On n-Normal Operators

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Abstract

In this paper we introduce n-normal operators on a Hilbert space $H$. We give some basic properties of these operators. In general an n-normal operators need not be a normal operator, a hyponormal operator.

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1 Introduction

Throughout this paper, $B(H)$ denotes to the algebra of all bounded linear operators acting on a complex Hilbert space $H$. An operator $T$ is said to be normal if $T^*T = TT^*$, (it is well known that normal operators have translation-invariant property, i.e., if $T$ is a normal operator, then $(T - \lambda)$ is a normal operator for every $\lambda \in \mathbb{C}$); self adjoint if $T^* = T$; positive if $T^* = T$ and $\langle Tx, x \rangle \geq 0$ for all $x \in H$; and projection if $T^2 = T = T^*$. For an operator $T \in H$, if $\|Tx\| = \|x\|$ for all $x \in H$ (or equivalently $T^*T = I$), then $T$ is called an isometry. An onto isometry is called unitary. An operator $T \in B(H)$ is called partial isometry if $T^*T$ is projection. An operator $T$ on $H$ is called subnormal if there exists a Hilbert space $K$ with $H$ is a subspace of $K$ and a normal operator $N$ on $K$ such that $NH \subseteq H$ and $N|H = T$; $T$ is hyponormal if $T^*T \geq TT^*$. Let $T \in B(H)$ and $x \in H$. The sequence $\{T^n x\}_{n=0}^{\infty}$ is called orbit of $x$ under $T$, and is denoted by $\text{orb}(T, x)$. If $\text{orb}(T, x)$ is dense in $H$, then $x$ is called a hypercyclic vector for $T$. An operator $T \in B(H)$ is called scalar
of order $m$ if it possesses a spectral distribution of order $m$, i.e., if there is a continuous unital morphism $\phi : C_0^m(\mathbb{C}) \to B(H)$ such that $\phi(z) = T$ where $z$ stands for the identity function on $\mathcal{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order $m$, $0 \leq m \leq \infty$.

An operator $T \in B(H)$ is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

2 n-normal operators

Definition 2.1. $T \in B(H)$ is called an $n$-normal operator if $T^nT^* = T^*T^n$.

Proposition 2.2. Let $T \in B(H)$. Then $T$ is $n$-normal if and only if $T^n$ is normal where $n \in \mathbb{N}$.

Proof. Let $T$ is $n$-normal, $T^nT^* = T^*T^n$. Therefore

$$T^n(T^*)^n = T^nT^*(T^*)^{n-1} = T^n(T^*T^n)(T^*)^{n-2} = (T^*)^2T^n(T^*)^{n-2} = (T^*)^nT^n.$$ 

Then $T^n$ is normal. Now, let $T^n$ is normal. Since $T^nT = TT^n$, by Fuglede theorem [8], $T^*T^n = T^nT^*$. Therefore $T$ is $n$-normal.

It is clear that a bounded normal operator is $n$-normal for any $n$. The converse is not true. Indeed if $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$, then $T$ is 2-normal which is not normal. And all nonzero nilpotent operators are $n$-normal operators, for $n \geq k$ where $k$ the index of nilpotence, but they are not normal. It is well known that if $T$ is normal, then it is hyponormal. And if $T$ is normal and $T^k$ is compact for some $k$, then $T$ is compact by [8]. The following example shows that these need not be true in case of $n$-normal operator.

Example 2.3. Let $H = \ell^2$ and $e_1, e_2, \ldots$ be standard orthogonal basis for $\ell^2$.

Define $T$ on $H$ by $Te_i = \begin{cases} e_1, & i=1 \\ e_{i+1}, & i=2j, j = 1, 2, \ldots \\ 0, & i=2j+1 \end{cases}$

Then $T^2 = P$, where $P$ is the orthogonal projection on the space span by $e_1$. So $T$ is 2-normal but neither $T$ nor $T^*$ is hyponormal.

Now, since $T^2$ is a projection on one-dimensional space, it is compact. However, since range of $T$ contains an infinite orthonormal set $\{e_i, i = 1, 3, 5, \ldots\}$, $T$ is not compact.

The following example shows that there exists an operator which is subnormal but not $n$-normal for any $n \in \mathbb{N}$.

Example 2.4. Let $U$ be unilateral shift on $\ell^2$ (i.e., $U(\alpha_0, \alpha_1, \cdots) = (0, \alpha_0, \alpha_1, \cdots)$.

Then $U$ is subnormal but for any $n \in \mathbb{N}$, $U^n$ is not normal.
It is well known that if $T$ is hyponormal and compact, then $T$ is normal. But we note that the nilpotent operator $T = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ is an $n$-normal operator, which is compact but not normal. Thus $T$ is not hyponormal.

**Theorem 2.5.** The set of all $n$-normal operators on $H$ is closed subset of $B(H)$ which is closed under scalar multiplication.

