Bertrand Mate of Biharmonic Reeb Curves in 3-Dimensional Kenmotsu Manifold

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Abstract

In this article, we study biharmonic Reeb curves in 3-dimensional Kenmotsu manifold. Moreover, we apply biharmonic Reeb curves in special 3-dimensional Kenmotsu manifold $\mathbb{K}$. Finally, we characterize Bertrand mate of the biharmonic Reeb curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

Keywords: Kenmotsu manifold, biharmonic curve, Bertrand curve, Reeb vector field.

1 Introduction

In the theory of space curves in differential geometry, the associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves. The well-known examples of such curves are Bertrand curves. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. These curves have an important role in the theory of curves.

Let $(N, h)$ and $(M, g)$ be Riemannian manifolds. A smooth map $\phi: N \rightarrow M$
$M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 dv_h,$$

where the section $T(\phi) := \text{tr} \nabla \phi d\phi$ is the tension field of $\phi$.

The Euler–Lagrange equation of the bienergy is given by $T_2(\phi) = 0$. Here the section $T_2(\phi)$ is defined by

$$T_2(\phi) = -\Delta \phi T(\phi) + \text{tr} R(T(\phi), d\phi) d\phi,$$

and called the bitension field of $\phi$. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this article, we study biharmonic Reeb curves in 3-dimensional Kenmotsu manifold. Moreover, we apply biharmonic Reeb curves in special 3-dimensional Kenmotsu manifold $\mathcal{K}$. Finally, we characterize Bertrand mate of the biharmonic Reeb curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold $\mathcal{K}$.

## 2 Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form $\eta$, the associated vector field $\xi$, $(1,1)$-tensor field $\phi$ and the associated Riemannian metric $g$. It is well known that [2]

$$\phi \xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0,$$

$$\phi^2(X) = -X + \eta(X) \xi,$$

$$g(X, \xi) = \eta(X),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y),$$

for any vector fields $X, Y$ on $M$. Moreover,

$$(\nabla_X \phi) Y = -\eta(Y) \phi(X) - g(X, \phi Y) \xi, \quad X, Y \in \chi(M),$$

$$\nabla_X \xi = X - \eta(X) \xi,$$

where $\nabla$ denotes the Riemannian connection of $g$, then $(M, \phi, \xi, \eta, g)$ is called an Kenmotsu manifold [2].

In Kenmotsu manifolds the following relations hold [2]:

$$\eta(R(X, Y) Z) = \eta(Y) g(X, Z) - \eta(X) g(Y, Z),$$

$$R(X, Y) \xi = \eta(X) Y - \eta(Y) X,$$

$$R(\xi, X) Y = \eta(Y) X - g(X, Y) \xi,$$

$$R(\xi, X) \xi = X - \eta(X) \xi,$$

where $R$ is the Riemannian curvature tensor.
3 Biharmonic Reeb Curves in the 3-Dimensional Kenmotsu Manifold

Let $\gamma$ be a curve on the 3-dimensional Kenmotsu manifold parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along $\gamma$ defined as follows:

$T$ is the unit vector field $\gamma'$ tangent to $\gamma$, $N$ is the unit vector field in the direction of $\nabla_T T$ (normal to $\gamma$), and $B$ is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

\[
\begin{align*}
\nabla_T T &= \kappa N, \\
\nabla_T N &= -\kappa T + \tau B, \\
\nabla_T B &= -\tau N,
\end{align*}
\]

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

\[
\begin{align*}
g(T, T) &= 1, \quad g(N, N) = 1, \quad g(B, B) = 1, \\
g(T, N) &= g(T, B) = g(N, B) = 0.
\end{align*}
\]

**Lemma 3.1.** (see [13]) If $\gamma$ is a biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold, then $\gamma$ is a helix.

