A Study on \((Q, L)\) - Fuzzy Subsemiring of a Semiring

S. Sampathu\(^1\), S. Anita Shanthi\(^2\) and A. Praveen Prakash\(^3\)

\(^1\)Department of Mathematics, Sri Muthukumaran College of Education
Chikkarayapuram, Chennai – 600069, Tamil Nadu, India
E-mail: sampathugokul@yahoo.in

\(^2\)Department of Mathematics, Annamalai University
Annamalainagar – 608002, Tamil Nadu, India
E-mail: shanthi.Anita@yahoo.com

\(^3\)Department of Mathematics, Hindustan University
Padur, Chennai - 603103, Tamil Nadu, India
E-mail: apraveenprakash@gmail.com

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Abstract

In this paper, we introduce the concept of \((Q,L)\)- fuzzy subsemirings of a semiring and establish some results on these. We also made an attempt to study the properties of \((Q,L)\)-fuzzy subsemirings of semiring under homomorphism and anti-homomorphism, and study the main theorem for this. We shall also give new results on this subject.

Keywords: \((Q,L)\)-fuzzy subset, \((Q,L)\)-fuzzy subsemiring, \((Q,L)\)-fuzzy relation, Product of \((Q,L)\)-fuzzy subsets, pseudo \((Q,L)\)-fuzzy coset, \((Q,L)\)-anti-fuzzy subsemiring.
Introduction

There are many concepts of universal algebras generalizing an associative ring \((R; +, \cdot)\). Some of them in particular, nearrings and several kinds of semirings have been proven very useful. An algebra \((R; +, \cdot)\) is said to be a semiring if \((R; +)\) and \((R; \cdot)\) are semigroups satisfying \(a.(b+c)=a.b+a.c\) and \((b+c).a=b.a+c.a\) for all \(a, b, c\) in \(R\). A semiring \(R\) is said to be additively commutative if \(a+b=b+a\) for all \(a, b\) in \(R\). A semiring \(R\) may have an identity \(1\), defined by \(1 \cdot a = a = a \cdot 1\) and a zero \(0\), defined by \(0+a=a=a+0\) and \(a.0=0=0.a\) for all \(a\) in \(R\). After the introduced of fuzzy sets by L.A. Zadeh [7], several researchers explored on the generalization of the notion of fuzzy set. Azriel Rosenfeld [2] defined a fuzzy group. Asok Kumer Ray [1] defined a product of fuzzy subgroups and Fuzzy subgroups and Anti-fuzzy subgroups have introduced R. Biswas [14] A. Solairaju and R. Nagarajan [3] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. We introduce the concept of \((Q, L)\)-fuzzy subsemiring of a semiring and established some results.

1 Preliminaries:

1.1 Definition: Let \(X\) be a non–empty set. A fuzzy subset \(A\) of \(X\) is a function \(A : X \rightarrow [0, 1]\).

1.2 Definition: Let \(X\) be a non-empty set and \(L = (L, \leq)\) be a lattice with least element 0 and greatest element 1 and \(Q\) be a non-empty set. A \((Q, L)\)-fuzzy subset \(A\) of \(X\) is a function \(A: X \times Q \rightarrow L\).

1.3 Definition: Let \((R, +, \cdot)\) be a semiring and \(Q\) be a non-empty set. A \((Q, L)\)-fuzzy subset \(A\) of \(R\) is said to be a \((Q, L)\)-fuzzy subsemiring (QLFSSR) of \(R\) if the following conditions are satisfied:

(i) \(A(x+y, q) \geq A(x, q) \land A(y, q),\)
(ii) \(A(xy, q) \geq A(x, q) \land A(y, q),\) for all \(x\) and \(y\) in \(R\) and \(q\) in \(Q\).

1.4 Definition: Let \(A\) and \(B\) be any two \((Q, L)\)-fuzzy subsets of sets \(R\) and \(H\), respectively. The product of \(A\) and \(B\), denoted by \(A \times B\), is defined as \(A \times B = \{(x, y) \in \{x \in R \land y \in H\}, q \in Q\}, \) where \(A \times B((x, y), q) = A(x, q) \land B(y, q)\).