**Proof.** First if $T$ is $n$-normal, and $\alpha$ is scalar, then $(\alpha T)^n(\alpha T)^* = \alpha^n \pi(T^n T^*) = \overline{\alpha} \alpha^n(T^* T^n)$ and $(\pi T^*)(\alpha^n T^n) = (\alpha T)^*(\alpha T)^n$. Hence $\alpha T$ is $n$-normal. Now, suppose that $(T_k)$ is sequence of $n$-normal operators converging to $T$ in $B(H)$. Then

$$\|T^n T^* - T^* T^n\| \leq \|T^n T^* - T^n_k T^n_k\| + \|T^n_k T^n_k - T^* T^n\| \to 0 \text{ as } k \to \infty.$$ 

Hence $T^n T^* = T^* T^n$. Thus $T$ is $n$-normal.

**Proposition 2.6.** Let $T \in B(H)$ be $n$-normal. Then

1. $T^*$ is $n$-normal.
2. If $T^{-1}$ exists, then $(T^{-1})$ is $n$-normal.
3. If $S \in B(H)$ is unitary equivalent to $T$, then $S$ is $n$-normal.
4. If $M$ is a closed subspace of $H$ such that $M$ reduces $T$, then $S = T/M$ is an $n$-normal operator.

**Proof.** (1) Since $T$ is $n$-normal, $T^n$ is normal. So $(T^n)^* = (T^*)^n$ is normal, $T^*$ is an $n$-normal operator.
(2) Since $T$ is $n$-normal, $T^n$ is normal. Since $(T^n)^{-1} = (T^{-1})^n$ is normal, $T^{-1}$ is an $n$-normal operator.
(3) Let $T$ be an $n$-normal operator and $S$ be unitary equivalent of $T$. Then there exists unitary operator $U$ such that $S = U T U^*$ so $S^n = U T^n U^*$. Since $T^n$ is normal, $S^n$ is normal. Therefore $S$ is $n$-normal.
(4) Since $T$ is $n$-normal, $T^n$ is normal. So $T^n/M$ is normal. And since $M$ is invariant under $T$, $T^n/M = (T/M)^n$. Thus $(T/M)^n$ is normal. So $T/M$ is $n$-normal.

Now, the following example shows that the class of $2$-normal operators may not have the translation-invariant property.

**Example 2.7.** Let $T = \begin{pmatrix} 0 & T_1 \\ 0 & 0 \end{pmatrix}$, where $T_1 : H_1 \to H$. Then $T$ is $2$-normal operator. But $[(T - \lambda)^2, (T - \lambda)^2] = \begin{pmatrix} -4 |\lambda|^2 T_1 T_1^* \\ 0 \\ 4 |\lambda|^2 T_1^* T_1 \end{pmatrix}$ not necessarily equal to zero unless $\lambda = 0$. Hence $(T - \lambda)^2$ is not normal. So $(T - \lambda)$ is not necessarily $2$-normal operator.

**Theorem 2.8.** If $S$, $T$ are commuting $n$-normal operators, then $ST$ is an $n$-normal operator.
Proof. Since $S$, $T$ are commuting $n$-normal operators, $S^n$, $T^n$ are commuting normal operator. So $S^nT^n$ is a normal operator. Since $S^nT^n = (ST)^n$, $(ST)^n$ is normal. Hence $ST$ is $n$-normal.

The following example shows that Theorem 2.8 is not necessarily true if $S$, $T$ are not commuting.

**Example 2.9.** Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ be operators on the Hilbert space $\mathbb{C}^2$. Then $S$ and $T$ are 2-normal. We note that $ST = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix} \neq \begin{pmatrix} i & -2 \\ 0 & i \end{pmatrix} = TS$. But as $(ST)^2 = \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}$ is not normal, $ST$ is not 2-normal.

**Corollary 2.10.** If $T$ is $n$-normal, Then $T^m$ is $n$-normal for any positive integer $m$.

The following example shows that sum of two commuting $n$-normal operators need not be $n$-normal.

**Example 2.11.** Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $S$ and $T$ are commuting 2-normal. But $S + T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $(S + T)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not normal. Thus $S + T$ is not 2-normal. We note here $S$ is a selfadjoint operator.

**Proposition 2.12.** Let $T$, $S$ be commuting $n$-normal operator, such that $(S + T)^* \text{ commutes with } \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k$. Then $(S + T)$ is an $n$-normal operator.

