Spectrum of Positive Definite Functions on Product Hypergroups

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Abstract

This paper aims to show that the amenability of $K_1 \times K_2$ is equivalent to the following condition: “If $\varphi$ is a continuous positive definite function defined on $K_1 \times K_2$ and $\varphi \geq 0$ then the constant function $1_{K_1 \times K_2}$ belongs to the spectrum of $\varphi$”, which $K_1$ and $K_2$ are locally compact hypergroups as defined by R. Jewett \cite{1}, with convolutions $*_1, *_2$ respectively. Our study deals with the cases of exponentially bounded product hypergroups and discrete solvable product hypergroups. And study of conditionally exponential convex functions.

Keywords: Product hypergroups, Positive definite functions, Exponentially bounded, Discrete solvable, Conditionally exponential convex functions.

1 Introduction

Let $K$ be a locally compact Hausdorff space, $M(K)$ denote the space of all bounded radon measures, $M^1(K)$ be the subset of all probability measures and $\varepsilon_x$ be the point mass measure of $x \in K$. The support of a measure $\mu$ is
denoted by \( \text{supp} \, \mu \). \( C(K) \) denotes the space of continuous functions on \( K \).

The space \( K \) is called a hypergroup if the following conditions are satisfied:

(H1) There exists a map: \( K \times K \to M^1(K), (x, y) \to \varepsilon_x * \varepsilon_y \), called convolution, which is continuous, where \( M^1(K) \) bears the vague topology.

(H2) \( \text{supp} \, \varepsilon_x * \varepsilon_y \) is compact.

(H3) There exists a homomorphism \( K \to K, x \to x^- \), called involution, such that \( x = (x^-)^- \) and \( (\varepsilon_x * \varepsilon_y)^- = \varepsilon_y^- * \varepsilon_x^- \).

(H4) There exists an element \( e \in K \), called unit element, such that \( \varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x \).

(H5) \( e \in \text{supp} \, \varepsilon_x * \varepsilon_y \) if and only if \( x = y \).

(H6) The map \( (x, y) \to \text{supp} \, \varepsilon_x * \varepsilon_y \) of \( K \times K \) into the space of nonvoid compact subset of \( K \) is continuous, the latter space with topology as given in [2,7].

Let \( K_1 \) and \( K_2 \) are locally compact hypergroups, with convolutions \( *_1, *_2 \) respectively. The cartesian product of \( K_1 \) and \( K_2 \) will take the form

\[
K_1 \times K_2 = \{(x_1, x_2) : x_1 \in K_1, \text{and } x_2 \in K_2\}
\]

with convolution \( * \) defined on \( M(K_1 \times K_2) \) by

\[
\varepsilon(x_1, x_2) * \varepsilon(y_1, y_2) = (\varepsilon_{x_1} *_1 \varepsilon_{y_1}) \times (\varepsilon_{x_2} *_2 \varepsilon_{y_2})
\]

where \( \varepsilon(x_1, x_2) \) is the one point mass measure. And the involution of the product hypergroups is defined by

\[
(x_1, x_2)^- = (x_1^-, x_2^-), \forall (x_1, x_2) \in K_1 \times K_2
\]

finally, the identity element of the product hypergroups is \((e_1, e_2)\), which \( e_1 \) and \( e_2 \) are the identities of \( K_1 \) and \( K_2 \) respectively.

A map \( \varphi \) define on \((K_1 \times K_2)^2\) on to \( \mathbb{R}^+ \) is called positive definite function if

\[
\sum_{i,j=1}^n c_i c_j \varphi((x_1, x_2)_i * (x_1, x_2)_j^-) \geq 0.
\]

where \( \{c_1, c_2, ..., c_n\} \in \mathbb{C}, \{(x_1, x_2)_1, (x_1, x_2)_2, ..., (x_1, x_2)_n\} \in K_1 \times K_2 \).

