An Extension of Fisher’s Theorem

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Abstract

A result of Brain Fisher is extended to two pairs of self-maps through the notions of weak compatibility and property EA.

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1 Introduction

In 1976 Brian Fisher [2] proved the following:

**Theorem 1.1:** Let $A$ be a self-map on a complete metric space $X$ satisfying the contractive type inequality

$$d^2(Ax, Ay) \leq b d(x, Ax)d(y, Ay) + cd(x, Ay)d(y, Ax) \text{ for all } x, y \in X, \ldots \quad (1.1)$$

where $0 \leq b, c < 1$. Then $A$ has a unique fixed point.

In this paper we extend Theorem 1.1 to two pairs of self-maps using the notion of property EA and weakly compatible maps (cf. Section 2 below).
2 Preliminaries

In this paper \( X \) denotes a metric space with metric \( d \). Self-maps \( A \) and \( S \) are commuting if \( ASx = SAx \) for all \( x \in X \).

**Definition 2.1:** \( A \) and \( S \) are compatible [3] if

\[
\lim_{n \to \infty} d(ASx_n, SAx_n) = 0 \quad \ldots \quad (2-a)
\]

whenever \( \{x_n\}_{n=1}^{\infty} \) is a sequence in \( X \) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \quad \ldots \quad (2-b)
\]

for some \( z \in X \).

Note that every commuting pair is compatible. That is compatibility is weaker than the commutativity. However, a compatible pair is commuting (cf. [3]).

By altering the asymptotic condition (2-a), later various types of compatibility like \( A \)- and \( S \)-compatibilities [9], Compatibility of type \( A \) (cf. [5]), type \( B \) (cf. [8]), type \( C \) (cf. [7]), type \( E \) (cf. [11]) and type \( P \) (See [6]) were developed in solving certain functional equations that arise dynamical programming. A nice comparative survey among these types of compatibility was done in [9] and [12].

**Definition 2.2:** Self maps \( A \) and \( S \) on \( X \) satisfy property EA [1] if there exists a sequence \( \{x_n\}_{n=1}^{\infty} \in X \) with the choice (2-b)

Obviously compatible and noncompatible pairs satisfy the property EA.

**Definition 2.3:** Self maps \( A \) and \( S \) are weakly compatible [4] if they commute at their coincidence points.

It was shown that every compatible pair is weakly compatible but the converse is not true [4], and the notions of weakly compatibility and property EA are independent [10].

3 Main Result and Remarks

**Theorem 3.1:** Let \( A, B, S \) and \( T \) be self-maps on \( X \) satisfying the inclusions

\[
A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X) \quad \ldots \quad (3)
\]

and the inequality
\[ d^2(Ax, By) \leq b d(Ax, Sx) d(By, Ty) + cd(Sx, By) d(Ty, Ax) \]

for all \( x, y \in X \), \( \ldots \) \( (4) \)

with the same choice of the constants \( b \) and \( c \) as in Theorem 1.

If one of \( S(X) \) and \( T(X) \) is complete and

(a) Either \((A, S)\) or \((B, T)\) satisfies property EA
(b) The pairs \((A, S)\) and \((B, T)\) are weakly compatible.

Then \( A, B, S \) and \( T \) have a unique common fixed point.

Proof. Suppose that \( A \) and \( S \) satisfy the property EA. By the inclusion \( A(X) \subseteq T(X) \), we can find another sequence \( \{y_n\}_{n=1}^{\infty} \) in \( X \) such that

\[ Ax_n = Ty_n \text{ for all } n \text{ so that from (2-b)} \]

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = z. \]

Let \( q = \lim_{n \to \infty} By_n \). We prove below that \( q = z \).

Writing \( x = x_n \) and \( y = y_n \) in the inequality (4), we get

\[ d^2(Ax_n, By_n) \leq b d(Ax_n, Sx_n) d(By_n, Ty_n) + cd(Sx_n, By_n) d(Ty_n, Ax_n). \]

Applying the limit as \( n \to \infty \) in this and using (5) it follows that

\[ d^2(z, q) \leq b \cdot 0 + c \cdot 0 \text{ so that } d^2(z, q) = 0 \text{ or } d(z, q) = 0. \text{ That is, } q = z. \]

Hence

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} By_n = z. \]

\( \ldots \) \( (6) \)

Similarly we can prove (6) if the pair \((B, T)\) satisfies the property EA.

**Case A:** Suppose that \( T(X) \) is complete subspace of \( X \).

Note that \( \{Ty_n\}_{n=1}^{\infty} \) is Cauchy and convergent sequence in \( T(X) \). Therefore \( z \in T(X) \).

That is \( z = Tq \) for some \( q \in X \). Now we show that \( q \) is a coincidence point of \( B \) and \( T \).

Taking \( x = x_n \) and \( y = q \) in the inequality (4) and using (6) we get
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\[ d^2(Ax_n, Bq) \leq b.d(Ax_n, Sx_n) d(Bq, Tq) + c.d(Sx_n, Bq)d(Tq, Ax_n) \]

or \[ d^2(Tq, Bq) \leq b.0 + c.0 = 0. \]

Hence \( Tq = Bq \), that is \( q \) is a coincidence point of \( T \) and \( B \).