We consider the special 3-dimensional manifold

$$K = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\},$$

where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}$$

are linearly independent at each point of $K$. Let $g$ be the Riemannian metric defined by

\[
\begin{align*}
g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1, \\
g(e_1, e_2) &= g(e_2, e_3) = g(e_1, e_3) = 0.
\end{align*}
\]

The characterising properties of $\chi(K)$ are the following commutation relations:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Let $\eta$ be the 1-form defined by

$$\eta(Z) = g(Z, e_3) \text{ for any } Z \in \chi(M).$$
Let \( \phi \) be the \((1,1)\) tensor field defined by
\[
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]

Then using the linearity of and \( g \) we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2(Z) = -Z + \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any \( Z, W \in \chi(\mathbb{K}) \). Thus for \( e_3 = \xi, (\phi, \xi, \eta, g) \) defines an almost contact metric structure on \( \mathbb{K} \).

Now, we consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold \( \mathbb{K} \).

**Theorem 3.4.** (see [13]) Let \( \gamma : I \rightarrow \mathbb{K} \) be a unit speed biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold \( \mathbb{K} \). Then, the parametric equations of \( \gamma \) are
\[
x(s) = \frac{C_1 \sin^5 \varphi}{\kappa^2 + \sin^4 \varphi \cos^2 \varphi} e^{-\cos \varphi s} \left( \frac{\kappa}{\sin^2 \varphi} \cos \left( \frac{\kappa}{\sin^2 \varphi} s + \sigma \right) \right.
\]
\[
+ \cos \varphi \sin \left( \frac{\kappa}{\sin^2 \varphi} s + \sigma \right) + C_2,
\]
\[
y(s) = \frac{C_1 \sin^5 \varphi}{\kappa^2 + \sin^4 \varphi \cos^2 \varphi} e^{-\cos \varphi s} \left( -\cos \varphi \cos \left( \frac{\kappa}{\sin^2 \varphi} s + \sigma \right) \right) 3.14 \]
\[
+ \frac{\kappa}{\sin^2 \varphi} \sin \left( \frac{\kappa}{\sin^2 \varphi} s + \sigma \right) \right) + C_3,
\]
\[
z(s) = C_1 e^{-\cos \varphi s},
\]
where \( C, C_1, C_2, C_3 \) are constants of integration.

4 **Bertrand Mate of Biharmonic Reeb Curves in the Special Three-Dimensional Kenmotsu Manifold \( \mathbb{K} \)**

A curve \( \gamma : I \rightarrow \mathbb{K} \) with \( \kappa \neq 0 \) is called a Bertrand curve if there exist a curve \( \gamma_B : I \rightarrow \mathbb{K} \) such that the principal normal lines of \( \gamma \) and \( \gamma_B \) at \( s \in I \) are equal. In this case \( \gamma_B \) is called a Bertrand mate of \( \gamma \).
On the other hand, let \( \gamma : I \to \mathbb{K} \) be a Bertrand curve parametrized by arc length. A Bertrand mate of \( \gamma \) is as follows:

\[
\gamma_B(s) = \gamma(s) + \lambda N(s), \quad \forall s \in I,
\]

where \( \lambda \) is constant.

**Theorem 4.1.** Let \( \gamma : I \to \mathbb{K} \) be a biharmonic curve parametrized by arc length. If \( \gamma_B \) is a Bertrand mate of \( \gamma \), then the parametric equations of \( \gamma_B \) are

\[
x_B(s) = \frac{\lambda \sin \varphi}{\kappa} \left( \frac{\kappa}{\sin^2 \varphi} \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) + \cos \varphi \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \right) (\bar{C}_1 s + \bar{C}_2) \\
+ \frac{C_1 \sin^3 \varphi}{\kappa} e^{-\cos \varphi s} \left( -\cos \sigma \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) + \sin \sigma \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \right) + C_2, \tag{4.10}
\]

\[
y_B(s) = \frac{\lambda \sin \varphi}{\kappa} \left( -\frac{\kappa}{\sin^2 \varphi} \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) + \cos \varphi \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \right) (\bar{C}_1 s + \bar{C}_2) \\
+ \frac{C_1 \sin^3 \varphi}{\kappa} e^{-\cos \varphi s} \left( \sin \sigma \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) + \cos \sigma \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \right) + C_3, \tag{4.10}
\]

\[
z_B(s) = \frac{\lambda}{\kappa} (\bar{C}_1 s + \bar{C}_2) + C_1 e^{-\cos \varphi s},
\]

where \( \sigma, \bar{C}_1, \bar{C}_2, C_1, C_2, C_3 \) are constants of integration.

