A Note on Fuzzy Almost Resolvable Spaces

G. Thangaraj¹ and D. Vijayan²

¹Department of Mathematics, Thiruvalluvar University
Vellore- 632 115, Tamilnadu, India
E-mail: g.thangaraj@rediffmail.com
²Department of Mathematics
Muthurangam Government Arts College (Autonomous)
Vellore- 632002, Tamilnadu, India
E-mail: jyoshijeev@gmail.com

(Received: 12-9-14 / Accepted: 10-11-14)

Abstract

In this paper we study the conditions under which a fuzzy topological space becomes a fuzzy almost resolvable space and the inter-relations between fuzzy almost resolvable, fuzzy almost irresolvable spaces, fuzzy submaximal spaces, fuzzy first category spaces, fuzzy Baire spaces, fuzzy weakly Volterra spaces are also investigated.

Keywords: Fuzzy almost resolvable, fuzzy almost irresolvable, fuzzy submaximal, fuzzy first category, fuzzy second category, fuzzy weakly Volterra, fuzzy Baire.

1 Introduction

In order to deal with uncertainties, the idea of fuzzy sets fuzzy set operations was introduced by L.A. Zadeh [16] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. Among the first fields of Mathematics
to be considered in the content of fuzzy sets was general topology. The concept of fuzzy topology was defined by C.L. Chang [3] in the year 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. E. Hewitt [6] introduced the concepts of resolvability and irresolvability in topological spaces. A.G. El’kin [4] introduced open hereditarily irresolvable spaces in the classical topology. The concept of almost resolvable spaces was introduced by Richard Bolstein [7] as a generalization of resolvable spaces of E. Hewitt [6]. The concept of almost resolvable spaces in fuzzy setting was introduced and studied by G. Thangaraj and D. Vijayan [15]. In this paper several characterizations of fuzzy almost resolvable spaces are studied and the inter-relations between fuzzy almost resolvable, fuzzy almost irresolvable spaces, fuzzy submaximal spaces, fuzzy first category spaces, fuzzy Baire spaces, fuzzy weakly Volterra spaces are also investigated.

2 Preliminaries

By a fuzzy topological space we shall mean a non-empty set $X$ together with a fuzzy topology $T$ (in the sense of Chang) and denote it by $(X,T)$.

**Definition 2.1:** Let $\lambda$ and $\mu$ be any two fuzzy sets in $(X,T)$. Then we define $\lambda \lor \mu : X \rightarrow [0,1]$ as follows: $(\lambda \lor \mu) (x) = \max \{\lambda (x), \mu (x)\}$. Also we define $\lambda \land \mu : X \rightarrow [0,1]$ as follows: $(\lambda \land \mu) (x) = \min \{\lambda (x), \mu (x)\}$.

For a family $\{\lambda_i \mid i \in I\}$ of fuzzy sets in $(X,T)$, the union $= \lor \{\lambda_i \mid i \in I\}$ and the intersection $\cap = \land \{\lambda_i \mid i \in I\}$ are defined respectively as $\psi(x) = \sup_i \{\lambda_i(x) \mid x \in X\}$ and $\delta(x) = \inf_i \{\lambda_i(x) \mid x \in X\}$.

**Definition 2.2:** Let $(X,T)$ be a fuzzy topological space and $\lambda$ be any fuzzy set in $(X,T)$. We define the closure and the interior of $\lambda$ as follows:

(i) $\text{Int} (\lambda) = \lor \{\mu \mid \mu \leq \lambda , \mu \in T\}$,
(ii) $\text{Cl} (\lambda) = \land \{\lambda \leq \mu , 1-\mu \in T\}$.

**Lemma 2.1[1]:** For a fuzzy set $\lambda$ of a fuzzy topological space $X$,

(i) $1- \text{Int} (\lambda) = \text{Cl} (1-\lambda)$,
(ii) $1- \text{Cl} (\lambda) = \text{Int} (1-\lambda)$.

**Definition 2.3 [8]:** A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy dense if there exists no fuzzy closed set $\mu$ in $(X,T)$ such that $\lambda < \mu < 1$. 

Definition 2.4 [8]: A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy nowhere dense if there exists no non-zero fuzzy open set $(\mu)\subseteq T$ such that $\mu \subseteq \text{cl}(\lambda)$. That is, $\text{int}\text{cl}(\lambda) = 0$.

Definition 2.5 [2]: A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called a fuzzy $F_\sigma$-set in $(X,T)$ if $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$, where $(1 - \lambda_i) \in T$ for $i \in I$.