For an example of positive, positive definite functions on a product hypergroups \( K_1 \times K_2 \) are given by a functions of the form \( f * f^- \), where \( f \) is a positive function on \( K_1 \times K_2 \) with compact support, \( f^- \) is defined by \( f^{-}(x_1, x_2) = \overline{f(x_1, x_2)}^{-1} \) and \( * \) is the convolution, it is easy to see that the function \( f * f^- \) is positive definite.
If $P(K_1 \times K_2)$ be the convex set of all continuous positive-definite functions $\varphi$ on $K_1 \times K_2$ with $\varphi(e_1, e_2) = 1$. The spectrum $sp\varphi$ of $\varphi \in P(K_1 \times K_2)$ can be defined as the set of all indecomposable $\psi \in P(K_1 \times K_2)$ which are limits, in the sense of the topology of uniform converges on compact subsets of $K_1 \times K_2$, of functions of the form

$$(x_1, x_2) \rightarrow \sum_{i,j=1}^{n} c_i c_j^* \epsilon_{(x_1, x_2)i} \ast \epsilon_{(x_1, x_2)j}^\psi \psi(x_1, x_2)$$

where $\{c_1, ..., c_n\} \in \mathbb{C}, \{(x_1, x_2)_1, (x_1, x_2)_2, ..., (x_1, x_2)_n\} \in K_1 \times K_2$.

If $\pi$ denotes the cyclic unitary representation of $K_1 \times K_2$ associated with $\varphi$, then $sp\varphi$ consists of all $\psi \in P(K_1 \times K_2)$ for which $\pi_\psi$ is irreducible and weakly contained in $\pi_\psi$ [2].

Our main subject here is to prove that exponentially bounded product hypergroups and solvable discrete hypergroups satisfy the following property (which we denote by (P)):

(P) If $\varphi \in P(K_1 \times K_2)$ and if $\varphi$ is positive in usual sense, then the constant positive-definite function 1 on $K_1 \times K_2$, $1_{K_1 \times K_2}$, belongs to $sp\varphi$. For connected hypergroups we show that the condition that the hypergroup is amenable is equivalent to the following weaker version $(P^*)$ of P:

$(P^*)$ if $\varphi \in P(K_1 \times K_2)$ and if $\varphi$ is positive, then $1_{K_1 \times K_2} \in sp_d(\varphi)$, where $sp_d(\varphi)$ is the spectrum of $\varphi$ when the domain of $\varphi$ is $(K_1 \times K_2)_d$ (the discrete product hypergroups).

## 2 Exponentially Bounded Hypergroups

Let $\pi$ be a continuous unitary representation of $K_1 \times K_2$ in the Hilbert space $(H_\pi, \langle ., . \rangle)$. A unit vector $\xi \in H_\pi$ will be called a positive vector for $\pi$, if

$$\text{Re} \ \langle \pi(x_1, x_2) \xi, \xi \rangle \geq 0$$

for all $(x_1, x_2) \in K_1 \times K_2$.

So,

$$\text{Re} \ \langle \pi(.) \xi, \xi \rangle \in P(K_1 \times K_2)$$

Now, it is easy to translate (P) into a property of unitary representations with positive vectors. In fact, consider the following property $(P')$ of $K_1 \times K_2$ which is formally stronger than (P):

$(P')$ If $\pi$ is a unitary representation of $K_1 \times K_2$ with a positive vector, then $\pi$ contains weakly $1_{K_1 \times K_2}$. 
Theorem 2.1 \((P)\) and \((P')\) are equivalent for every product hypergroups \(K_1 \times K_2\).

Proof: Let \(\pi\) be a unitary representation of \(K_1 \times K_2\) with a positive vector \(\xi \in H_\pi\). Let \(\varphi (x_1, x_2) = \text{Re} \left( \pi (x_1, x_2) \xi, \xi \right)\), \((x_1, x_2) \in K_1 \times K_2\). If \((P)\) holds, then \(1_{K_1 \times K_2}\) is weakly contained in \(\pi_\varphi\) which is the subrepresentation of \(\pi \oplus \pi\) and this implies that \(1_{K_1 \times K_2}\) is weakly contained in \(\pi\).