**Proof.** Since $(S + T)^n(S + T)^* = \left( \sum_{k=0}^{n-1} \binom{n}{k} S^{n-k} T^k \right) (S^* + T^*)$, $(S + T)^n(S + T)^* = S^n S^* + \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k (S + T)^* + T^n S^* + S^n T^* + T^n T^*$. And since $(S + T)^*$ is commuting with $\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k$, $(S + T)^n(S + T)^* = S^n S^* + (S + T)^* \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k + S^* T^n + T^* S^n + T^* T^n$. So $(S + T)^n(S + T)^* = (S + T)^*(S^n + T^n) + (S + T)^* \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k$. Hence $(S + T)^n(S + T)^* = (S + T)^*(\sum_{k=0}^{n-1} \binom{n}{k} S^{n-k} T^k) = (S + T)^*(S + T)^n$. □
Lemma 2.13. If \( S, T \in \mathbb{B}(H) \) are 2-normal operators and \( ST + TS = 0 \), then \( T + S \) and \( ST \) are 2-normal.

Proof. Since \( ST + TS = 0 \), \( S^2T^2 = T^2S^2 \). So \( (S + T)^2 = S^2 + T^2 \) is normal. Thus \((S + T)\) is a 2-normal operator.

Now since \( ST + TS = 0 \), \((ST)^2 = -S^2T^2 = -T^2S^2 \). Hence by Theorem 2.8, \( ST \) is a 2-normal operator. \(\square\)

Now we state some well known lemmas which we shall need.

Lemma 2.14. If \( T \) is normal, then \( Tx = \lambda x \) if and only if \( T^*x = \overline{\lambda}x \).

Lemma 2.15. If \( P \) is the projection on a closed subspace \( M \), \( N \) respectively. Then \( MN \) if and only if \( PQ = 0 \).

Lemma 2.16. If \( P \) is the projection on a closed subspace \( M \) of \( H \), then \( M \) reduces of \( T \) if and only if \( TP = PT \).

Theorem 2.17. Let \( T \) be an operator on finite dimensional Hilbert space \( H \), \( \lambda_1, \ldots, \lambda_m \) be eigenvalues of \( T \) such that \( \lambda_i \neq \lambda_j \), \( i \neq j \), \( M_1, \ldots, M_m \) the corresponding eigenspaces, and \( P_1, \ldots, P_m \) the projections on \( M_1, \ldots, M_m \) respectively. Then \( M_i \)'s are pairwise orthogonal and they span \( H \) if and only if \( T \) is n-normal operator.

Proof. Assume \( M_i \)'s are pairwise orthogonal and they span \( H \). Then for \( x \in H \), \( x = x_1 + x_2 + \ldots + x_m, \) \( x_i \in M_i, T^n x = T^n x_1 + \ldots + T^n x_m = \lambda^n_1 x_1 + \ldots + \lambda^n_m x_m \).

Since \( P_i \)'s are projection on eigenspace \( M_i \)'s which are pairwise orthogonal, by lemma 2.14 \( P_i x = x_i \). Hence \( Ix = x_1 + \ldots + x_m = P_1 x + \ldots + P_m x = (P_1 + \ldots + P_m)x \) for every \( x \in H \). Thus \( I = \sum_{i=1}^n P_i \). Since \( T^n x = \lambda_1^n x_1 + \ldots + \lambda_m^n x_m = \lambda_1^n P_1 x + \ldots + \lambda_m^n P_m x = (\lambda_1^n P_1 + \ldots + \lambda_m^n P_m)x \) for all \( x \in H \). So \( T^n = \sum_{i=1}^n \lambda_i^n P_i \). Hence \( T^{*n} = \sum_{i=1}^n \lambda_i^n P_i \). Since \( M_i \)'s are pairwise orthogonal, \( P_i P_j = \begin{cases} P_i & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \) So \( T^nT^{*n} = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m \) and \( T^{*n}T^n = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m \). Thus \( T^n \) is normal, i.e., \( T \) is an n-normal operator.

Suppose \( T \) is an n-normal operator. Then \( T^n \) is a normal operator. We claim that \( M_i \)'s are pairwise orthogonal. Let \( x_i, x_j \) be vectors in \( M_i, M_j \) \( (i \neq j) \) such that \( T^n x_i = \lambda^n_i x_i \) and \( T^n x_j = \lambda^n_j x_j \). Then \( \lambda_i^n \langle x_i, x_j \rangle = \langle \lambda_i^n x_i, x_j \rangle = \langle T^n x_i, x_j \rangle = \langle x_i, T^n x_j \rangle = \langle x_i, \lambda_j^n x_j \rangle = \lambda_j^n \langle x_i, x_j \rangle \). So \( (\lambda_i^n - \lambda_j^n) \langle x_i, x_j \rangle = 0 \). Since \( \lambda_i^n \neq \lambda_j^n \), \( \langle x_i, x_j \rangle = 0 \). This shows that \( M_i \)'s are pairwise orthogonal. Let \( M = M_1 + \ldots + M_m \). Then \( M \) is a closed subspace of \( H \). Let \( P \) be associated projection onto \( M \). Then \( P = P_1 + \ldots + P_m \). Since \( T^n \) is normal, each \( M_i \) reduces \( T^n \). It follows that \( T^n P = PT^n \). Consequently \( M_i \) is invariant under \( T^n \). Suppose that \( M_i \neq \{0\} \). Let \( T_1 = T^n/M_i \). Then \( T_1 \) is an operator on non-trivial finite dimensional complex Hilbert space \( M_i \) with empty point spectrum which is impossible. Therefore \( M_i \neq \{0\} \), i.e., \( M = H \). \(\square\)
Theorem 2.18. Let $T_1, \ldots, T_m$ be $n$-normal operators in $B(H)$. Then $(T_1 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes \ldots \otimes T_m)$ are $n$-normal operators.