1.5 Definition: Let \((R, +, \cdot)\) and \((R', +, \cdot)\) be any two semirings and \(Q\) be a non empty set. Let \(f: R \rightarrow R'\) be any function and \(A\) be a \((Q, L)\)-fuzzy subsemiring in \(R\), \(V\) be a \((Q, L)\)-fuzzy subsemiring in \(f(R)=R'\), defined by \(V(y, q) = \sup_{x \in f^{-1}(y)} A(x, q)\), for all \(x\) in \(R\) and \(y\) in \(R'\) and \(q\) in \(Q\). Then \(A\) is called a pre-image of \(V\) under \(f\) and is denoted by \(f^{-1}(V)\).
1.6 Definition: Let $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $(R, +, \cdot)$ and $a$ in $R$. Then the pseudo $(Q, L)$-fuzzy coset $(aA)_p^q$ is defined by $(aA)_p^q(x, q) = p(a)A(x, q)$, for every $x$ in $R$ and for some $p$ in $P$ and $q$ in $Q$.

1.7 Definition: Let $A$ be a $(Q, L)$-fuzzy subset in a set $S$, the strongest $(Q, L)$-fuzzy relation on $S$, that is a $(Q, L)$-fuzzy relation $V$ with respect to $A$ given by $V((x, y), q) = A(x, q) \land A(y, q)$, for all $x$ and $y$ in $S$ and $q$ in $Q$.

1.8 Definition: Let $(R, +, \cdot)$ be a semiring and $Q$ be a non empty set. A $(Q, L)$-fuzzy subset $A$ of $R$ is said to be a $(Q, L)$-anti-fuzzy subsemiring (QLAFSSR) of $R$ if the following conditions are satisfied:

(i) $A(x+y, q) \leq A(x, q) \lor A(y, q)$,
(ii) $A(xy, q) \leq A(x, q) \lor A(y, q)$, for all $x$ and $y$ in $R$ and $q$ in $Q$.

1.9 Definition: Let $X$ be a non-empty set and $A$ be a $(Q, L)$-fuzzy subsemiring of a semiring $R$. Then $A^0$ is defined as $A^0(x, q) = A(x, q)/A(0, q)$, for all $x$ in $R$ and $q$ in $Q$, where $0$ is the identity element of $R$.

1.10 Definition: Let $A$ be a $(Q, L)$-fuzzy subset of $X$. For $\alpha$ in $L$, a $Q$-level subset of $A$ is the set $A_\alpha = \{ x \in X : A(x, q) \geq \alpha \}$.

2 Properties of $(Q, L)$-Fuzzy Subsemiring of a Semiring

2.1 Theorem: If $A$ and $B$ are two $(Q, L)$-fuzzy subsemiring of a semiring $R$, then their intersection $A \cap B$ is a $(Q, L)$-fuzzy subsemiring of $R$.

Proof: Let $x$ and $y$ belongs to $R$ and $q$ in $Q$, $A = \{ <(x, q), A(x, q) > \}$ in $R$ and $q$ in $Q$ and $B = \{ <(x, q), B(x, q) > \}$ in $R$ and $q$ in $Q$. Let $C = A \cap B$ and $C = \{ <(x, q), C(x, q) > \}$ in $R$ and $q$ in $Q$.

(i) $C(x+y, q) = A(x+y, q) \land B(x+y, q) \geq \{ A(x, q) \land A(y, q) \} \land \{ B(x, q) \land B(y, q) \} \geq \{ A(x, q) \land B(x, q) \} \land \{ A(y, q) \land B(y, q) \} = C(x, q) \land C(y, q)$.

Therefore, $C(x+y, q) \geq C(x, q) \land C(y, q)$, for all $x$ and $y$ in $R$ and $q$ in $Q$.

(ii) $C(xy, q) = A(xy, q) \land B(xy, q) \geq \{ A(x, q) \land A(y, q) \} \land \{ B(x, q) \land B(y, q) \} \geq \{ A(x, q) \land B(x, q) \} \land \{ A(y, q) \land B(y, q) \} = C(x, q) \land C(y, q)$.

Therefore, $C(xy, q) \geq C(x, q) \land C(y, q)$, for all $x$ and $y$ in $R$ and $q$ in $Q$. Hence $A \cap B$ is a $(Q, L)$-fuzzy subsemiring of a semiring $R$.