A locally compact product hypergroups is called Exponentially bounded if

\[
\lim_{n} |G^n|^\frac{1}{n} = 1
\]

for each compact neighbourhood \(G\) of \((e_1, e_2)\), where \(|.|\) denotes the Haar measure and \(G^n = \{g_1, ..., g_n; g_i \in G\}\). Exponentially bounded hypergroups are amenable[4].

Theorem 2.2 Exponentially bounded product hypergroups satisfy property \((P)\).

Proof: Let \(K_1 \times K_2\) be an exponentially bounded product hypergroups and let \(\varphi \in P (K_1 \times K_2)\), with \(\varphi \geq 0\). Let \(G\) be a compact neighbourhood of \((e_1, e_2)\) with the condition \(G = G^{-1}\), and \(\epsilon > 0\). Then there is an \(n \in N\) such that

\[
\int_{G^{n+1} \times G^{n+1}} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-} (\varphi) \, d(y_1, y_2) d(z_1, z_2) \\
\leq (1 + \epsilon) \int_{G^n \times G^n} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-} (\varphi) \, d(y_1, y_2) d(z_1, z_2) \quad (1)
\]

where \(d(y_1, y_2)\), and \(d(z_1, z_2)\) are Haar measures on \(K_1 \times K_2\).

In fact, otherwise

\[
|G^{n+1}|^2 \geq \int_{G^{n+1} \times G^{n+1}} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-} (\varphi) \, d(y_1, y_2) d(z_1, z_2) \\
> (1 + \epsilon) \int_{G^n \times G^n} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-} (\varphi) \, d(y_1, y_2) d(z_1, z_2)
\]

for all \(n \in N\).

Since

\[
\int_{G^n \times G^n} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-} (\varphi) \, d(y_1, y_2) d(z_1, z_2) > 0,
\]

this would be a contradiction with

\[
\lim_{n} |G^n|^\frac{1}{n} = 1.
\]
Now choose \( n \in \mathbb{N} \) such that (1) holds.

Let \( f = \chi_{G^n} \) be the characteristic function of \( G^n \). Let \( \pi \) be the unitary representation of \( K_1 \times K_2 \) associated to \( \varphi \) with Hilbert space \( H_\pi \). Let \( \xi \in H_\pi \) be such that \( \varphi (x_1, x_2) = \langle \pi (x_1, x_2) \xi, \xi \rangle \), \((x_1, x_2) \in K_1 \times K_2 \).

Then
\[
\| \pi (f) \xi \|^2 = \int_{K_1} \int_{K_2} f^- f (x_1, x_2) \varphi (x_1, x_2) \ d(x_1, x_2) > 0,
\]
since \( f^- f (e_1, e_2) \varphi (e_1, e_2) > 0 \) and \( f^- f (x_1, x_2) \varphi (x_1, x_2) \geq 0 \) for all \((x_1, x_2) \in K_1 \times K_2 \).

Now let
\[
\psi (x_1, x_2) = \frac{1}{\| \pi (f) \xi \|^2} \pi (x_1, x_2) \pi (f) \xi, \quad (x_1, x_2) \in K_1 \times K_2.
\]
Then \( \psi \) is associated to \( \pi \). moreover, for each \((x_1, x_2) \in K_1 \times K_2 \)
\[
|\psi (x_1, x_2) - 1|^2 = \frac{1}{\| \pi (f) \xi \|^4} |\langle \pi ((x_1, x_2) f - f) \xi, \pi (f) \xi \rangle|^2 \leq \frac{\| \pi ((x_1, x_2) f - f) \xi \|^2}{\| \pi (f) \xi \|^2}
\]
\[
= \frac{\int_{(K_1 \times K_2)^2} ((x_1, x_2) f - f) (y_1, y_2) ((x_1, x_2) f - f) (z_1, z_2) \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) \varphi (z_1, z_2) d(y_1, y_2) d(z_1, z_2)}{\int_{(K_1 \times K_2)^2} f (y_1, y_2) f (z_1, z_2) \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) \varphi (z_1, z_2) d(y_1, y_2) d(z_1, z_2)}
\]
\[
= \frac{\int_{G^n \Delta G^n} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) \varphi (z_1, z_2) d(y_1, y_2) d(z_1, z_2)}{\int_{G^n} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) \varphi (z_1, z_2) d(y_1, y_2) d(z_1, z_2)}
\]
where \( \Delta \) is the symmetric difference.

