Intuitionistic Fuzzy $\omega$-Extremally Disconnected Spaces

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Abstract

In this paper, a new class of intuitionistic fuzzy topological spaces called intuitionistic fuzzy $\omega$ extremally disconnected spaces is introduced and several other properties are discussed.

Keywords: Intuitionistic fuzzy $\omega$ extremally disconnected spaces, lower (resp.upper) intuitionistic fuzzy $\omega$ continuous functions.

1 Introduction

After the introduction of the concept of fuzzy sets by Zadeh [13], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of “Intuitionistic fuzzy sets” was first published by Atanassov [2] and many works by the same author and his colleagues appeared in the literature [3-5]. Later this concept was generalized to “Intuitionistic L-fuzzy sets” by Atanassov and stoeva [6]. An introduction to intuitionistic fuzzy topological space was
introduced by Dogan Coker [8]. Several types of fuzzy connectedness in intuitionistic fuzzy topological spaces defined by Coker (1997). The construction is based on the idea of intuitionistic fuzzy set developed by Atanassov (1983, 1986; Atanassov and Stoeva, 1983). The concept of fuzzy extremally disconnected spaces was studied in [7]. In this paper a new class of intuitionistic fuzzy topological spaces namely, intuitionistic fuzzy \( \omega \) extremally disconnected spaces is introduced by using the notions introduced in [7, 11, 12]. The concept of fuzzy \( \omega \)-open set was studied in [10]. Tietze extension theorem for intuitionistic fuzzy \( \omega \)-extremally disconnected spaces has been discussed as in [1]. Some interesting properties and characterizations are studied.

2 Preliminaries

Definition 2.1[4]: Let \( X \) be a non empty fixed set. An intuitionistic fuzzy set (IFS for short) \( A \) is an object having the form \( A = \left\{ (x, \mu_A(x), \gamma_A(x)) : x \in X \right\} \) where the functions \( \mu_A : X \to I \) and \( \gamma_A : X \to I \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of non membership (namely \( \gamma_A(x) \)) of each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \).

Remark 2.1[8]: For the sake of simplicity, we shall use the symbol \( A = \langle x, \mu_A, \gamma_A \rangle \).

Definition 2.2[4]: Let \( X \) be a non empty set and the IFSs \( A \) and \( B \) be in the form \( A = \left\{ (x, \mu_A(x), \gamma_A(x)) : x \in X \right\} \), \( B = \left\{ (x, \mu_B(x), \gamma_B(x)) : x \in X \right\} \). Then

\( (a) \) \( A \subseteq B \) iff \( \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \);

\( (b) \) \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \);

\( (c) \) \( \overline{A} = \left\{ (x, \gamma_A(x), \mu_A(x)) : x \in X \right\} \);

\( (d) \) \( A \cap B = \left\{ (x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) : x \in X \right\} \);

\( (e) \) \( A \cup B = \left\{ (x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x)) : x \in X \right\} \);

\( (f) \) \( \mathcal{A} = \left\{ (x, \mu_A(x), 1-\mu_A(x)) : x \in X \right\} \);

\( (g) \) \( \check{\mathcal{A}} = \left\{ (x, 1-\gamma_A(x), \gamma_A(x)) : x \in X \right\} \).

Definition 2.3[8]: Let \( X \) be a non empty set and let \( \{ A_i : i \in I \} \) be an arbitrary family of IFSs in \( X \). Then

\( (a) \) \( \cap \mathcal{A} = \left\{ (x, \land \mu_A(x), \lor \gamma_A(x)) : x \in X \right\} \);

\( (b) \) \( \cup \mathcal{A} = \left\{ (x, \lor \mu_A(x), \land \gamma_A(x)) : x \in X \right\} \).
Definition 2.4\cite{citeauthor}: Let $X$ be a non empty fixed set. Then, $0_\omega = \{ (x, 0, 1) : x \in X \}$ and $1_\omega = \{ (x, 1, 0) : x \in X \}$.