2.2 Theorem: The intersection of a family of $(Q, L)$-fuzzy subsemiring of a semiring $R$ is a $(Q, L)$-fuzzy subsemiring of $R$. 
Proof: Let \( \{A_i\}_{i \in I} \) be a family of \((Q, L)\)-fuzzy subsemiring of a semiring \( R \) and \( A = \bigcap_{i \in I} A_i \). Then for \( x \) and \( y \) belongs to \( R \) and \( q \) in \( Q \), we have \( A(x+y, q) = \inf_{i \in I} A_i(x, q) \wedge A_i(y, q) \). Therefore, \( A(x+y, q) \geq \inf_{i \in I} A_i(x, q) \wedge A_i(y, q) \). Hence the intersection of a family of \((Q, L)\)-fuzzy subsemiring of a semiring \( R \) is a \((Q, L)\)-fuzzy subsemiring of \( R \).

2.3 Theorem: If \( A \) and \( B \) are \((Q, L)\)-fuzzy subsemiring of a semiring \( R \) and \( H \), respectively, then \( A \times B \) is a \((Q, L)\)-fuzzy subsemiring of \( R \times H \).

Proof: Let \( A \) and \( B \) be \((Q, L)\)-fuzzy subsemiring of a semiring \( R \) and \( H \), respectively. Let \( x_1 \) and \( x_2 \) be in \( R \), \( y_1 \) and \( y_2 \) be in \( H \). Then \( (x_1, y_1) \) and \( (x_2, y_2) \) are in \( R \times H \) and \( q \) in \( Q \). Now, \( A \times B[(x_1, y_1) + (x_2, y_2), q] = A \times B((x_1 + x_2, y_1 + y_2), q) = A(x_1 + x_2, q) \wedge B(y_1 + y_2, q) \geq \{ A(x_1, q) \wedge B(y_1, q) \} \wedge \{ A(x_2, q) \wedge B(y_2, q) \} = A \times B((x_1, y_1), q) \wedge A \times B((x_2, y_2), q). \)

Therefore, \( A \times B[(x_1, y_1) + (x_2, y_2), q] \geq A \times B((x_1, y_1), q) \wedge A \times B((x_2, y_2), q). \)

Hence \( A \times B \) is a \((Q, L)\)-fuzzy subsemiring of \( R \times H \).

2.4 Theorem: Let \( A \) be a \((Q, L)\)-fuzzy subset of a semiring \( R \) and \( V \) be the strongest \((Q, L)\)-fuzzy relation of \( R \). Then \( A \) is an \((Q, L)\)-fuzzy subsemiring of \( R \) if and only if \( V \) is an \((Q, L)\)-fuzzy subsemiring of \( R \times R \).

Proof: Suppose that \( A \) is an \((Q, L)\)-fuzzy subsemiring of a semiring \( R \). Then for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are in \( R \times R \) and \( q \) in \( Q \). We have,

\[
V(x+y, q) = V((x_1, x_2) + (y_1, y_2), q) = V((x_1 + y_1, x_2 + y_2), q) = A((x_1 + y_1, q) \wedge A((x_2 + y_2, q) \geq \{ A(x_1, q) \wedge A(y_1, q) \} \wedge \{ A(x_2, q) \wedge A(y_2, q) \} = V((x_1, x_2), q) \wedge V(y_1, y_2), q) = V(x, q) \wedge V(y, q).
\]
Therefore, $V(x+y,q) \geq V(x,q) \land V(y,q)$, for all $x$ and $y$ in $R \times R$.

And,

$V(xy,q) = V((x_1y_1, x_2y_2),q) = V((x_1,y_1, x_2, y_2),q) = A(x_1,y_1,q) \land A(x_2,y_2,q) \geq [A(x_1,q) \land A(x_2,q)] \land [A(y_1,q) \land A(y_2,q)] = V((x_1,y_1),q) \land V((x_2,y_2),q) = V(x,q) \land V(y,q)$.