Definition 2.6 [2]: A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called a fuzzy $G_\delta$-set in $(X,T)$ if $\lambda = \bigwedge_{i=1}^{\infty} (\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.7 [16]: A fuzzy topological space $(X,T)$ is called an fuzzy open hereditarily irresolvable space if $\text{int}\text{cl}(\lambda) \neq 0$, then $\text{int}(\lambda) \neq 0$, for any non-zero fuzzy set $\lambda$ in $(X,T)$.

Definition 2.8 [2]: A fuzzy topological space $(X,T)$ is called a fuzzy submaximal space if $\text{cl}(\lambda) = 1$ for any non-zero fuzzy set $\lambda$ in $(X,T)$, then $\mathcal{A} \subseteq T$.

Definition 2.9 [9]: A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy first category if $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$, where $\lambda_i$'s are fuzzy nowhere dense sets in $(X,T)$. Any other fuzzy set in $(X,T)$ is said to be of second category.

Definition 2.10 [8]: A fuzzy topological space $(X,T)$ is called fuzzy first category if $\bigvee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets $\lambda_i$'s are fuzzy nowhere dense sets in $(X,T)$. A topological space which is not of fuzzy first category, is said to be of fuzzy second category.

Lemma 2.2 [1]: For a family of $\mathcal{A} = \{\lambda_a\}$ of fuzzy sets of a fuzzy topological space $(X,T)$, $\vee \text{cl}(\lambda_a) \leq \text{cl}(\vee(\lambda_a))$. In case $\mathcal{A}$ is a finite set, $\vee \text{cl}(\lambda_a) = \text{cl}(\vee(\lambda_a))$. Also $\vee\text{int}(\lambda_a) \leq \text{int}(\vee(\lambda_a))$.

Definition 2.11 [12]: Let $(X,T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X,T)$ is called a fuzzy $\sigma$-nowhere dense set, if $\lambda$ is a fuzzy $F_\sigma$-set in $(X,T)$ such that $\text{int}(\lambda) = 0$.

Definition 2.12 [12]: A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called a fuzzy $\sigma$–first category set if $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$, where the fuzzy sets $\lambda_i$'s are fuzzy $\sigma$ – nowhere dense sets in $(X,T)$. Any other fuzzy set in $(X,T)$ is said to be of fuzzy $\sigma$ – second category.

3 Fuzzy Almost Resolvable Spaces

Definition 3.1 [15]: A fuzzy topological space $(X,T)$ is called a fuzzy almost resolvable space if $\bigvee_{i=1}^{\infty} (\lambda_i) = 1$, where the fuzzy sets $\lambda_i$'s in $(X,T)$ are such that $\text{int}(\lambda_i) = 0$. Otherwise $(X, T)$ is called a fuzzy almost irresolvable space.
Proposition 3.1: If \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where the fuzzy sets \((\mu_i)\)'s are fuzzy dense sets in a fuzzy topological space \((X,T)\), then \((X,T)\) is a fuzzy almost resolvable space.

Proof: Suppose that \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where \( \text{cl}(\mu_1) = 1 \text{in}(X,T) \). Then we have \( 1 - (\bigwedge_{i=1}^{\infty} (\mu_i)) = 1 - 0 = 1 \), where \( 1 - \text{cl}(\mu_i) = 0 \). This implies that \( \bigvee_{i=1}^{\infty} (1 - \mu_i) = 1 \), where \( \text{int}(1 - \mu_i) = 0 \). Let \( 1 - \mu_i = \lambda_i \). Then, we have \( \bigvee_{i=1}^{\infty} (\lambda_i) = 1 \), where \( \text{int}(\lambda_i) = 0 \), in \((X,T)\). Hence \((X,T)\) is a fuzzy almost resolvable space.

Definition 3.2 [5]: A fuzzy topological space \((X,T)\) is called a fuzzy hyper connected space if every fuzzy open set is fuzzy dense in \((X,T)\). That is, \( \text{cl} (\mu_i) = 1 \), for all \( \mu_i \in T \).

Proposition 3.2: If \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where the fuzzy sets \((\mu_i)\)'s are fuzzy open sets in a fuzzy hyper connected space \((X,T)\), then \((X,T)\) is a fuzzy almost resolvable space.

Proof: Suppose that \((\bigwedge_{i=1}^{\infty} (\mu_i)) = 0 \), where \( \mu_i \in T \). Since the fuzzy topological space \((X,T)\) is a fuzzy hyper connected space, the fuzzy open set \( \mu_i \) is a fuzzy dense set in \((X,T)\) for each \( i \). Hence we have \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where \( \text{cl}(\mu_1) = 1 \) in \((X,T)\). Then by proposition 3.1, \((X,T)\) is a fuzzy almost resolvable space.