Definition 2.5\cite{citeauthor}: Let $X$ and $Y$ be two non empty fixed sets and $f : X \rightarrow Y$ be a function. Then

(a) If $B = \{ (y, \mu_B(y), \gamma_B(y)) : y \in Y \}$ is an IFS in $Y$, then the pre image of $B$ under $f$, denoted by $f^{-1}(B)$, is the IFS in $X$ defined by $f^{-1}(B) = \{ (x, f^{-1}(-\mu_B(x), f^{-1}(\gamma_B(x))) : x \in X \}$.

(b) If $A = \{ (x, \lambda_A(x), \nu_A(x)) : x \in X \}$ is an IFS in $X$, then the image of $A$ under $f$, denoted by $f(A)$, is the IFS in $Y$ defined by $f(A) = \{ (y, f(\lambda_A(y)), (1-f(1-\nu_A))(y)) : y \in Y \}$ where,

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(1-f(1-\nu_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

for the IFS $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$.

Definition 2.6\cite{citeauthor}: Let $X$ be a non empty set. An intuitionistic fuzzy topology (IFT for short) on a non empty set $X$ is a family $\tau$ of intuitionistic fuzzy sets (IFSs for short) in $X$ satisfying the following axioms: $(T_1)$ $0, 1 \in \tau$, $(T_2)$ $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

$$(T_3) \bigcup_{i \in J} G_i \in \tau \text{ for any arbitrary family } \{G_i : i \in J\}.$$ In this case the pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS for short) in $X$.

Definition 2.7\cite{citeauthor}: Let $X$ be a non empty set. The complement $\overline{A}$ of an IFOS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS for short) in $X$. 

Definition 2.8: Let \((X, \tau)\) be an IFTS and \(A=\langle x, \mu_A, \gamma_A \rangle\) be an IFS in \(X\). Then the fuzzy interior and fuzzy closure of \(A\) are defined by

\[
\text{cl}(A) = \bigcap \{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\},
\]

\[
\text{int}(A) = \bigcup \{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.
\]

Remark 2.2: Let \((X, \tau)\) be an IFTS. \(\text{cl}(A)\) is an IFCS and \(\text{int}(A)\) is an IFOS in \(X\), and (a) \(A\) is an IFCS in \(X\) iff \(\text{cl}(A) = A\); (b) \(A\) is an IFOS in \(X\) iff \(\text{int}(A) = A\).

Proposition 2.1: Let \((X, \tau)\) be an IFTS. For any IFS \(A\) in \((X, \tau)\), we have

(a) \(\text{cl}(\overline{A}) = \overline{\text{int}(A)}\), (b) \(\text{int}(\overline{A}) = \overline{\text{cl}(A)}\).

Definition 2.9: Let \((X, \tau)\) and \((Y, \phi)\) be two IFTSs and let \(f : X \to Y\) be a function. Then \(f\) is said to be fuzzy continuous iff the pre image of each IFS in \(\phi\) is an IFS in \(\tau\).

Definition 2.10: Let \((X, \tau)\) and \((Y, \phi)\) be two IFTSs and let \(f : X \to Y\) be a function. Then \(f\) is said to be fuzzy open(resp. closed) iff the image of each IFS in \(\tau\) (resp. \((1-\tau)\)) is an IFS in \(\phi\) (resp. \((1-\phi)\)).

Definition 2.11: A subset \(A\) of an IFTS \((X, \tau)\) is called an IF semi-open set if \(\text{IFcl}(A), \text{IFint}(A)\) and \(\overline{A}\) denote an intuitionistic fuzzy closure of \(A\), an intuitionistic fuzzy interior of \(A\) and the complement of \(A\) in \(X\) respectively.

Definition 2.12: A subset of a topological space \((X, T)\) is called \(\omega\)-closed in \((X, T)\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \((X, T)\). A subset \(A\) is called \(\omega\)-open if \(\text{IFcl}(A)\) is \(\omega\)-closed.