Proof. Since $(T_1 \oplus \ldots \oplus T_m)^n(T_1 \oplus \ldots \oplus T_m)^* = (T_1^* \oplus \ldots \oplus T_m^*)(T_1^* \oplus \ldots \oplus T_m) = T_1^* T_1 \oplus \ldots \oplus T_m^* T_m = (T_1 \otimes \ldots \otimes T_m)^n(T_1^* \otimes \ldots \otimes T_m^*) = (T_1 \otimes \ldots \otimes T_m)^*(T_1^* \otimes \ldots \otimes T_m)^n$. Then $(T_1 \otimes \ldots \otimes T_m)$ is an $n$-normal operator. Now, for $x_1, \ldots, x_m \in H$, $(T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^*(x_1 \otimes \ldots \otimes x_m)$

\[
= (T_1^* \oplus \ldots \oplus T_m^*)(T_1^* \otimes \ldots \otimes T_m^*)(x_1 \otimes \ldots \otimes x_m) = T_1^* T_1^* x_1 \otimes \ldots \otimes T_m^* T_m^* x_m,
\]

\[
= T_1^* T_1^* x_1 \otimes \ldots \otimes T_m^* T_m^* x_m = (T_1^* \otimes \ldots \otimes T_m^*)(T_1^* \otimes \ldots \otimes T_m^*)(x_1 \otimes \ldots \otimes x_m),
\]

\[
= (T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^*(x_1 \otimes \ldots \otimes x_m).
\]

So $(T_1 \otimes \ldots \otimes T_m)^n(T_1 \otimes \ldots \otimes T_m)^* = (T_1 \otimes \ldots \otimes T_m)^*(T_1 \otimes \ldots \otimes T_m)^n$. Thus $(T_1 \otimes \ldots \otimes T_m)$ is $n$-normal.

Proposition 2.19. $(T - \lambda)$ is an $n$-normal operator for every $\lambda \in \mathbb{C}$ if and only if $T$ is a normal operator.

Proof. Since $(T - \lambda)$ is $n$-normal for every $\lambda \in \mathbb{C}$, $(T - \lambda)^n(T - \lambda)^* = (T - \lambda)^* (T - \lambda)^n$. Hence $(T^* - \bar{\lambda})(\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k) = (\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k)T^* - \bar{\lambda})$. So $(\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k) - (\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k)\bar{\lambda} = (\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k) - (\sum_{k=1}^{n}(-1)^k(T^{n-k}\lambda^k)\bar{\lambda} = 0$. From the left side of the last equation we get the term which $k = n$ is zero. Hence $\sum_{k=1}^{n-1}(-1)^k(T^{n-k}\lambda^k) - (T^{n-k}\lambda^k) = 0$. Thus $(-1)^{n-1} n(\lambda(\lambda^2T - TT^*) + \sum_{k=1}^{n-2}(-1)^k(T^{n-k}\lambda^k) - (T^{n-k}\lambda^k) = 0$. Put $\lambda = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $r > 0$, we get

\[
(-1)^{n-1} n(re^{i\theta})^{n-1}(T^*T - TT^*) + \sum_{k=1}^{n-2}(-1)^k(n(re^{i\theta})^k(T^{n-k} - T^{n-k}T^*) = 0.
\]

So $(-1)^{n-1}(T^*T - TT^*) + \frac{1}{n(re^{i\theta})^{n-2}}\sum_{k=1}^{n-2}(-1)^k(n(re^{i\theta})^k(T^{n-k} - T^{n-k}T^*) = 0$.

Let $r \rightarrow \infty$. Then $T^*T - TT^* = 0$. Hence $T$ is normal. The converse is trivial.

Proposition 2.20. Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where $A$ and $B$ are selfadjoint operators. Then $T$ is 2-normal operator if and only if $B^2$ commutes with $A$, and $A^2$ commutes with $B$.

$iBA^2 + BAB$ and $T^*T^2 = A^3 - AB^2 + iA^2B + iABA - iBA^2 + iB^3 + BAB + B^2A$.

Since $B^2A = AB^2$ and $A^2B = BA^2$, $T^*T^2 = T^*T^2$. Hence $T$ is 2-normal.