Therefore, $V(xy,q) \geq V(x,q) \land V(y,q)$, for all $x$ and $y$ in $R \times R$. This proves that $V$ is an $(Q,L)$-fuzzy subsemiring of $R \times R$. Conversely assume that $V$ is an $(Q,L)$-fuzzy subsemiring of $R \times R$, then for any $x=(x_1,x_2)$ and $y=(y_1,y_2)$ are in $R \times R$, we have

$A((x_1+y_1),q) \land A(x_2,y_2,q) = V((x_1+y_1),x_2, y_2),q) = V((x_1,y_1, x_2, y_2),q) = [A(x_1,q) \land A(x_2,q)] \land [A(y_1,q) \land A(y_2,q)]$

If $A((x_1+y_1),q) \leq A(x_2,y_2),q) = A(x_1,q) \land A(y_1,q) \leq A(x_2,q) \land A(y_2,q)$, we get,

$A((x_1+y_1),q) \geq A((x_1,q) \land A(y_1,q)$, for all $x_1$ and $y_1$ in $R$.

And,

$A((x_1y_1),q) \land A(x_2y_2,q) = V((x_1y_1, x_2y_2),q) = V((x_1,y_1, x_2, y_2),q) = [A(x_1,q) \land A(x_2,q)] \land [A(y_1,q) \land A(y_2,q)]$

If $A(x_1y_1,q) \leq A(x_2y_2),q) = A(x_1,q) \land A(x_2,q) \leq A(y_1,q) \land A(y_2,q)$, we get $A(x_1y_1,q) \geq A(x_1,q) \land A(y_1,q)$, for all $x_1$, $y_1$ in $R$. Therefore $A$ is an $(Q,L)$-fuzzy subsemiring of $R$.

2.5 Theorem: $A$ is an $(Q,L)$-fuzzy subsemiring of a semiring $(R, +, \cdot)$ if and only if $A((x+y),q) \geq A(x,q) \land A(y,q), A(xy,q) \geq A(x,q) \land A(y,q)$, for all $x$ and $y$ in $R$.

Proof: It is trivial.

2.6 Theorem: If $A$ is an $(Q,L)$-fuzzy subsemiring of a semiring $(R, +, \cdot)$, then $H = \{ x \in R : A(x,q) = 1 \}$ is either empty or is a subsemiring of $R$.

Proof: If no element satisfies this condition, then $H$ is empty. If $x$ and $y$ in $H$, then $A((x+y),q) \geq A(x,q) \land A(y,q) = 1 \land 1 = 1$. Therefore, $A((x+y),q) = 1$. And, $A(xy,q) \geq A(x,q) \land A(y,q) = 1 \land 1 = 1$. Therefore, $A(xy,q) = 1$. We get $x+y, xy$ in $H$. Therefore, $H$ is a subsemiring of $R$. Hence $H$ is either empty or is a subsemiring of $R$.

2.7 Theorem: If $A$ be an $(Q,L)$-fuzzy subsemiring of a semiring $(R, +, \cdot)$, then if $A((x+y),q) = 0$, then either $A(x,q) = 0$ or $A(y,q) = 0$, for all $x$ and $y$ in $R$ and $q$ in $Q$.

Proof: Let $x$ and $y$ in $R$ and $q$ in $Q$. By the definition $A((x+y),q) \geq A(x,q) \land A(y,q)$, which implies that $0 \geq A(x,q) \land A(y,q)$. Therefore, either $A(x,q) = 0$ or $A(y,q) = 0$. 
2.8 Theorem: Let A be a \((Q, L)\)-fuzzy subsemiring of a semiring \(R\). Then \(A^0\) is a \((Q, L)\)-fuzzy subsemiring of a semiring \(R\).

Proof: For any \(x \in R\) and \(q \in Q\), we have \(A^0(x+y,q) = A^0(x,q) \wedge A^0(y,q) \geq (1/A(0,q)) [A(x,q) \wedge A(y,q)] = [A(x,q)/A(0,q)] \wedge [A(y,q)/A(0,q)] = A^0(x,q) \wedge A^0(y,q)\).

That is \(A^0(x+y,q) \geq A^0(x,q) \wedge A^0(y,q)\) for all \(x, y \in R\) and \(q \in Q\).

\[A^0(xy,q) = A(x,y,q)/A(0,q) \geq (1/A(0,q)) [A(x,q) \wedge A(y,q)] = [A(x,q)/A(0,q)] \wedge [A(y,q)/A(0,q)] = A^0(x,q) \wedge A^0(y,q)\]

That is \(A^0(xy,q) \geq A^0(x,q) \wedge A^0(y,q)\) for all \(x, y \in R\) and \(q \in Q\). Hence \(A^0\) is a \((Q,L)\)-fuzzy subsemiring of a semiring \(R\).