An IFTS \((X, T)\) represent intuitionistic fuzzy topological spaces and for a subset \(A\) of a space \((X, T)\), \(\text{IFcl}(A)\), \(\text{IFint}(A)\) and \(\overline{A}\) denote an intuitionistic fuzzy closure of \(A\), an intuitionistic fuzzy interior of \(A\) and the complement of \(A\) in \(X\) respectively.

Notation 2.1: Let \(X\) be any non-empty set and \(A \in \zeta^X\). Then for \(x \in X\), \(\{\mu_A(x), \gamma_A(x)\}\) is denoted by \(A^\sim\).

Definition 2.13: An intuitionistic fuzzy real line \(\mathbb{R}_f(I)\) is the set of all monotone decreasing intuitionistic fuzzy set \(A \in \xi^R\) satisfying \[
\{A(t) : t \in R\} = 1^\sim\]
and \[
\{A(t) : t \in R\} = 0^\sim\] after the identification of an intuitionistic fuzzy sets.
A, B ∈ ℝ_2(I) if and only if A(t−) = B(t−) and A(t+) = B(t+) for all t ∈ ℝ
where A(t−) = ∩ \{ A(s) : s < t \} and A(t+) = ∪ \{ A(s) : s > t \}.

The intuitionistic fuzzy unit interval ℝ_2(I) is a subset of ℝ_2(I) such that [A] ∈ ℝ_2(I) if the membership and non-membership of A are defined by

\[ \mu_A(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases} \quad \text{and} \quad \gamma_A(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases} \]

respectively.

The intuitionistic fuzzy topology on ℝ_2(I) is generated from the subbasis

\{ L_t^I, R_t^I, t ∈ ℝ \}

where L_t^I, R_t^I : ℝ_2(I) → ℝ_2(I) are given by L_t^I(A) = A(t−) and R_t^I(A) = A(t+) respectively.

**Definition 2.14[1]:** Let (X, T) be an intuitionistic fuzzy topological space. The characteristic function of intuitionistic fuzzy set A in X is the function Ψ_A : X → ℝ_2(I) defined by Ψ_A(x) = A^x, for each x ∈ X.

**Notation 2.2[1]:** Let (X, T) be intuitionistic fuzzy topological space and let A ⊂ X. Then an intuitionistic fuzzy ω_c is of the form \( \langle x, χ_A(x), 1 - χ_A(x) \rangle \).

## 3 Intuitionistic Fuzzy ω-Extremally Disconnected Spaces

In this section, the concept of ω extremally disconnectedness in intuitionistic fuzzy topological space is introduced besides proving several other propositions.

**Definition 3.1:** A subset A of an IFTS (X, T) is called intuitionistic fuzzy ω closed(IF ω closed for short) if IFcl(A) ⊆ U whenever A ⊆ U and U is IF semi-open in (X, T).

**Definition 3.2:** A subset A of an IFTS (X, T) is called intuitionistic fuzzy ω open (IF ω open for short) if A is IF ω closed.

**Definition 3.3:** Let (X, T) be an intuitionistic fuzzy topological space and A be an intuitionistic fuzzy set in X. Then the intuitionistic fuzzy ω closure of A (IFω cl(A) for short) and intuitionistic fuzzy ω interior of A (IFω int(A) for short) are defined by

IFω cl(A) = ∩ \{ K : K is an intuitionistic fuzzy ω closed set in X and A ⊆ K \},
\( \text{IF} \omega \text{ int}(A) = \bigcup \{ G : G \text{ is an intuitionistic fuzzy } \omega \text{ open set in } X \text{ and } G \subseteq A \} \).

**Definition 3.4:** A function \( f : (X,T) \to (Y,S) \) is called intuitionistic fuzzy \( \omega \) continuous if \( f^{-1}(V) \) is an intuitionistic fuzzy \( \omega \) closed set of \( (X,T) \) for every intuitionistic fuzzy closed set \( V \) of \( (Y,S) \).