**Proof.** Assume that \( \mathbf{T} \) is

\[
\mathbf{T} = T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2 + T_3 \mathbf{e}_3, \tag{4.3}
\]

where \( T_1, T_2, T_3 \) are differentiable functions on \( I \).

From [13], we obtain

\[
\mathbf{T} = \sin \varphi \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \mathbf{e}_1 + \sin \varphi \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right) \mathbf{e}_2 + \cos \varphi \mathbf{e}_3. \tag{4.4}
\]

Using (3.3) in (4.4), we obtain

\[
\mathbf{T} = (z \sin \varphi \sin\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right), z \sin \varphi \cos\left(\frac{\kappa}{\sin^2 \varphi} s + \sigma\right), -z \cos \varphi). \tag{4.5}
\]

Because, by making use of (3.3), we have

\[
\nabla_{\mathbf{T}} \mathbf{T} = (T'_1 + T_1 T_3) \mathbf{e}_1 + (T'_2 + T_2 T_3) \mathbf{e}_2 + T'_3 \mathbf{e}_3. \tag{4.6}
\]

From (3.1) and (4.4), we get
\[ \nabla_T T = \sin \varphi \left( \frac{\kappa}{\sin^2 \varphi} \cos \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \sin \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] \right) e_1 + \cos \varphi \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \cos \left[ \frac{\kappa}{\sin^2 \varphi} s + C \right] e_2. \]

Then, by using Frenet formulas (3.1), we get

\[ \mathbf{N} = \frac{1}{\kappa} \nabla_T T \]

\[ = \frac{1}{\kappa} \left[ \sin \varphi \left( \frac{\kappa}{\sin^2 \varphi} \cos \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \sin \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] \right) e_1 \right] + \cos \varphi \left[ \frac{\kappa}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \cos \left[ \frac{\kappa}{\sin^2 \varphi} s + C \right] e_2. \]

Finally, we substitute (3.5) and (4.8) into (4.1), we get (4.2). The proof is completed.

**Corollary 4.2.** Let \( \gamma : I \rightarrow \mathbb{K} \) be a biharmonic curve parametrized by arc length. If \( \gamma_B \) is a Bertrand mate of \( \gamma \), then the parametric equations of \( \gamma_B \) in terms of \( \tau \) are

\[
x_B(s) = \frac{\lambda \sin \varphi}{\sqrt{1 - \tau^2}} \left( \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} \cos \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \sin \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] \right) \left( \bar{C}_1 s + \bar{C}_2 \right) + \frac{C_1 \sin^3 \varphi e^{-\cos \varphi s}}{\sqrt{1 - \tau^2}} \left( - \cos \sigma \cos \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s \right] + \sin \sigma \sin \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s \right] \right) + C_2,
\]

\[
y_B(s) = \frac{\lambda \sin \varphi}{\sqrt{1 - \tau^2}} \left( - \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} \sin \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] + \cos \varphi \cos \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] \right) \left( \bar{C}_1 s + \bar{C}_2 \right) + \frac{C_1 \sin^3 \varphi e^{-\cos \varphi s}}{\sqrt{1 - \tau^2}} \left( \sin \sigma \cos \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] + \cos \sigma \sin \left[ \frac{\sqrt{1 - \tau^2}}{\sin^2 \varphi} s + \sigma \right] \right) + C_3,
\]

\[
z_B(s) = \frac{\lambda}{\sqrt{1 - \tau^2}} (\bar{C}_1 s + \bar{C}_2) + C_1 e^{-\cos \varphi s},
\]

where \( \sigma, \bar{C}_1, \bar{C}_2, C_1, C_2, C_3 \) are constants of integration.

**References**


