cup (\bar{D}_2 \setminus Z_a) = Z_\alpha, Z_a \setminus l(\bar{D}_2, Z_a) = Z_a \setminus Z_\alpha \neq \emptyset; \)
\( l(\bar{D}_2, Z_b) = \cup (\bar{D}_2 \setminus Z_b) = Z_\beta, Z_b \setminus l(\bar{D}_2, Z_b) = Z_b \setminus Z_\beta \neq \emptyset; \)
\( l(\bar{D}_2, Z_c) = \cup (\bar{D}_2 \setminus Z_c) = Z_\gamma, Z_c \setminus l(\bar{D}_2, Z_c) = Z_c \setminus Z_\gamma \neq \emptyset; \)
\( l(\bar{D}_2, Z_d) = \cup (\bar{D}_2 \setminus Z_d) = Z_\delta, Z_d \setminus l(\bar{D}_2, Z_d) = Z_d \setminus Z_\delta \neq \emptyset; \)
Maximal Subgroups of the Semigroup \( B_X(D) \)

We have the elements \( Z_\alpha, Z_\beta, Z_\gamma \) and \( Z_\delta \) are nonlimiting elements of the sets \( \bar{B}(\alpha)_\alpha, \bar{B}(\alpha)_\beta, \bar{B}(\alpha)_\gamma \) and \( \bar{B}(\alpha)_\delta \) respectively. By statement of the Theorem 1.1.3 it follows, that the conditions \( Y_\alpha \cap Z_\alpha \neq \emptyset, Y_\beta \cap Z_\beta \neq \emptyset \) and \( Y_\gamma \cap Z_\gamma \neq \emptyset \) hold.

Therefore the following conditions are hold:

\[
\begin{align*}
Y_\alpha \supseteq Z_\alpha, & \quad Y_\beta \supseteq Z_\beta, & \quad Y_\gamma \supseteq Z_\gamma, \\
Y_\alpha \cap Z_\alpha \neq \emptyset, & \quad Y_\beta \cap Z_\beta \neq \emptyset, & \quad Y_\gamma \cap Z_\gamma \neq \emptyset, \\
Y_\beta \cap Z_\beta \neq \emptyset, & \quad Y_\gamma \cap Z_\gamma \neq \emptyset, & \quad Y_\delta \cap Z_\delta \neq \emptyset. \\
\end{align*}
\]

In the same manner it can be proved that any subsemilattice of the semilattice \( D \) having diagrams 13 and 14.

Theorem is proved.

**Corollary 2.2:** Let \( D \in \Sigma \times (X, 8), \ Z_\gamma = \emptyset \) and \( \alpha \in B_X(D) \). Binary relation \( \alpha \) is an idempotent relation of the semigroup \( B_X(D) \) iff binary relation \( \alpha \) satisfies only one conditions of the following conditions:

1) \( \alpha = \emptyset; \)

2) \( \alpha = (Y_\alpha \times \emptyset) \cup (Y_\alpha \times T), \) where \( \emptyset \neq T \in D, \ Y_\alpha \not\in \emptyset, \) and satisfies the conditions: \( Y_\alpha \supseteq \emptyset, \ Y_\alpha \cap T \neq \emptyset; \)

3) \( \alpha = (Y_\beta \times \emptyset) \cup (Y_\beta \times T) \cup (Y_\gamma \times T'), \) where \( \emptyset \neq T \subset T' \subset D, \ Y_\beta, Y_\gamma \not\in \emptyset, \) and satisfies the conditions: \( Y_\beta \not\subseteq \emptyset, \ Y_\beta \cup Y_\gamma \not\subseteq T, \ Y_\beta \cap T \neq \emptyset; \)

4) \( \alpha = (Y_\alpha \times \emptyset) \cup (Y_\alpha \times T) \cup (Y_\alpha \times T') \cup (Y_\gamma \times T'), \) where \( \emptyset \neq T \subset T' \subset D, \ Y_\alpha, Y_\beta, Y_\gamma \not\in \emptyset, \) and satisfies the conditions: \( Y_\alpha \not\subseteq \emptyset, \ Y_\alpha \cup Y_\beta \not\subseteq T, \ Y_\alpha \cup Y_\gamma \not\subseteq T', \ Y_\alpha \cap T \neq \emptyset; \)