**Proposition 3.1:** For any intuitionistic fuzzy set \( A \) of an intuitionistic fuzzy topological space \( (X,T) \), the following statements hold:

(a) \( \text{IF} \omega \text{ cl}(A) = \text{IF} \omega \text{ int}(A) \)
(b) \( \text{IF} \omega \text{ int}(A) = \text{IF} \omega \text{ cl}(A) \)

**Definition 3.5:** Let \( (X,T) \) be an intuitionistic fuzzy topological space. Let \( A \) be any intuitionistic fuzzy \( \omega \) open set in \( (X,T) \). If an intuitionistic fuzzy \( \omega \) closure of \( A \) is intuitionistic fuzzy \( \omega \) open, then \( (X,T) \) is said to be an intuitionistic fuzzy \( \omega \) extremally disconnected space.

**Proposition 3.2:** For an intuitionistic fuzzy topological space \( (X,T) \) the following statements are equivalent:

(a) \( (X,T) \) is intuitionistic fuzzy \( \omega \) extremally disconnected.
(b) For each intuitionistic fuzzy \( \omega \) closed set \( A \), \( \text{IF} \omega \text{ int}(A) \) is intuitionistic fuzzy \( \omega \) closed.
(c) For each intuitionistic fuzzy \( \omega \) open set \( A \),
\( \text{IF} \omega \text{ cl}(\text{IF} \omega \text{ int}(A)) = \text{IF} \omega \text{ cl}(A) \)
(d) For each pair of intuitionistic fuzzy \( \omega \) open sets \( A \) and \( B \) in \( (X,T) \) with
\( \text{IF} \omega \text{ cl}(A) = B \), \( \text{IF} \omega \text{ cl}(B) = \text{IF} \omega \text{ cl}(A) \).

**Proposition 3.3:** Let \( (X,T) \) be an intuitionistic fuzzy topological space. Then \( (X,T) \) is an intuitionistic fuzzy \( \omega \) extremally disconnected space if and only if for any intuitionistic fuzzy \( \omega \) open set \( A \) and intuitionistic fuzzy \( \omega \) closed set \( B \) such that \( A \subseteq B \), \( \text{IF} \omega \text{ cl}(A) \subseteq \text{IF} \omega \text{ int}(B) \).

**Notation 3.1:** An intuitionistic fuzzy set which is both intuitionistic fuzzy \( \omega \) open set and intuitionistic fuzzy \( \omega \) closed set is called intuitionistic fuzzy \( \omega \) clopen set.

**Remark 3.1:** Let \( (X,T) \) be an intuitionistic fuzzy \( \omega \) extremally disconnected space. Let \( \{ A_i, B_i / i \in N \} \) be a collection such that \( A_i \)'s are intuitionistic fuzzy \( \omega \) open sets, \( B_i \)'s are intuitionistic fuzzy \( \omega \) closed sets and let \( A, B \) be intuitionistic fuzzy \( \omega \) clopen sets respectively. If \( A_i \subseteq A \subseteq B_j \) and \( A_i \subseteq B \subseteq B_j \) for all \( i, j \in N \) then there exists an intuitionistic fuzzy \( \omega \) clopen set \( C \) such that \( \text{IF} \omega \text{ cl}(A_i) \subseteq C \subseteq \text{IF} \omega \text{ int}(B_j) \) for all \( i, j \in N \).

**Proposition 3.4:** Let \( (X,T) \) be an intuitionistic fuzzy \( \omega \) extremally disconnected space. Let \( \left( A_q \right)_{q \in Q} \) and \( \left( B_q \right)_{q \in Q} \) be the monotone increasing collections of
intuitionistic fuzzy $\omega$ open sets and intuitionistic fuzzy $\omega$ closed sets of $(X, T)$ respectively and suppose that $A_{q_1} \subseteq B_{q_2}$ whenever $q_1 < q_2$ ($Q$ is the set of rational numbers). Then there exists a monotone increasing collection $\{C_q\}_{q \in Q}$ of intuitionistic fuzzy $\omega$ clopen sets of $(X, T)$ such that $1F\omega \text{cl}(A_{q_1}) \subseteq C_{q_2}$ and $C_{q_2} \subseteq 1F\omega \text{int}(B_{q_2})$ whenever $q_1 < q_2$.