2.9 Theorem: Let \(A\) be a \((Q, L)\)-fuzzy subsemiring of a semiring \(R\). \(A^+\) be a fuzzy set in \(R\) defined by \(A^+(x,q) = A(x,q) + 1 - A(0,q)\), for all \(x \in R\) and \(q \in Q\), where 0 is the identity element. Then \(A^+\) is an \((Q, L)\)-fuzzy subsemiring of a semiring \(R\).

Proof: Let \(x\) and \(y\) in \(R\) and \(q\) in \(Q\). We have,

\[A^+(x+y,q) = A^+(x,q) + A^+(y,q) \geq A(x,q) + A^0(y,q) = A(x,q) + 1 - A(0,q) = A(x,q) \wedge A^0(y,q)\]

which implies that \(A^+(x+y,q) \geq A^+(x,q) \wedge A^+(y,q)\) for all \(x, y \in R\) and \(q \in Q\).

\[A^+(xy,q) = A(x,y,q)/A(0,q) \geq A(x,q) \wedge A^0(y,q) = A(x,q)/A(0,q) \wedge A^0(y,q)\]

Therefore, \(A^+(xy,q) \geq A^+(x,q) \wedge A^+(y,q)\) for all \(x, y \in R\) and \(q \in Q\). Hence \(A^+\) is an \((Q,L)\)-fuzzy subsemiring of a semiring \(R\).

2.10 Theorem: Let \(A\) be an \((Q, L)\)-fuzzy subsemiring of a semiring \(R\), \(A^+\) be a fuzzy set in \(R\) defined by \(A^+(x,q) = A(x,q) + 1 - A(0,q)\), for all \(x \in R\) and \(q \in Q\), where 0 is the identity element. Then there exists 0 in \(R\) such that \(A(0,q) = 1\) if and only if \(A^+(x,q) = A(x,q)\).

Proof: It is trivial.

2.11 Theorem: Let \(A\) be an \((Q, L)\)-fuzzy subsemiring of a semiring \(R\), \(A^+\) be a fuzzy set in \(R\) defined by \(A^+(x,q) = A(x,q) + 1 - A(0,q)\), for all \(x \in R\) and \(q \in Q\), where 0 is the identity element. Then there exists \(x\) in \(R\) such that \(A^+(x,q) = 1\) if and only if \(x = 0\).

Proof: It is trivial.

2.12 Theorem: Let \(A\) be an \((Q, L)\)-fuzzy subsemiring of a semiring \(R\), \(A^+\) be a fuzzy set in \(R\) defined by \(A^+(x,q) = A(x,q) + 1 - A(0,q)\), for all \(x \in R\) and \(q \in Q\), where 0 is the identity element. Then \((A^+)^+ = A^+\).
**Proof:** Let $x$ and $y$ in $R$ and $q$ in $Q$. We have, $(A^+)^+(x,q) = A^+(x,q) + I - A^+(0,q) = \{A(x,q) + I - A(0,q)\} + I - \{A(0,q) + I - A(0,q)\} = A(x,q) + I - A(0,q) = A^+(x,q)$.

Hence $(A^+)^+ = A^+$. 

### 2.13 Theorem:
Let $A$ and $B$ be $(Q,L)$-fuzzy subsets of the sets $R$ and $H$ respectively, and let $\alpha$ in $L$. Then $(A \times B)_\alpha = A_\alpha \times B_\alpha$.

**Proof:** Let $\alpha$ in $L$. Let $(x, y)$ be in $(A \times B)_\alpha$ if and only if $A \times B((x,y),q) \geq \alpha$, if and only if $A(x,q) \wedge B(x,q) \geq \alpha$, if and only if $x \in A_\alpha$ and $y \in B_\alpha$, if and only if $(x, y) \in A_\alpha \times B_\alpha$. Therefore, $(A \times B)_\alpha = A_\alpha \times B_\alpha$.

In the following Theorem $\circ$ is the composition operation of functions:

### 2.14 Theorem:
Let $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $H$ and $f$ is an isomorphism from a semiring $R$ onto $H$. Then $A \circ f$ is an $(Q, L)$-fuzzy subsemiring of $R$.