5) \( \alpha = (Y_\beta \times \emptyset) \cup (Y_\beta \times T) \cup (Y_\gamma \times T') \cup (Y_\gamma \times T'), \) where \( \emptyset \neq T \subset T' \subset \bar{D}, \ Y_\beta, Y_\gamma \not\in \emptyset, \) and satisfies the conditions: \( Y_\beta \not\subseteq \emptyset, \ Y_\beta \cup Y_\gamma \not\subseteq T, \ Y_\beta \cup Y_\gamma \not\subseteq T', \ Y_\beta \cap T \neq \emptyset; \)

6) \( \alpha = (Y_\alpha \times \emptyset) \cup (Y_\alpha \times T) \cup (Y_\alpha \times T') \cup (Y_\alpha \times T \cup T'), \) where
$\emptyset \neq T, T' \in D$, $T \cap T' \neq \emptyset$, $Y_6^a, Y_7^a \in \{\emptyset\}$ and satisfies the conditions: $Y_6^a \cup Y_7^a \supseteq T$, $Y_6^a \cup Y_7^a \supseteq T'$, $Y_6^a \cap T \neq \emptyset$, $Y_7^a \cap T' \neq \emptyset$;

7) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times T \} \cup \{Y_6^a \times T' \} \cup \{Y_7^a \times (T \cup T')\}$, where, $\emptyset \neq T \subset T'$, $\emptyset \neq T \subset T'$, $T \cap T' \neq \emptyset$, $T \cap T' \neq \emptyset$, $Y_6^a, Y_7^a \in \emptyset$ and satisfies the conditions: $Y_6^a \supseteq \emptyset$, $Y_6^a \cup Y_7^a \supseteq T$, $Y_6^a \cup Y_7^a \supseteq T'$, $Y_6^a \cap T \neq \emptyset$, $Y_7^a \cap T' \neq \emptyset$, $Y_6^a \cap T' \neq \emptyset$.

8) $\alpha=\{Y_6^a \times Z_1\} \cup \{Y_6^a \times T\} \cup \{Y_6^a \times Z_2\} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times D\}$, where

$T \in \{Z_6, Z_7\}$, $Y_6^a, Y_7^a, Y_8^a, Y_9^a \in \emptyset$ and satisfies the conditions: $Y_6^a \supseteq \emptyset$, $Y_6^a \cup Y_7^a \supseteq T$, $Y_6^a \cup Y_7^a \supseteq T'$, $Y_6^a \cup Y_7^a \supseteq T$, $Y_8^a \cup Y_9^a \cup Y_7^a \supseteq Z_1$, $Y_8^a \cup Y_9^a \cup Y_7^a \supseteq Z_1$, $Y_8^a \cup Y_9^a \cup Y_7^a \neq Z_1$, $Y_6^a \cup T \neq \emptyset$, $Y_6^a \cap Z_4 \neq \emptyset$, $Y_6^a \cap Z_5 \neq \emptyset$.

9) $\alpha=\{Y_6^a \times Z_1\} \cup \{Y_6^a \times Z_2\} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times D\}$, where

$\emptyset \neq Z_4 \subset Z_4$, $\emptyset \neq Z_5 \subset Z_5$, $Z_5 \cap Z_5 \neq \emptyset$, $Z_5 \cap Z_5 \neq \emptyset$, $Y_6^a, Y_7^a, Y_8^a, Y_9^a \in \emptyset$ and satisfies the conditions: $Y_6^a \supseteq \emptyset$, $Y_6^a \cup Y_7^a \supseteq Z_5$, $Y_6^a \cup Y_7^a \supseteq Z_5$, $Y_6^a \cup Y_7^a \supseteq Z_5$, $Y_6^a \cap Z_6 \neq \emptyset$, $Y_6^a \cap Z_7 \neq \emptyset$, $Y_6^a \cap D \neq \emptyset$.

10) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times T\} \cup \{Y_6^a \times T'\} \cup \{Y_7^a \times (T \cup T')\} \cup \{Y_7^a \times T\}$, where

$\emptyset \neq T, T', T \cap T' \neq \emptyset$, $T \cap T' \subset T'$, $Y_6^a, Y_7^a \in \emptyset$ and satisfies the conditions: $Y_6^a \cup Y_7^a \cup T$, $Y_6^a \cup Y_7^a \cup T'$, $Y_6^a \cup T \neq \emptyset$, $Y_6^a \cup T' \neq \emptyset$.

11) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times Z_2\} \cup \{Y_6^a \times T\} \cup \{Y_7^a \times \emptyset\}$, where

$T \in \{Z_6, Z_7\}$, $Y_6^a, Y_7^a, Y_8^a, Y_9^a \in \emptyset$ and satisfies the conditions: $Y_7^a \cup Y_8^a \cup Z_4$, $Y_6^a \cup Y_7^a \supseteq Z_4$, $Y_6^a \cup Y_7^a \supseteq Z_4$, $Y_6^a \cup Y_7^a \supseteq Z_4$, $Y_6^a \cap Z_6 \neq \emptyset$, $Y_6^a \cap Z_7 \neq \emptyset$, $Y_6^a \cap D \neq \emptyset$.

12) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times T\} \cup \{Y_6^a \times T'\} \cup \{Y_7^a \times (T \cup T')\} \cup \{Y_7^a \times T\} \cup \{Y_7^a \times (T \cup T')\}$, where

$\emptyset \neq T, T', T \cap T' \neq \emptyset$, $T \cap T' \subset T'$, $(T \cup T') \neq \emptyset$, $T \cap T' \neq \emptyset$, $Y_6^a, Y_7^a \in \emptyset$ and satisfies the conditions: $Y_6^a \cup Y_7^a \cup T$, $Y_6^a \cup Y_7^a \cup T'$, $Y_6^a \cup Y_7^a \cup T'$, $Y_6^a \cap T' \neq \emptyset$, $Y_6^a \cup T \neq \emptyset$, $Y_6^a \cap T' \neq \emptyset$.

13) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times Z_2\} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times D\}$, where

$Z_5 \subset Z_5$, $Z_5 \subset Z_5$, $Z_5 \cap Z_5 \neq \emptyset$, $Z_5 \subset Z_5$, $Z_5 \cap Z_5 \neq \emptyset$, $Y_6^a, Y_7^a, Y_8^a, Y_9^a \in \emptyset$ and satisfies the conditions: $Y_6^a \supseteq \emptyset$, $Y_6^a \cup Y_7^a \supseteq Z_5$, $Y_6^a \cup Y_7^a \supseteq Z_5$, $Y_6^a \cup Y_7^a \supseteq Z_7$, $Y_6^a \cup Y_7^a \supseteq Z_7$, $Y_6^a \cap Z_5 \neq \emptyset$, $Y_6^a \cap Z_7 \neq \emptyset$, $Y_7^a \cap Z_6 \neq \emptyset$.

14) $\alpha=\{Y_6^a \times \emptyset \} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times Z_2\} \cup \{Y_6^a \times Z_1\} \cup \{Y_6^a \times D\}$, where,
Proof: This corollary immediately follows from the theorem 2.1.

The corollary is proved.

Lemma 3.1: The number of automorphisms of those semilattices, which are defined by the diagrams 1), 2), 3), 4), 5), 12), 13), 14) and 16) in fig. 2 is equal to 1, those semilattices which are defined diagrams 6), 7), 8), 9), 10) and 11) in fig. 2 is equal to 2 and that semilattice which is defined by the diagram 15) in fig. 2 is equal to 4.

Proof: Let us prove the given lemma in case of the semilattice which is defined by the diagram 16) in fig. 2. The proofs of the other cases are almost identical of the ongoing one.

