On Some Integral Inequalities Analogs to Hilbert's Inequality

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Abstract

In this paper we give some further extensions of well-known Hilbert's inequality. We give equivalent form in two dimensions as application.

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1 Introduction

The well-known Hilbert's inequality and its equivalent form are presented first:

**Theorem A:** [4] If \(f\) and \(g\) \(\in L^2[0, \infty)\), then the following inequalities hold and are equivalent

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy \leq \pi \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2},
\]

(1)

and

\[
\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y} \, dx \right)^2 \, dy \leq \pi^2 \int_0^\infty f^2(x) \, dx,
\]

(2)
where $\pi$ and $\pi^2$ are the best possible constants.

The classical Hilbert's integral inequality (1) had been generalized by Hardy-Riesz (see [2]) in 1925 as the following result.

If $f, g$ are nonnegative functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and

$$0 < \int_0^\infty g^q(x)dx < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \csc\left(\frac{\pi}{p}\right) \left(\int_0^\infty f^p(x) \, dx\right)^{1/p} \left(\int_0^\infty g^q(y) \, dy\right)^{1/q},$$

(3)

where the constant factor $\pi \csc(\pi/p)$ is the best possible. When $p=q=2$, inequality (3) is reduced to (1).

In recent years, a number of mathematicians had given lots of generalizations of these inequalities. We mention here some of these contributions in this direction:

Li et al. [5] have proved the following Hardy-Hilbert's type inequality using the hypotheses of (1):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max(x,y)} \, dx \, dy < c \left(\int_0^\infty f^2(x) \, dx\right)^{1/2} \left(\int_0^\infty g^2(y) \, dy\right)^{1/2},$$

(4)

Where the constant factor $c=\sqrt{2}(\pi - 2 - tan^{-1}\sqrt{2}) = 1.7408\ldots$ is the best possible.

Y. Li, Y. Qian, and B. He [6] deduced the following result:

**Theorem B:** If $f, g \geq 0, 0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then one has

$$\int_0^\infty \int_0^\infty \frac{|nx-ny|}{x+y+|x-y|} \, f(x)g(y) \, dx \, dy < 4 \left(\int_0^\infty f^2(x) \, dx\right)^{1/2} \left(\int_0^\infty g^2(x) \, dx\right)^{1/2},$$

(5)

where the constant factors 4 is the best possible.

More and more results regarding this direction on Hilbert's type inequalities can be found for example in [3, 7, 8].
2 Main Results

In this paper, we give some analogs of Hilbert’s type inequality. We will use the following lemma in establishing the main result.

**Lemma 2.1:** [1] Let \( y, \alpha, \beta \) be three non-negative real numbers. Then we have the following equations

\[
\int_0^\infty \frac{|\ln x - \ln y|^r}{ax + \beta y + |x - y|} \left( \frac{x}{y} \right)^{1/2} \, dy = \int_0^\infty \frac{|\ln x - \ln y|^r}{ax + \beta y + |x - y|} \left( \frac{x}{y} \right)^{1/2} \, dx
\]

\[
= \int_0^1 \frac{2^{r+1} |\ln t|^r}{(\alpha + 1) + t^2 (\beta - 1)} \, dt + \int_0^1 \frac{2^{r+1} |\ln t|^r}{t^2 (\alpha - 1) + (\beta + 1)} \, dt = A,
\]

where \( A := A(y, \alpha, \beta) \in [0, \infty] \).

Another result stated in the following theorem [1] is under consideration.

**Theorem 2.1:** If \( f, g \) are real functions such that \( 0 < \int_0^\infty f^2(x) \, dx < \infty \), \( 0 < \int_0^\infty g^2(x) \, dx < \infty \), then we have

\[
\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^r}{ax + \beta y + |x - y|} f(x) g(y) \, dx \leq A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2},
\]

where \( A \) is defined in Lemma 2.1 and is the best possible.

In the following theorem, we introduce an equivalent form to inequality (6).

**Theorem 2.2:** Suppose \( f \geq 0 \) and \( 0 < \int_0^\infty f^2(x) \, dx < \infty \), then

\[
\int_0^\infty \left[ \int_0^\infty \frac{|\ln x - \ln y|^r}{ax + \beta y + |x - y|} f(x) \, dx \right]^2 \, dy \leq A^2 \int_0^\infty f^2(x) \, dx,
\]

where \( A \) is defined in Lemma 2.1. Furthermore, Inequality (7) is equivalent to (6).

**Proof:** Let

\[
I = \int_0^\infty \left[ \int_0^\infty \frac{|\ln x - \ln y|^r}{ax + \beta y + |x - y|} f(x) \, dx \right]^2 \, dy.
\]
Setting $x = yz$, $dx = ydz$, then we get

$$I = \int_0^\infty \left[ \int_0^\infty \frac{|\ln z|^\nu}{az + \beta + |z - 1|} f(yz) \, dz \right]^2 \, dy.$$  

By Minkowski's inequality for integrals,

$$I \leq \left( \int_0^\infty \left[ \int_0^\infty \frac{|\ln z|^\nu}{az + \beta + |z - 1|} f(yz) \, dy \right]^2 \, dz \right)^{1/2}.$$  

Thus Inequality (7) holds.