**Proof:** Let $x$ and $y$ in $R$ and $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $H$ and $Q$ be a non-empty set. Then we have,

$(A \circ f)((x+y), q) = A(f(x+y), q) = A(f(x,q) + f(y,q)) \geq A(f(x,q)) \wedge A(f(y,q)) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$, which implies that $(A \circ f)((x+y), q) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$.

And $(A \circ f)(xy,q) = A(f(xy), q) = A(f(x,q)f(y,q)) \geq A(f(x,q)) \wedge A(f(y,q)) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$, which implies that $(A \circ f)(xy,q) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$.

Therefore $(A \circ f)$ is an $(Q, L)$-fuzzy subsemiring of a semiring $R$.

### 2.15 Theorem:
Let $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $H$ and $f$ is an anti-isomorphism from a semiring $R$ onto $H$. Then $A \circ f$ is an $(Q, L)$-fuzzy subsemiring of $R$.

**Proof:** Let $x$ and $y$ in $R$ and $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $H$ and $Q$ be a non-empty set. Then we have,

$(A \circ f)((x+y), q) = A(f(x+y), q) = A(f(y,q) + f(x,q)) \geq A(f(y,q)) \wedge A(f(x,q)) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$, which implies that $(A \circ f)((x+y), q) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$.

And $(A \circ f)(xy,q) = A(f(xy), q) = A(f(y,q)f(x,q)) \geq A(f(x,q)) \wedge A(f(y,q)) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$, which implies that $(A \circ f)(xy,q) \geq (A \circ f)(x,q) \wedge (A \circ f)(y,q)$. Therefore $A \circ f$ is an $(Q, L)$-fuzzy subsemiring of a semiring $R$.

### 2.16 Theorem:
Let $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $(R, +, \cdot)$, then the pseudo $(Q, L)$-fuzzy coset $(aA)^p$ is an $(Q, L)$-fuzzy subsemiring of a semiring $R$, for $a$ in $R$ and $p$ in $P$. 

Proof: Let $A$ be an $(Q, L)$-fuzzy subsemiring of a semiring $R$. For every $x$ and $y$ in $R$ and $q$ in $Q$, we have,

$$((aA)^p)(x+y,q)=p(a)\cap A(x+y,q) \geq p(a)\cap (A(x,q) \wedge A(y,q)) = (((aA)^p)(x,q) \wedge ((aA)^p)(y,q)).$$

Therefore, $((aA)^p)(x+y,q) \geq (((aA)^p)(x,q) \wedge ((aA)^p)(y,q))$.

Hence $(aA)^p$ is an $(Q, L)$-fuzzy subsemiring of a semiring $R$.

2.17 Theorem: Let $(R, +, .)$ and $(R', +, .)$ be any two semirings $Q$ be a non-empty set. The homomorphic image of an $(Q, L)$-fuzzy subsemiring of $R$ is an $(Q, L)$-fuzzy subsemiring of $R'$.

Proof: Let $(R, +, .)$ and $(R', +, .)$ be any two semirings. Let $f: R \rightarrow R'$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $V = f(A)$, where $A$ is an $(Q,L)$-fuzzy subsemiring of $R$. We have to prove that $V$ is an $(Q,L)$-fuzzy subsemiring of $R'$. Now, for $f(x)$, $f(y)$ in $R'$, $V(f(x)+f(y),q) = V(f(x+y),q) \geq A((x+y),q) \wedge A(y,q)$ which implies that $V(f(x)+f(y),q) \geq V(f(x),q) \wedge V(f(y),q)$.

Again, $V(f(x)f(y),q) = V(f(xy),q) \geq A(x,q) \wedge A(y,q)$ which implies that $V(f(x)f(y),q) \geq V(f(x),q) \wedge V(f(y),q)$. Hence $V$ is an $(Q, L)$-fuzzy subsemiring of $R'$.

2.18 Theorem: Let $(R, +, .)$ and $(R', +, .)$ be any two semirings $Q$ be a non-empty set. The homomorphic preimage of an $(Q, L)$-fuzzy subsemiring of $R'$ is an $(Q, L)$-fuzzy subsemiring of $R$.