Now let $T$ be 2-normal. So $T^2T^* = T^*T^2$. Hence $-B^2A + iBA^2 - iA^2B + AB^2 = -AB^2 + iABA + iB^3 - BAB + B^2A$. Let $T_1 = AB^2 - B^2A$ and $T_2 = BA^2 - A^2B$. Then $T_1^* = -T_1$, $T_2^* = -T_2$ (i.e., $T_1$, $T_2$ are skew hermitian) and $T_1 + iT_2 = 0$. So $-T_1 + iT_2 = 0$. This gives $-T_1 = AB^2 - B^2A = 0$. Similarly, $B^2A = AB^2$.

It is clear that a 2-normal operator is a 2-k-normal operator and a 3-normal operator is a 3-k-normal operator. The following examples show that a 2-normal operator need not be 3-normal operator and vice versa.

Example 2.21. Let $T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ is a normal operator. But $T^3 = \begin{pmatrix} 8 & 4 \\ 0 & -8 \end{pmatrix}$ is not normal. So $T$ is 2-normal but it is not 3-normal.

Example 2.22. Let $T = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}$. Then $T^3 = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$ is a normal operator. But $T^2 = \begin{pmatrix} 0 & 4 \\ -4 & -4 \end{pmatrix}$ is not normal. So $T$ is 3-normal but it is not 2-normal.

Proposition 2.23. Suppose $T$ is both k-normal and $(k + 1)$-normal for some positive integer $k$. Then $T$ is $(k + 2)$-normal. And hence $T$ is n-normal for all $n \geq k$.

Proof. Since $T$ is k-normal, $T^{k+1}T^* = T^*T^k$. Hence $TT^{k+1}T^* = TT^kT^*T^*$. So $T^{k+1}T^*T^k = TT^kT^*T^{k+1}$. Since $T$ is $(k + 1)$-normal, $T^*T^{k+2} = T^{k+2}T^*$. Thus $T$ is $(k + 2)$-normal.

Corollary 2.24. If $T$ is 2-normal and 3-normal, then $T$ is an n-normal for all $n \geq 2$.

The following example shows a 2-normal and 3-normal operator may not be normal.

Example 2.25. Let $T = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ be an operator acting in two-dimensional complex Hilbert space. Then $T$ is 2-normal, 3-normal, and hence it is n-normal for all $n \geq 2$ but it is not normal.

Proposition 2.26. Suppose $T$ is a k-normal operator for a positive integer $k$ and it is a partial isometry. Then $T$ is a $(k + 1)$-normal operator. And hence $T$ is n-normal for all $n \geq k$. 
Proof. Since \( T \) is partial isometry, \( TT^*T = T \) by [5, p.250]. Hence \( TT^*T^k = T^k \) and \( T^kT^*T = T^k \). Since \( T \) is \( k \)-normal, \( T^{k+1}T^* = T^k \) and \( T^*T^{k+1} = T^k \). Thus \( T^{k+1}T^* = T^*T^{k+1} \). Therefore \( T \) is \( (k+1) \)-normal. And hence by Proposition 2.23 \( T \) is \( n \)-normal for all \( n \geq k \).

Corollary 2.27. If \( T \) is \( 2 \)-normal and partial isometry, then \( T \) is \( n \)-normal for all integer \( n \geq 2 \).

We note that, in Example 2.25 if \( a \) equal to 1, then \( T \) is a \( 2 \)-normal operator and a partial isometry but not normal.

Lemma 2.28. Let \( T \) be \( k \)-normal and \( (k+1) \)-normal. If either \( T \) or \( T^* \) is injective, then \( T \) is normal.

Proof. Since \( T \) is \( (k+1) \)-normal, \( T^{k+1}T^* = T^*T^{k+1} \). And since \( T \) is \( k \)-normal, \( T^{k+1}T^* = T^kT^*T \). Hence \( T^k(TT^* - T^*T) = 0 \). Since \( T \) is injective, \( TT^* - T^*T = 0 \). Thus \( T \) is normal. In case \( T^* \) is injective, since \( T^* \) is \( k \)-normal and \( (k+1) - normal \), \( T^* \) is normal. Hence \( T \) is normal.

Proposition 2.29. Let \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a, b, c, d \in \mathbb{C} \). Then \( T \) is \( 2 \)-normal if and only if \( (a+d) = 0 \) and \( (|b| = |c| \) or \( b(d - \bar{\alpha}) = \bar{\alpha}(d - a) \).

Proof. Suppose \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( 2 \)-normal. Then \( T^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{pmatrix} \) is normal. Hence \( |ab + bc| = |ac + dc| \) and \( (ab + bd)(cd + d^2) - (a^2 + bc) = (ac + dc)((cb + d^2) - (a^2 + bc)). \) Since \( |b(a+d)| = |c(a+d)| \) and \( |(b(a+d))(\bar{c} + \bar{d})| = \bar{c}(a + d)(cb + d^2 - a^2 - bc) \), \( |b||a+d| = |c||a+d| \) and \( |b(a+d)|(\bar{d} - \bar{\alpha})(d - \bar{\alpha}) = \bar{\alpha}(d - \bar{\alpha})(d-a)(a+d). \) Hence \( |b||a+d| = |c||a+d| \) and \( b(a+d)(\bar{d} - \bar{\alpha})(d - \bar{\alpha}) = \bar{\alpha}(\bar{d} - \bar{\alpha})(d-a)(a+d). \) Thus \( |b| = |c| \) or \( |a+d| = 0 \) and \( b(d - \bar{\alpha}) = \bar{\alpha}(d - a) \) or \( |a+d|^2 = 0 \).