Proposition 3.3: If \( \bigvee_{i=1}^{\infty} (\lambda_i) = 1 \), where the fuzzy sets \((\lambda_i)\)'s are fuzzy \( \sigma \)-nowhere dense sets in a fuzzy topological space \((X,T)\), then \((X,T)\) is a fuzzy almost resolvable space.

Proof: Let \( (\lambda_i) \)'s \( (i = 1 \text{ to } \infty) \) be fuzzy \( \sigma \)-nowhere dense sets in \((X,T)\). Then \( (\lambda_i)\)'s are fuzzy \( F_\sigma \)-sets with \( \text{int}(\lambda_i) = 0 \). Now \( \bigvee_{i=1}^{\infty} (\lambda_i) = 1 \), where \( \text{int}(\lambda_i) = 0 \), implies that \((X,T)\) is a fuzzy almost resolvable space.

Definition 3.3 [14]: A fuzzy topological space \((X,T)\) is called fuzzy P-space, if countable intersection of fuzzy open sets in \((X,T)\) is fuzzy open. That is, every non-zero fuzzy \( G_\delta \)-set in \((X,T)\), is fuzzy open in \((X,T)\).

Proposition 3.4: If \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where the fuzzy sets \((\mu_i)\)'s are fuzzy \( G_\delta \)-sets in a fuzzy hyper connected and P-space, then \((X,T)\) is a fuzzy almost resolvable space.

Proof: Let \( (\mu_i) \)'s \( (i = 1 \text{ to } \infty) \) be fuzzy \( G_\delta \)-sets in the fuzzy P-space \((X,T)\). Then \( (\mu_i) \)'s are fuzzy open sets in \((X,T)\). Hence, we have \( \bigwedge_{i=1}^{\infty} (\mu_i) = 0 \), where the fuzzy sets \((\mu_i)\)'s are fuzzy open sets in a fuzzy hyper connected space \((X,T)\). Therefore, by proposition 3.2, \((X,T)\) is a fuzzy almost resolvable space.
**Proposition 3.5:** In a fuzzy almost resolvable space \((X,T)\), if \((\lambda_i)’s\) are fuzzy \(F_\sigma\)–sets, then \((X,T)\) is a fuzzy \(\sigma\)-first category space.

**Proof:** Let \((X,T)\) be a fuzzy almost resolvable space. Then we have \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where \(\text{int}(\lambda_i) = 0\). Since \((\lambda_i)’s\) are fuzzy \(F_\sigma\)-sets in \((X,T)\) and \(\text{int}(\lambda_i) = 0\), we have \((X,T)\) is fuzzy \(\sigma\)-first category space.

**Definition 3.4 [11]:** A fuzzy topological space \((X, T)\) is called a fuzzy nodec space, if every non-zero fuzzy nowhere dense set in \((X,T)\), is a fuzzy closed set in \((X,T)\).

**Proposition 3.6:** If \((X,T)\) is a fuzzy first category space and fuzzy nodec space, then \((X,T)\) is fuzzy almost resolvable space.

**Proof:** Let \((X,T)\) be a fuzzy first category space. Then we have \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where the fuzzy sets \((\lambda_i)’s\) are fuzzy nowhere dense sets in \((X,T)\). Since \((X,T)\) is a fuzzy nodec space, the fuzzy nowhere dense sets are fuzzy closed sets in \((X,T)\). Hence \((\lambda_i)’s\) are fuzzy closed sets in \((X,T)\). That is, \(\text{cl}(\lambda_i) = \lambda_i\). Now \(\text{intcl}(\lambda_i) = 0\), implies that \(\text{int}(\lambda_i) = 0\). Hence we have \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where the fuzzy sets \((\lambda_i)’s\) in \((X,T)\) are such that \(\text{int}(\lambda_i) = 0\). Hence \((X,T)\) is a fuzzy almost resolvable space.

**Proposition 3.7:** If the fuzzy topological space \((X,T)\) is a fuzzy second category space, then \((X,T)\) is a fuzzy almost irresolvable space.

**Proof:** Let \((X,T)\) be a fuzzy second category space. Then \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), where the fuzzy sets \((\lambda_i)’s\) are fuzzy nowhere dense sets in \((X,T)\). That is, \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), where \(\text{int cl}(\lambda_i) = 0\). Now \(\text{int cl}(\lambda_i) \leq \text{int cl}(\lambda_i)\), implies that \(\text{int cl}(\lambda_i) = 0\). Hence \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), where \(\text{int cl}(\lambda_i) = 0\) and therefore \((X,T)\) is a fuzzy almost irresolvable space.