Proof: Let $(R, +, .)$ and $(R', +, .)$ be any two semirings. Let $f: R \rightarrow R'$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $V = f(A)$, where $V$ is an $(Q,L)$-fuzzy subsemiring of $R'$. We have to prove that $A$ is an $(Q,L)$-fuzzy subsemiring of $R$. Let $x$ and $y$ in $R$ and $q$ in $Q$. Then, $A(x+y,q) = V(f(x+y),q) = V(f(x)+f(y),q) \wedge V(f(y),q) = A(x,q) \wedge A(y,q)$ which implies that $A(x+y,q) \geq A(x,q) \wedge A(y,q)$.

Again, $A(xy,q) = V(f(xy),q) = V(f(x)f(y),q) \geq V(f(x),q) \wedge V(f(y),q) = A(x,q) \wedge A(y,q)$ which implies that $A(xy,q) \geq A(x,q) \wedge A(y,q)$.

Hence $A$ is an $(Q, L)$-fuzzy subsemiring of $R$. 
2.19 Theorem: Let \((R, +, \cdot)\) and \((R', +, \cdot)\) be any two semirings \(Q\) be a non-empty set. The anti-homomorphic image of an \((Q, L)\)-fuzzy subsemiring of \(R\) is an \((Q, L)\)-fuzzy subsemiring of \(R'\).

Proof: Let \((R, +, \cdot)\) and \((R', +, \cdot)\) be any two semirings. Let \(f: R \rightarrow R'\) be an anti-homomorphism. Then, \(f(x+y) = f(y)+f(x)\) and \(f(xy) = f(y)f(x)\), for all \(x, y \in R\) and \(q\) in \(Q\). Let \(V = f(A)\), where \(A\) is an \((Q, L)\)-fuzzy subsemiring of \(R\). We have to prove that \(V\) is an \((Q, L)\)-fuzzy subsemiring of \(R'\). Now, for \(x, y\) in \(R\), \(V(f(x)+f(y), q) \geq V(f(y+x), q) \geq A(y+x, q) \geq A(y, q) \land A(x, q) = A(x, q) \land A(y, q)\) which implies that \(V(f(x)+f(y), q) \geq V(f(x), q) \land V(f(y), q)\).

Again, \(V(f(x)f(y), q) = V(f(xy), q) \geq A(xy, q) \geq A(y, q) \land A(x, q) = A(x, q) \land A(y, q)\) which implies that \(V(f(x)f(y), q) \geq V(f(x), q) \land V(f(y), q)\). Hence \(V\) is an \((Q, L)\)-fuzzy subsemiring of \(R'\).

2.20 Theorem: Let \((R, +, \cdot)\) and \((R', +, \cdot)\) be any two semirings \(Q\) be a non-empty set. The anti-homomorphic preimage of an \((Q, L)\)-fuzzy subsemiring of \(R'\) is an \((Q, L)\)-fuzzy subsemiring of \(R\).

Proof: Let \((R, +, \cdot)\) and \((R', +, \cdot)\) be any two semirings. Let \(f: R \rightarrow R'\) be an anti-homomorphism. Then, \(f(x+y) = f(y)+f(x)\) and \(f(xy) = f(y)f(x)\), for all \(x, y \in R\) and \(q\) in \(Q\). Let \(V = f(A)\), where \(V\) is an \((Q, L)\)-fuzzy subsemiring of \(R'\). We have to prove that \(A\) is an \((Q, L)\)-fuzzy subsemiring of \(R\). Let \(x, y \in R\) and \(q\) in \(Q\).

Then \(A(x+y, q) = V(f(x+y), q) = V(f(x)+f(y), q) \geq V(f(y), q) \land V(f(x), q) = V(f(y), q) \land V(f(x), q) = V(f(x), q) \land V(f(y), q) = A(x, q) \land A(y, q)\), which implies that \(A(x+y, q) \geq A(x, q) \land A(y, q)\).

Again, \(A(xy, q) = V(f(xy), q) = V(f(y)f(x), q) \geq V(f(x), q) \land V(f(y), q) = V(f(x), q) \land V(f(y), q) = A(x, q) \land A(y, q)\) which implies that \(A(xy, q) \geq A(x, q) \land A(y, q)\).

Hence \(A\) is an \((Q, L)\)-fuzzy subsemiring of \(R\).

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References