By giving similar arguments that in the last Proposition one can prove the following.

Proposition 2.30. Let \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a, b, c, d \in \mathbb{C} \). Then \( T \) is \( 3 \)-normal if and only if \( (a^2 + bc + ad + d^2) = 0 \) and \( |b| = |c| \) or \( \bar{\alpha}(d-a) = b(d-\bar{\alpha}) \).

Next, we characterize when a two-dimensional upper triangular complex matrix is \( n \)-normal.

Proposition 2.31. For \( n \geq 2 \) we have \( T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) is \( n \)-normal if and only if \( b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) = 0 \).
Proof. Let \( T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \). Then \( T \) is \( n \)-normal if and only if

\[
T^n = \begin{pmatrix} a^n & b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) \\ 0 & c^n \end{pmatrix}.
\]

is normal if and only if \(| b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) | = 0 \) if and only if \( b(a^{n-1} + a^{n-2}c + \ldots + c^{n-1}) = 0 \).

\[\square\]

Example 2.32. Consider \( n = 3 \) in the last Proposition. Then \( T \) is a \( 3 \)-normal operator if and only if \( b(a^2 + ac + c^2) = 0 \). Take \( a = 2 \), \( b = 1 \), and \( c = -1 + \sqrt{3}i \). Then \( T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix} \) is \( 3 \)-normal. Note that \( T^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \) is normal. Thus \( T \) is \( 3 \)-normal.

We note that by use the last Proposition we may get an \( n \)-normal operator but not normal.

Proposition 2.33. Let \( T \in B(H) \), \( F = T^n + T^* \), and \( G = T^n - T^* \). Then \( T \) is an \( n \)-normal operator if and only if \( G \) commutes with \( F \).

Proof. \( FG = GF \) if and only if \((T^n + T^*)(T^n - T^*) = (T^n - T^*)(T^n + T^*)\) if and only if \( T^{2n} - T^nT^* + T^nT^n - T^{*2} = T^{2n} + T^nT^* - T^nT^n - T^{*2} \) if and only if \( T^nT^* - T^nT^n = 0 \) if and only if \( T \) is an \( n \)-normal.

\[\square\]

Proposition 2.34. Let \( T \in B(H) \), \( B = T^nT^* \), \( F = T^n + T^* \), and \( G = T^n - T^* \). If \( T \) is an \( n \)-normal, then \( B \) commutes with \( F \) and \( G \).

Proof. Since \( T \) is an \( n \)-normal, \( BF = T^nT^*(T^n + T^*) = T^nT^nT^* + T^nT^*T^n = T^nT^nT^n + T^nT^*T^n = (T^n + T^*)T^nT^* = FB \). By similar way we can prove that \( BG = GB \).

\[\square\]

Proposition 2.35. Let \( T \) be a weighted shift with nonzero weights \( \{\alpha_k\}_{k=0}^\infty \). Then \( T \) is \( n \)-normal if and only if \(| \alpha_k | \leq | \alpha_{k+1} | \leq \ldots \leq | \alpha_{k+n+1} | \) for \( k = n, n + 1, \ldots \).

Proof. Let \( \{e_k\}_{k=0}^\infty \) be an orthogonal basis of Hilbert space \( H \). Since \( T^n e_k = \alpha_k \ldots \alpha_{k+n-1} e_k \) and \( T^n e_k = \alpha_k \ldots \alpha_{k+n-1} e_k \), \( T^nT^n e_k = | \alpha_k |^2 \ldots | \alpha_{k+n-1} |^2 e_k \) and \( T^nT^n e_k = | \alpha_k |^2 \ldots | \alpha_{k+n-1} |^2 e_k \). Thus \( T^n \) is normal if and only if \(| \alpha_k |^2 \ldots | \alpha_{k+n-1} |^2 = | \alpha_{k-1} |^2 \ldots | \alpha_{k-n} |^2 \) for \( k = n, n + 1, \ldots \).

\[\square\]

Proposition 2.36. Let \( T \in B(H) \) be an \( n \)-normal operator and invertible. Then \( T \) and \( T^{-1} \) have a common nontrivial closed invariant subspace.
Proof. Since $T$ is $n$-normal and invertible, $T^n$ and $(T^{-1})^n$ are normal. Hence by [1, Corollary 4.5] $T^n$ and $(T^{-1})^n$ both have no hypercyclic vector. Thus by [7], $T$ and $T^{-1}$ both have no hypercyclic vector. Therefore by [2], $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace.