Now (1) implies that for \((x_1, x_2) \in G \)
\[
\int_{(x_1, x_2) G^n \Delta G^n} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) d(y_1, y_2) d(z_1, z_2)
\]
\[
\leq \int_{(G^n)^2} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) d(y_1, y_2) d(z_1, z_2)
\]
\[
+ \int_{((x_1, x_2) G^n \Delta G^n)^2} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) d(y_1, y_2) d(z_1, z_2)
\]
\[
\leq \varepsilon \int_{G^n} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) d(y_1, y_2) d(z_1, z_2)
\]
\[
+ \int_{((x_1, x_2) G^n \Delta G^n)^2} \varepsilon (y_1, y_2) \varepsilon (z_1, z_2) \varphi (y_1, y_2) d(y_1, y_2) d(z_1, z_2)
\]
\[
\leq 2\epsilon \int_{(G')^2} \varepsilon(y_1, y_2) * \varepsilon(z_1, z_2)^{-1} (\varphi) d(y_1, y_2)d(z_1, z_2)
\]
since \((x_1, x_2)^{-1} \in G\). Hence \(|\psi (x_1, x_2) - 1|^2 \leq 2\epsilon\) for all \((x_1, x_2) \in G\).

It is to be noted that last Theorem can be reformulate in the form: "If \(\varphi\) is positive and \(\varphi \in P(K_1 \times K_2)\) where \((K_1 \times K_2)\) is an exponentially bounded product hypergroups, then the constant function \(1_{K_1 \times K_2}\) is the uniform limit on compact subsets of \(K_1 \times K_2\) of functions of the form

\[
(x_1, x_2) \rightarrow \sum_{i,j=1}^{n} \varepsilon(x_1, x_2)_i * \varepsilon(x_1, x_2)_j^{-1} (\varphi (x_1, x_2)) c_i \varphi_j
\]

where \(c_i \geq 0\) and \((x_1, x_2)_i \in K_1 \times K_2\) for all \(1 \leq i \leq n\).

**Theorem 2.3** Discrete solvable product hypergroups satisfy property \((P)\).

**Proof:** Let \(K_1 \times K_2\) be a discrete solvable product hypergroups and let \(\varphi \in P(K_1 \times K_2)\) with \(\varphi \geq 0\). Let \((K_1 \times K_2) = (K_1 \times K_2)_n \supseteq (K_1 \times K_2)_{n-1} \supseteq .... \supseteq (K_1 \times K_2)_0 = \{(e_1, e_2)\}\), be a composition series with abelian factor \((K_1 \times K_2)_i/(K_1 \times K_2)_{i-1}\), \(1 \leq i \leq n\). First we show by induction on \(i\) that: for each \(0 \leq i \leq n\) there is a net \((\psi)_{\alpha}\) in \(P(K_1 \times K_2)\) with \(\psi \geq 0\) such that \(\lim \psi(1, x_2) = 1\) for all \((x_1, x_2) \in (K_1 \times K_2)_i\) and such that \(\pi_{\psi} \alpha\) is weakly contained in \(\pi\) for all \(\alpha\).