Now, to prove that Inequality (7) is equivalent to (6): Suppose that Inequality (6) holds, and let

$$g(y) = \int_0^\infty \frac{|\ln x - \ln y|^\nu}{ax + \beta y + |x - y|} f(x) \, dx.$$  

Hence

$$0 < \int_0^\infty g^2(y) \, dy = \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^\nu}{ax + \beta y + |x - y|} f(x) \, dx \right) g(y) \, dy.$$  

By Fubini’s Theorem and Inequalities (6),

$$\int_0^\infty g^2(y) \, dy = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\nu}{ax + \beta y + |x - y|} f(x) \, g(y) \, dx \, dy \leq A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}.$$
Notice that by Inequality (7), \( g \in L^2 \). So the last integral is finite, and hence
\[
\left( \int_0^\infty g^2(y) \, dy \right)^{1/2} \leq A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2}.
\]

Thus
\[
\int_0^\infty \left[ \int_0^\infty \frac{|\ln x - \ln y|^r}{\alpha x + \beta y + |x - y|} f(x) \, dx \right]^2 \, dy \leq A^2 \int_0^\infty f^2(x) \, dx.
\]

Conversely, if Inequality (7) holds, then
\[
\int_0^\infty \left[ \int_0^\infty \frac{|\ln x - \ln y|^r}{\alpha x + \beta y + |x - y|} f(x) g(y) \, dx \, dy \right]
= \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^r}{\alpha x + \beta y + |x - y|} f(x) \, dx \right) g(y) \, dy.
\]

By Cauchy-Schwarz inequality we get
\[
\int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^r}{\alpha x + \beta y + |x - y|} f(x) \, dx \right) g(y) \, dy
\leq \left( \int_0^\infty \left[ \int_0^\infty \frac{|\ln x - \ln y|^r}{\alpha x + \beta y + |x - y|} f(x) \, dx \right]^2 \, dy \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}
\leq A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}.
\]

**Lemma 2.2:** [2] Let \( f \) be a nonnegative integrable function, and \( F(x) = \int_0^x f(t) \, dt \), then
\[
\int_0^\infty \left( \frac{F(x)}{x} \right)^p \, dx < \left( \frac{p}{p - 1} \right)^p \int_0^\infty f(x) \, dx, \quad p > 1.
\]

Using the above lemma and together with Theorem 2.1, we introduce the following result.

**Theorem 2.3:** Let \( f, g \geq 0 \),
\[
F(x) = \int_0^x f(t) \, dt, \quad G(y) = \int_0^y g(t) \, dt,
\]
and assume that $0 < \int_0^\infty f^2(x) \, dx < \infty$ and $0 < \int_0^\infty g^2(y) \, dy < \infty$, then we have

$$
\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \frac{F(x) \, G(y)}{x \, y} \, dx \, dy
\leq \mu \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}.
$$

(9)

**Proof:** Let

$$
I = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \frac{F(x) \, G(y)}{x \, y} \, dx \, dy.
$$

By Holder's inequality, we obtain

$$
I \leq \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \left( \frac{x}{y} \right)^{1/2} \, dx \right) \left( \frac{F(x)}{x} \right)^2 \, dx \right\}^{1/2} \times \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \left( \frac{y}{x} \right)^{1/2} \, dy \right) \left( \frac{G(y)}{y} \right)^2 \, dy \right\}^{1/2}.
$$

By using Lemma 2.1,

$$
I \leq \left\{ \int_0^\infty A \left( \frac{F(x)}{x} \right)^2 \, dx \right\}^{1/2} \times \left\{ \int_0^\infty A \left( \frac{G(y)}{y} \right)^2 \, dy \right\}^{1/2},
$$

$$
I \leq A \left\{ \int_0^\infty \left( \frac{F(x)}{x} \right)^2 \, dx \right\}^{1/2} \left\{ \int_0^\infty \left( \frac{G(y)}{y} \right)^2 \, dy \right\}^{1/2}.
$$

Finally, by Lemma 2.2, for $p=2$, we have

$$
I \leq 4A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}.
$$

Letting $\mu = 4A$, and inequality (9) is proved.