**Definition 3.5 [13]:** A fuzzy topological space \((X,T)\) is called a fuzzy Volterra space if \(\text{cl}(\bigwedge_{i=1}^{N} (\lambda_i)) = 1\), where \((\lambda_i)’s\) are fuzzy dense and fuzzy \(G_\delta\)-sets in \((X,T)\).

**Definition 3.6 [13]:** A fuzzy topological space \((X,T)\) is called a fuzzy weakly Volterra space if \(\text{cl}(\bigwedge_{i=1}^{N} (\lambda_i)) \neq 0\), where \((\lambda_i)’s\) are fuzzy dense and fuzzy \(G_\delta\)-sets in \((X,T)\).

**Proposition 3.8:** If a fuzzy topological space \((X,T)\) is not a fuzzy weakly Volterra space, then \((X,T)\) is a fuzzy almost resolvable space.

**Proof:** Let \((X,T)\) be a fuzzy non-weakly Volterra space. Then, we have \(\text{cl}(\bigwedge_{i=1}^{N} (\lambda_i)) = 0\), where \((\lambda_i)’s\) are fuzzy dense and fuzzy \(G_\delta\)-sets in \((X,T)\).
Now $cI(\Lambda_{i=1}^{N} (\lambda_{i})) = 0$, implies that $\text{int}(V_{i=1}^{N} (1-\lambda_{i}) = 1$ and $\text{cl}(\lambda_{i}) = 1$, implies that $\text{int}(1-\lambda_{i}) = 0$. Let $(\mu_{j})$’s $(j = 1$ to $\infty)$ be fuzzy sets in $(X, T)$ such that $\text{int}(\mu_{j}) = 0$ and take the first $N$ $(\mu_{j})$’s as $(1-\lambda_{i})$’s. Now $V_{i=1}^{N} (1-\lambda_{i}) \subseteq V_{i=1}^{\infty} (\mu_{i})$, implies that $\text{int}(V_{i=1}^{N} (1-\lambda_{i})) \leq \text{int}(V_{i=1}^{\infty} (\mu_{i})) \leq V_{i=1}^{\infty} (\mu_{i}).$ Then, we have $1 \leq (V_{i=1}^{\infty} (\mu_{i})).$ That is, $V_{i=1}^{\infty} (\mu_{i})) = 1$, where the fuzzy sets $(\mu_{i})$’s in $(X, T)$ are such that $\text{int}((\mu_{i}))= 0$. Hence the fuzzy topological space $(X, T)$ is a fuzzy almost resolvable space.

4 Inter-Relations between Fuzzy almost Resolvable, Fuzzy almost Irresolvable Spaces and Fuzzy First Category, Fuzzy Second Category Spaces, Fuzzy Baire Spaces

**Proposition 4.1:** If the fuzzy almost resolvable space $(X, T)$ is a fuzzy submaximal space, then $(X, T)$ is a fuzzy first category space.

**Proof:** Let $(X, T)$ be a fuzzy almost resolvable space. Then $V_{i=1}^{\infty} (\lambda_{i}) = 1$, where the fuzzy sets $(\lambda_{i})$’s in $(X, T)$ are such that $\text{int}(\lambda_{i}) = 0$. Then we have $\Lambda_{i=1}^{\infty} (1-\lambda_{i}) = 0$, where $\text{cl}(1-\lambda_{i}) = 1$. Since the space $(X, T)$ is a fuzzy submaximal space, the fuzzy dense sets $(1-\lambda_{i})$’s are fuzzy open sets in $(X, T)$. Then $(\lambda_{i})$’s are fuzzy closed sets in $(X, T)$ and hence $\text{cl}(\lambda_{i}) = \lambda_{i}$. Now int $\text{cl}(\lambda_{i}) = \text{int}(\lambda_{i}) = 0$. Then $(\lambda_{i})$’s are fuzzy nowhere dense sets in $(X, T)$. Hence $V_{i=1}^{\infty} (\lambda_{i}) = 1$, where the fuzzy sets $(\lambda_{i})$’s are fuzzy nowhere dense sets in $(X, T)$ implies that $(X, T)$ is a fuzzy first category space.

**Remark:** In view of the above proposition, we have the following result. “If the fuzzy almost resolvable space $(X, T)$ is a fuzzy submaximal space, then $(X, T)$ is not a fuzzy second category space”.