Fig. 5

Suppose \( Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\} \) (see fig. 5). Our purpose is to prove that the number of automorphisms of the given semilattice \( Q \) is equal to 1. Indeed, if \( T_i(n, m) \) denote the element \( T_i \) of the semilattice \( Q \) such that \( n = |Q_i|, m = |Q_i| \) (see
(1.1)) and \( \varphi \) is arbitrary automorphism of the semilattice \( Q \), then \( \varphi(T_i) = T_j \) only if \( n_i = n_j \) and \( m_i = m_j \), i.e. \( (n_i, m_i) = (n_j, m_j) \).

For the semilattice \( Q \) we have:
\[
\begin{align*}
T_0 &= T_0(1,8), \quad T_1 = T_1(2,6), \quad T_2 = T_2(2,5), \quad T_3 = T_3(3,3), \\
T_4 &= T_4(4,4), \quad T_5 = T_5(6,2), \quad T_6 = T_6(5,2), \quad T_7 = T_7(8,1).
\end{align*}
\]

So, we have \( \varphi(T_i) = T_i \) for every \( i = 0, 1, 2, \ldots, 6 \), because \( (n_i, m_i) = (n_j, m_j) \) equality is satisfied if and only if \( i = j \).

Therefore the number of automorphisms of the given semilattice \( Q \) is equal to 1.

Lemma is proved.

Denote by the symbol \( G_x(D,\varepsilon) \) a maximal subgroup of the semigroup \( B_x(D) \) whose unit is an idempotent binary relation \( \varepsilon \) of the semigroup \( B_x(D) \).

**Theorem 3.1:** For any idempotent binary relation \( \varepsilon \) of the semigroup \( B_x(D) \), the order a subgroup \( G_x(D,\varepsilon) \) of the semigroup \( B_x(D) \) is one, or two, or four.

**Proof:** Let \( \varepsilon \) be an arbitrary binary relation of the semigroup \( B_x(D) \). Now, if we denote by \( \Phi \) the group of all complete automorphisms of the semilattice \( V(D,\varepsilon) \), then by virtue of Theorem 1.3. We have that the groups \( G_x(D,\varepsilon) \) and \( \Phi \) are anti-isomorphic.

To prove the theorem, we will consider the following cases with regard to the idempotent binary relation \( \varepsilon \):

1) The idempotent binary relation \( \varepsilon \) satisfies the conditions 1), 2), 3), 4), 5), 12), 13), 14) and 16) of the Theorem 2.1 and theorem 2.2., then the diagram of the semilattice \( V(D,\varepsilon) \) has form 1, 2, 3, 4, 5, 12, 13, 14 and 16 in Fig. 2. Therefore in this case the number of automorphisms of the semilattice \( V(D,\varepsilon) \) is equal to one (see Lemma 3.1).

Now, taking into account Theorem 1.3, we obtain \( \|G_x(D,\varepsilon)\| = 1 \).

2) The idempotent binary relation \( \varepsilon \) satisfies the conditions 6), 7), 8), 9), 10) and 11) of the Theorem 2.1 and theorem 2.2. So that the diagram of the semilattice \( V(D,\varepsilon) \) has form 6-11 in Fig. 2. Therefore in this case the number of automorphisms of the semilattice \( V(D,\varepsilon) \) is equal to two (see
Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $|G_x(D,\varepsilon)|=2$.

3) The idempotent binary relation is of type 15) so that the number of automorphisms of the semilattice $V(D,\varepsilon)$ has form 15 in Fig. 2. Clearly, in this case the number of automorphisms of the semilattice $V(D,\varepsilon)$ is four (see Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $|G_x(D,\varepsilon)|=4$.

Since the diagrams shown in Fig. 2 exhaust all the diagrams of the $XI$–subsemilattices of the semilattice $D$, the idempotent binary relations of the semigroup $B_x(D)$ are exhausted by types 1-16 from Theorem 2.1 and Theorem 2.2. Hence it follows that for any idempotent $\varepsilon$ of the semigroup $B_x(D)$, the order a subgroup $G_x(D,\varepsilon)$ of the semigroup $B_x(D)$ is one, or two, or four.

References

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