**Corollary 2.1:** Let $\alpha = \beta = 1$ in Theorem 2.3, then we obtain

$$
\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{x + y + |x-y|} \frac{F(x) \, G(y)}{x \, y} \, dx \, dy
$$
\[ \leq K_\gamma \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}, \]  
\[ (10) \]

where the constant

\[ K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1}. \]

Here, \( \gamma = 1, 2, 3, \ldots \) and \( K_0 = 2. \)

**Proof:** The proof of (10) is similar to that of (9), and here we only prove that:

\[ K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1}. \]  
\[ (11) \]

We have

\[ K_\gamma = \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{x+y+|x-y|} \left( \frac{x}{y} \right)^{1/2} \, dy = \int_0^\infty \frac{|\ln t|^\gamma}{1 + t + |1-t|} \left( \frac{1}{t} \right)^{1/2} \, dt \]
\[ = \int_0^1 \frac{|\ln t|^\gamma}{2} \left( \frac{1}{t} \right)^{1/2} \, dt + \int_1^\infty \frac{|\ln t|^\gamma}{2t} \left( \frac{1}{t} \right)^{1/2} \, dt. \]

For the last integral, take \( t = s^{-1} \) and rewrite this integral in term of \( t \), We obtain

\[ K_\gamma = \int_0^1 \frac{|\ln t|^\gamma}{2} \left( \frac{1}{t} \right)^{1/2} \, dt + \int_0^1 \frac{|\ln t|^\gamma}{2t} \left( \frac{1}{t} \right)^{1/2} \, dt = \int_0^1 |\ln t|^\gamma \left( \frac{1}{t} \right)^{1/2} \, dt. \]

Setting \( h = t^{1/2} \), we get

\[ K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1}. \]

### 3 Several Special Cases

We now introduce some special inequalities of (9) by choosing different values for \( \gamma, \alpha, \) and \( \beta. \)

1. If \( \gamma = \alpha = 0, \beta = 1, \) then we obtain

\[ \int_0^\infty \int_0^\infty \frac{1}{y + |x-y|} \frac{F(x)}{x} \frac{G(y)}{y} \, dx \, dy \]
\[ \leq \mu \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}, \]  
\[ (12) \]

where \( \mu = 4\lambda \) and from Lemma 2.1,
\[ A = \int_0^1 \frac{1}{1 - \frac{2}{t^2 + 2}} \, dt + \int_0^1 \frac{2}{\sqrt{2} - t + \frac{1}{2\sqrt{2}}} \, dt = 2 + 2 \int_0^1 \left( \frac{1/2\sqrt{2}}{\sqrt{2} - t} + \frac{1/2\sqrt{2}}{\sqrt{2} + t} \right) \, dt = 2 + \frac{1}{\sqrt{2}} \left( -\ln|\sqrt{2} - t| \big|_0^1 + \ln|\sqrt{2} + t| \big|_0^1 \right) = 3.24646. \]

(2) If \( \gamma = 0, \alpha = 1, \beta = 2, \) then
\[
\int_0^\infty \int_0^\infty \frac{1}{x + 2y + |x - y|} \frac{F(x) G(y)}{x} \, dx \, dy 
\leq \mu \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}, \tag{13}
\]
where \( \mu = 4A \) and from Lemma 2.1,
\[
A = \int_0^1 \frac{2}{2 + \frac{1}{t^2}} \, dt + \int_0^1 \frac{1}{3} \, dt = 2 \left[ \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \tan^{-1} 0 \right] + \frac{2}{3} = 2.2071.
\]

(3) If \( \gamma = 1, \alpha = \beta = 0, \) then
\[
\int_0^\infty \int_0^\infty \frac{\ln x - \ln y}{|x - y|} \frac{F(x) G(y)}{x y} \, dx \, dy 
\leq \mu \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}, \tag{14}
\]
where \( \mu = 4A, \) and from Lemma 2.1,
\[
A = \int_0^1 -4 \ln t \, dt + \int_0^1 -4 \ln t \, dt = -8 \int_0^1 \ln t \, dt.
\]

Since
\[
\int_0^1 \ln t \, t^{-1/2} \, dt = \pi^2.
\]

Then we have
\[
A = -8 \int_0^1 \ln t \, \frac{1}{1 - t^2} \, dt = 2\pi^2.
\]

(4) If \( \gamma = \alpha = 0, \beta = 2, \) then
\[ \int_0^\infty \int_0^\infty \frac{1}{2y + |x - y|} \frac{F(x)G(y)}{x} \frac{dx}{y} dy \leq \mu \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(y) \, dy \right)^{1/2}, \quad (15) \]

where \( \mu = 4A \) and from Lemma 2.1,

\[
A = \int_0^1 \frac{2}{1 + t^2} \, dt + \int_0^1 \frac{2}{-t^2 + 3} \, dt = \frac{\pi}{2} + 2 \int_0^1 \left( \frac{1/2\sqrt{3}}{\sqrt{3} - t} + \frac{1/2\sqrt{3}}{\sqrt{3} + t} \right) \, dt \\
= \frac{\pi}{2} + \frac{1}{\sqrt{3}} \left( -\ln|\sqrt{3} - t| \bigg|_0^1 + \ln|\sqrt{3} + t| \bigg|_0^1 \right) = 1.968. 
\]

References