4 Properties and Characterizations of Intuitionistic Fuzzy $\omega$ Extremally Disconnected Spaces

In this section, various properties and characterizations of intuitionistic fuzzy $\omega$ extremally disconnected spaces are discussed.

**Definition 4.1:** Let $(X, T)$ be an intuitionistic fuzzy topological space. A function $f : X \to R_1(I)$ is called lower (resp., upper) intuitionistic fuzzy $\omega$ continuous, if $f^{-1}(R_1^f)$ (resp., $f^{-1}(I_1^f)$) is an intuitionistic fuzzy $\omega$ open set (resp., intuitionistic fuzzy $\omega$ clopen) for each $t \in R$.

**Lemma 4.1:** Let $(X, T)$ be an intuitionistic fuzzy topological space. Let $A \in \xi^X$ and let $f : X \to R_1(I)$ be such that

$$f(x)(t) = \begin{cases} 1^- & \text{if } t < 0, \\ A^- & \text{if } 0 \leq t \leq 1, \\ 0^- & \text{if } t > 1, \end{cases}$$

for all $x \in X$ and $t \in R$. Then $f$ is lower (resp., upper) intuitionistic fuzzy $\omega$ continuous iff $A$ is intuitionistic fuzzy $\omega$ open (resp., intuitionistic fuzzy $\omega$ clopen) set.

**Proposition 4.1:** Let $(X, T)$ be an intuitionistic fuzzy topological space and let $A \in \xi^X$. Then $\psi_A$ is lower (resp., upper) intuitionistic fuzzy $\omega$ continuous iff $A$ is intuitionistic fuzzy $\omega$ open (resp., intuitionistic fuzzy $\omega$ clopen).

**Definition 4.2:** Let $(X, T)$ and $(Y, S)$ be two intuitionistic fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called intuitionistic fuzzy strongly $\omega$ continuous if $f^{-1}(A)$ is intuitionistic fuzzy $\omega$ clopen in $(X, T)$ for every intuitionistic fuzzy $\omega$ open set in $(Y, S)$.

**Proposition 4.2:** Let $(X, T)$ be an intuitionistic fuzzy topological space. Then the following statements are equivalent:

(a) $(X, T)$ is intuitionistic fuzzy $\omega$ extremally disconnected,
(b) If $g, h : X \to R_1(I)$, $g$ is lower intuitionistic fuzzy $\omega$ continuous, $h$ is upper intuitionistic fuzzy $\omega$ continuous and $g \leq h$, then there exists an
intuitionistic fuzzy strongly $\omega$ continuous function, $f: (X,T) \rightarrow R_f(I)$ such that $g \leq f \leq h$.

(c) If $A$ and $B$ are intuitionistic fuzzy $\omega$ open sets such that $B \subseteq A$ then there exists an intuitionistic fuzzy strongly $\omega$ continuous function $f: (X,T, \leq) \rightarrow I_f(I)$ such that $B \subseteq L_f f \subseteq R_f f \subseteq A$.

5 Tietze Extension Theorem for Intuitionistic Fuzzy $\omega$ Extremally Disconnected Spaces

In this section, Tietze extension theorem for intuitionistic fuzzy $\omega$ extremally disconnected space is studied.

**Proposition 5.1:** Let $(X,T)$ be an upper intuitionistic fuzzy $\omega$ extremally disconnected space and let $A \subseteq X$ be such that $\chi_A^*$ is an intuitionistic fuzzy $\omega$ open set in $(X,T)$. Let $f: (A,T/A) \rightarrow I_f(I)$ be an intuitionistic fuzzy strongly $\omega$ continuous function. Then $f$ has an intuitionistic fuzzy strongly $\omega$ continuous extension over $(X,T)$.

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**References**