Let $\lambda$ be the coordinate in $\mathbb{C}$ and $d_\mu(\lambda)$, denotes planar Lebesgue measure. Let $D$ be a bounded open subset of $\mathbb{C}$. We shall denote by $L^2(D, H)$ the Hilbert space of measurable function $f : D \rightarrow H$ such that

$$\|f\|_{2,D} = \left\{ \int_D \|f(\lambda)\|^2 d_\mu(\lambda) \right\}^{\frac{1}{2}} < \infty.$$  

The space of functions $f \in L^2(D, H)$ that are analytic in $D$ (i.e., $\overline{\partial}f = 0$) is denoted by

$$A^2(D, H) = L^2(D, H) \cap \hat{\mathcal{O}}(U, H).$$

$A^2(D, H)$ is called the Bergman space for $D$.

Let $D$ be a bounded open subset of $D$ and $m$ a fixed non-negative integer. The vector valued Sobolev space $W^m(D, H)$ with respect to $\overline{\partial}$ and of order $m$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\overline{\partial}f, ..., \overline{\partial}^m f$ in the sense of distributions also belong to $L^2(D, H)$. Endowed with the norm

$$\|f\|_{W^m} = \sum_{i=0}^m \|\overline{\partial}^i f\|^2_{2,D},$$

$W^m(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$.

**Theorem 2.37.** Let $D$ be an arbitrary bounded disk in $\mathbb{C}$. If $T \in B(H)$ is $2$-normal with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then the operator

$$\lambda - T : W^2(D, H) \rightarrow L^2(D, H)$$

is one to one.

**Proof.** Let $f \in W^2(D, H)$ such that $(\lambda - T)f = 0$ i.e.,

$$\| (\lambda - T) f \|_{W^2} = 0. \quad (1)$$

Then, for $i = 1, 2$, we have

$$\| (\lambda - T) \overline{\partial} f \|_{2,D} = 0. \quad (2)$$

Hence for $i = 1, 2$, we get $\| (\lambda^2 - T^2) \overline{\partial} f \|_{2,D} = 0$. For $i = 1, 2$. Since $T^2$ is normal,

$$\| (\lambda^2 - T^2) \overline{\partial} f \|_{2,D} = 0. \quad (3)$$

Since $\lambda - T$ is invertible for $\lambda \in D \setminus \sigma(T)$, the equation 2 implies that $\| \overline{\partial} f \|_{2,D \setminus \sigma(T)} = 0$. Therefore

$$\| (\lambda - T) \overline{\partial} f \|_{2,D \setminus \sigma(T)} = 0. \quad (4)$$
Since \( \sigma(T) \cap (-\sigma(T)) = \emptyset \) and \( \sigma(T^*) = \sigma(T)^* \), \( \bar{\lambda} + T^* \) is invertible for \( \lambda \in \sigma(T) \). Therefore, from equation 3, we have

\[
\| (\bar{\lambda} - T^*) \partial_i f \|_{2,\sigma(T)} = 0. 
\]  
(5)

Hence from 4 and 5, we get

\[
\| (\bar{\lambda} - T^*) \partial_i f \|_{2,D} = 0. 
\]  
(6)

By [6, Proposition 2.1], we obtain

\[
\| (I - P) f \|_{2,D} = 0, 
\]  
(7)

where \( P \) denotes the orthogonal projection of \( L^2(D, H) \) onto the Bergman space \( A^2(D, H) \). Hence \( (\lambda - T) P f = (\lambda - T) f = 0 \). Since \( T \) has SVEP, \( f = P f = 0 \). Hence \( \lambda - T \) is one to one.

**Lemma 2.38.** Let \( T \in B(H) \) be an 2-normal operator with property for \( \sigma(T) \cap (-\sigma(T)) = \emptyset \). If \( V \) is an isometry, then the operator \( \lambda - VTV^* : W^2(D, H) \longrightarrow L^2(D, H) \) is one to one. \( \square \)

**Proof.** Let \( f \in W^2(D, H) \) such that \( (\lambda - VTV^*) f = 0 \). Then \( (\lambda - T)V f = 0 \). Hence for \( i = 0, 1, 2 \), \( (\lambda - T)V^* \partial f = 0 \). By Theorem 2.37, for \( i = 0, 1, 2 \), \( V^* \partial f = 0 \). Hence for \( i = 0, 1, 2 \), \( VTV^* \partial f = 0 \). Thus \( \lambda \partial f = 0 \) for \( i = 0, 1, 2 \). By [6, Proposition 2.1] with \( T = (0) \), we get \( \| (I - P) f \|_{2,D} = 0 \), where \( P \) denotes the orthogonal projection of \( L^2(D, H) \) onto the Bergman space \( A^2(D, H) \). Hence \( \lambda f = \lambda P f = 0 \). By [4, Corollary 10.7], there exists a constant \( c > 0 \) such that

\[
c \| Pf \|_{2,D} \leq \| \lambda Pf \|_{2,D} = 0. \]  
So \( f = P f = 0 \). Thus \( \lambda - VTV^* \) is one to one.