Again \( B(X) \subset S(X) \) implies that \( Bq \in S(X) \) or \( Bq = Sr \) for some \( r \in X \).

Then from the inequality (4) with \( x = r, y = q \) we get

\[ d^2(Ar, Bq) \leq b.d(Ar, Sr)d(Bq, Tq) + c.d(Sr, Bq)d(Tq, Ar). \]

Using \( Bq = Tq = Sr \) in this, we see that \( d^2(Ar, Sr) \leq 0 \) or \( Ar = Sr \). Hence

\[ Ar = Sr = Bq = Tq. \] \( \ldots \) (7)

In other words, \( r \) is a coincidence point of \( A \) and \( S \) and \( q \) is a coincidence point of \( B \) and \( T \).

**Case B:** Suppose that \( S(X) \) is complete subspace of \( X \).

Since \( \{Sx_n\}_{n=1}^\infty \) is a Cauchy sequence and convergent sequence in \( S(X) \) we see that \( z \in S(X) \) or \( z = Tp \) for some \( p \in X \).

Now we write \( x = x_n \) and \( y = p \) in the inequality (4). Then

\[ d^2(Ax_n, Bp) \leq b.d(Ax_n, Sx_n) d(Bp, Tp) + c.d(Sx_n, Bp)d(Tp, Ax_n) \]

or \( d^2(Tp, Bp) \leq b.0 + c.0 = 0 \) so that \( Tp = Bp \) or that \( p \) is a coincidence point of \( T \) and \( B \).

Again \( B(X) \subset S(X) \) implies that \( Bp \in S(X) \) or \( Bp = Sv \) for some \( v \in X \).

Then from the inequality (4) with \( x = v \) and \( y = p \), we get

\[ d^2(Av, Bp) \leq b.d(Av, Sv) d(Bp, Tp) + c.d(Sv, Bp)d(Tp, Av). \]

Using \( Tp = Bp = Sv \), this gives

\[ d^2(Av, Sv) \leq b.d(Av, Sv)d(Tp, Tp) + c.d(Bp, Bp)d(Tp, Av) = 0 \) or \( Av = Sv \).

Thus \( v \) is a coincidence point of \( A \) and \( S \) and \( p \) is a coincidence point of \( B \) and \( T \).

Since the pairs \( (A, S) \) and \( (B, T) \) are weakly compatible, we find that
$\text{ASr} = S\text{Ar}$ and $BTq = TBq$. This implies $Az = Sz$ and $Bz = Tz$.

Now from the inequality (4) with $x = y = z$, it follows that
\[
d^2(Az, Bz) \leq b \cdot d(Az, Sz)d(Bz, Tz) + c \cdot d(Sz, Bz)d(Tz, Az)
\]
\[
\leq b \cdot d(Sz, Sz)d(Tz, Tz) + c \cdot d(Az, Bz)d(Bz, Az)
\]
\[
\Rightarrow (1 - c) \ d^2(Az, Bz) \leq 0 \quad \Rightarrow \quad d^2(Az, Bz) = 0 \quad \text{or} \quad Az = Bz.
\]

Thus $Az = Sz = Bz = Tz$ … (8)

Now we prove that $Az = z$.

From the inequality (4) with $x = z$ and $y = q$, we have
\[
d^2(Az, Bq) \leq b \cdot d(Az, Sz)d(Bz, Tq) + c \cdot d(Sz, Bq)d(Tq, Az) \leq b \cdot 0 + c \cdot d^2(Az, z)
\]
\[
\Rightarrow (1 - c) \ d^2(Az, z) \leq 0 \quad \Rightarrow \quad Az = z.
\]

Hence $Az = Sz = Bz = Tz = z$. Thus $z$ is a common fixed point of $A, S, B$ and $T$.

**Uniqueness:** Let $z, z'$ be two common fixed points of $A, S, B$ and $T$.

From the inequality (4) with $x = z$ and $y = z'$, we get
\[
d^2(Az, Sz') \leq b \cdot d(Az, Sz)d(Bz', Tz') + c \cdot d(Sz, Bz')d(Tz', Az) \leq 0 + c \cdot d(z, z')d(z', z)
\]
\[
\text{or} \quad d^2(z, z') \leq c \cdot d^2(z, z') \quad \text{so that} \quad z = z'.
\]

Hence the fixed point is unique.

**Remark 3.1:** Writing $B = A$ and $S = T = I$, the identity map on $X$ in Theorem 3.1, we get (1) from (4) as a special case. It is also known that the identity map commutes and hence is weakly compatible with every map. Further from the proof of Theorem 1.1, the sequence $\{A^n x\}_n$ is Cauchy for each $x \in X$. Therefore if $X$ is complete, this converges to some $z \in X$ and its convergence is equivalent to the property $EA$ of the pair $(A, I)$, that is the condition (a) of Theorem 3.1.

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References