**Proposition 4.2:** If the fuzzy almost irresolvable space $(X, T)$ is a fuzzy submaximal space, then $(X, T)$ is a fuzzy second category space.

**Proof:** Let $(X, T)$ be a fuzzy almost irresolvable space. Then $V_{i=1}^{\infty} (\lambda_{i}) \neq 1$, where the fuzzy sets $(\lambda_{i})$’s are such that $\text{int}(\lambda_{i}) = 0$. Now int $(\lambda_{i}) = 0$, implies that $\text{cl}(1-\lambda_{i}) = 1$. That is, $(1-\lambda_{i})$’s are fuzzy dense sets in $(X, T)$. Since $(X, T)$ is a fuzzy submaximal space, the fuzzy dense sets $(1-\lambda_{i})$’s are fuzzy open sets in $(X, T)$. Then $(\lambda_{i})$’s are fuzzy closed sets in $(X, T)$. That is, $\text{cl}(\lambda_{i}) = \lambda_{i}$. Now $\text{int}(\lambda_{i}) = 0$, implies that $\text{int} \text{cl}(\lambda_{i}) = 0$. Then $(\lambda_{i})$’s are fuzzy nowhere dense sets in $(X, T)$. Hence we have $V_{i=1}^{\infty} (\lambda_{i}) \neq 1$, where the fuzzy sets $(\lambda_{i})$’s are fuzzy nowhere dense sets in $(X, T)$. Therefore $(X, T)$ is a fuzzy second category space.

**Definition 4.1 [10]:** A fuzzy topological space $(X, T)$ is called a fuzzy Baire space if $\text{int}[V_{i=1}^{\infty} (\lambda_{i})] = 0$, where $(\lambda_{i})$’s are fuzzy nowhere dense sets in $(X, T)$. 


Proposition 4.3: If the fuzzy almost resolvable space \((X, T)\) is a fuzzy submaximal space, then \((X, T)\) is not a fuzzy Baire space.

**Proof:** Let the fuzzy almost resolvable space \((X, T)\) be a fuzzy submaximal space. Then, by proposition 4.1, \((X, T)\) is a fuzzy first category space and hence \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where the fuzzy sets \((\lambda_i)\)'s are fuzzy nowhere dense sets in \((X, T)\). Now \(\text{int} \left[ \bigvee_{i=1}^{\infty} (\lambda_i) \right] = \text{int} [1] = 1 \neq 0\). Hence \((X, T)\) is not a fuzzy Baire space.

Theorem 4.1 [9]: If the fuzzy topological space \((X, T)\) is a fuzzy open hereditarily irresolvable space, then \(\text{int}(\lambda) = 0\) for any non-zero fuzzy dense set \(\lambda\) in \((X, T)\) implies that \(\text{int} \text{cl}(\lambda) = 0\).

Proposition 4.4: If the fuzzy almost resolvable space \((X, T)\) is a fuzzy open hereditarily irresolvable space, then \((X, T)\) is not a fuzzy Baire space.

**Proof:** Let \((X, T)\) be a fuzzy almost resolvable space. Then, \(\bigvee_{i=1}^{\infty} (\lambda_i) = 1\), where the fuzzy sets \((\lambda_i)\)'s in \((X, T)\) are such that \(\text{int} (\lambda_i) = 0\). Since \((X, T)\) is a fuzzy open hereditarily irresolvable space, \(\text{int} (\lambda_i) = 0\), implies that \(\text{intcl} (\lambda_i) = 0\). Now \(\text{int} \left[ \bigvee_{i=1}^{\infty} (\lambda_i) \right] = \text{int} [1] = 1 \neq 0\). Hence \((X, T)\) is not a fuzzy Baire space.

Proposition 4.5: If the fuzzy almost irresolvable space \((X, T)\) is a fuzzy open hereditarily irresolvable space, then \((X, T)\) is a fuzzy second category space.

**Proof:** \((X, T)\) is fuzzy almost irresolvable space. Then \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), where the fuzzy sets \((\lambda_i)\)'s in \((X, T)\) are such that \(\text{int} (\lambda_i) = 0\). Since \((X, T)\) is a fuzzy open hereditarily irresolvable space, \(\text{int} (\lambda_i) = 0\), implies that \(\text{intcl} (\lambda_i) = 0\). Then \((\lambda_i)\)'s are fuzzy nowhere dense sets in \((X, T)\). Hence \(\bigvee_{i=1}^{\infty} (\lambda_i) \neq 1\), where the fuzzy sets \((\lambda_i)\)'s fuzzy nowhere dense sets in \((X, T)\), implies that \((X, T)\) is a fuzzy second category space.

References