\( \square \)

**Proposition 2.39.** Let \( T \in B(H) \) be an 2-normal operator. If \( T \) is quasinilpotent, then \( T \) is nilpotent, and hence \( T \) is subscalar.

**Proof.** Since \( T \) is quasinilpotent, \( \sigma(T) = \{0\} \). Hence by the spectral mapping theorem we get \( \sigma(T^n) = \sigma(T)^n = \{0\} \). Thus \( T^n \) is quasinilpotent and normal. So \( T^n = 0 \) i.e., \( T \) is nilpotent and \( T \) is algebraic operator and hence by [3], \( T \) is subscalar.

\( \square \)

**Proposition 2.40.** Let \( T \in B(H) \) be a 2-normal Operator with the property that \( \sigma(T) \cap (-\sigma(T)) = \emptyset \). Then \( T \) is subscalar of order 2.
Proof. Consider an arbitrary bounded disk $D \subset \mathbb{C}$ which contains $\sigma(T)$ and the quotient space $H(D) = W^2(D, H)/(\lambda - T)W^2(D, H)$ endowed with the Hilbert space norm. The class of a vector or an operator $A$ on $H(D)$ will be denoted respectively by $\tilde{f}$, $A$. Let $M$ be the operator of multiplication by $\lambda$ on $W^2(D, H)$. Then $M$ is a scalar operator of order 2 and has a spectral distribution $\phi$. Let $S = M$. Since $(\lambda - T)W^2(D, H)$ is invariant under every operator $M_f$, $f \in C^2_0(C)$, we infer that $S$ is a scalar operator of order 2 with spectral distribution $\phi$.

Consider the natural map $V : H \rightarrow H(D)$ denoted by $Vh = 1 \otimes h$, for $h \in H$, where $1 \otimes h$ denotes the constant function sending $\lambda \in D$ to $h$. Then $VT = SV$. In particular $R(V)$ is an invariant subspace for $S$. Now we shall prove that $V$ is one to one and has closed range.

Let $\{h_n\}, \{f_n\}$ be sequences respectively in $H$, $W^2(D, H)$ such that
\[\lim_{n\to\infty} \|(\lambda - T)f_n + 1 \otimes h\|_{W^2} = 0. \tag{8}\]

It suffices to show that $\lim_{n\to\infty} h_n = 0$.

By the definition of the norm of Sobolev space (8) implies that
\[\lim_{n\to\infty} \|(\lambda - T)\overline{f}_n\|_{2,D} = 0. \tag{9}\]

\[\lim_{n\to\infty} \|(\lambda - T)\overline{f}_n\|_{2,D} = 0 \text{ since } T^2 \text{ is normal, for } i = 1, 2\]

\[\lim_{n\to\infty} \|(\lambda^2 - T^*T)\overline{f}_n\|_{2,D} = 0. \tag{10}\]

Since $\lambda - T$ invertible for $\lambda \in D \setminus \sigma(T)$, (9) implies that $\lim_{n\to\infty} \|\overline{f}_n\|_{2,D \setminus \sigma(T)} = 0$. Therefore
\[\lim_{n\to\infty} \|(\lambda - T^*)\overline{f}_n\|_{2,D \setminus \sigma(T)} = 0. \tag{11}\]

Since for $\sigma(T) \cap (-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\lambda + T^*$ is invertible for $\lambda \in \sigma(T)$. Therefore from (10) we have
\[\lim_{n\to\infty} \|(\lambda - T^*)\overline{f}_n\|_{2,\sigma(T)} = 0. \tag{12}\]

Hence by (11) and (12) we get
\[\lim_{n\to\infty} \|(\lambda - T^*)\overline{f}_n\|_{2,D} = 0. \tag{13}\]

By [6, Proposition 2.1], we obtain
\[\lim \| (I - P) f_n \|_{2,D} = 0, \tag{14}\]

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Substituting (14) into (8), we get $\lim_{n\to\infty} \|(\lambda - T)Pf_n + 1 \otimes h_n\|_{2,D} = 0$.

Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T)$. Then for $\lambda \in \Gamma$
On $n$-normal operators

$$\lim_{n \to \infty} \|P f_n(\lambda) + (\lambda - T)^{-1}(1 \otimes h)\| = 0$$
uniformly. Hence by Riesz-Dunford functional

$$\lim_{n \to \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} P f_n(\lambda) d\lambda + h_n \right\| = 0.$$

But since $\frac{1}{2\pi i} \int_{\Gamma} P f_n(\lambda) d\lambda = 0$, by Cauchy’s theorem calculus, $\lim_{n \to \infty} h_n = 0$. Thus $V$ is one to one and has closed range. \qed

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