For \(i = 0\), the assertion is trivial (take \(\psi_\alpha = \varphi\)). For any \(i\) suppose that a net \((\psi)_{\alpha} \in N\) exists. Let \(\psi\) be a limit point of \(\{\psi_\alpha\}_{\alpha \in N}\) in the weak *-topology \(\sigma(l^\infty(K_1 \times K_2), l^1(K_1 \times K_2))\). Then \(\psi \in P(K_1 \times K_2)\) and \(\psi \geq 0\).

Moreover

\[
\psi (x_1, x_2) = \lim_{\alpha} \psi_\alpha (x_1, x_2) = 1
\]

for all \((x_1, x_2) \in (K_1 \times K_2)_i\).

Hence \(\psi \mid (K_1 \times K_2)_{i-1}\) factors to a positive definite function of \((K_1 \times K_2)_{i-1}/(K_1 \times K_2)_i\). Thus by last theorem in its reformulated form there is a net \((\psi_\beta)_{\beta}\) in \(P((K_1 \times K_2)_{i-1}/(K_1 \times K_2)_i)\) of the form

\[
\psi_\beta(x_1, x_2) = \sum c_k \varepsilon(x_1, x_2)_k * \varepsilon(x_1, x_2)_k^{-1} (\psi(x_1, x_2)), \quad (x_1, x_2) \in (K_1 \times K_2)_{i-1}
\]

where all \(c_k \geq 0\) and \((x_1, x_2) \in (K_1 \times K_2)_{i-1}\), such that

\[
\lim \psi_\beta' (x_1, x_2) = 1
\]

for all \((x_1, x_2) \in (K_1 \times K_2)_{i+1}\).

It is clear that \(\psi \beta' \in P(K_1 \times K_2)\) and \(\psi \beta' \geq 0\). Moreover \(\pi_{\psi_\alpha} = \pi_{\psi}\). Hence each \(\pi_{\psi_\beta}'\) is weakly contained in \(\{\pi_{\psi_\alpha} \mid \alpha \in A\}\) which is weakly contained in
\(\pi_\varphi\). So, we get a net \((\psi_\alpha)_\alpha \in P(K_1 \times K_2)\) such that \(\lim \psi_\alpha(x_1, x_2) = 1\) for all \((x_1, x_2) \in (K_1 \times K_2)_n = (K_1 \times K_2)\) and such that each \(\pi_{\psi_\alpha}\) is weakly contained in \(\pi_\varphi\). Hence \(1_{K_1 \times K_2}\) is weakly contained in \(\pi_\varphi\).

Now we reformulate property (P*), defined earlier, as follows: If \(\pi\) is a unitary representation of \(K_1 \times K_2\) with positive vectors, then \(1_{K_1 \times K_2}\) is weakly contained in \(\pi\), when \(\pi\) and \(1_{K_1 \times K_2}\) is viewed as representations of the discrete product hypergroups \(K_1 \times K_2\).

**Theorem 2.4** For a connected product hypergroups \(K_1 \times K_2\), the following statements are equivalent:

i) \(K_1 \times K_2\) has property (P*).

ii) \(K_1 \times K_2\) is amenable.

**Proof:** Suppose \(K_1 \times K_2\) is amenable. Let \(N\) be the closure of the commutative subhypergroup of \(K_1 \times K_2\), by \([8]\) proposition 3, \(N\) has polynomial growth hence it is exponentially bounded \([4]\). Let \(\varphi \in P(K_1 \times K_2)\), \(\varphi \geq 0\). By last theorem in its reformulated form there is a net \((\psi_\alpha)_\alpha \in P(K_1 \times K_2)\) with \(\psi_\alpha \geq 0\) such that \(\lim \psi_\alpha(x_1, x_2) = 1\) for all \((x_1, x_2) \in N\) and such that \(\pi_{\psi_\alpha}\) is weakly contained in \(\pi_\varphi\) for all \(\alpha\). Considering \(K_1 \times K_2\) as a discrete product hypergroups we can apply the method of proof of the last theorem to get some \(\psi \in P(K_1 \times K_2)\), \(\psi \geq 0\) with \(\psi \mid N = 1\) and such that \(\pi_\psi\) is weakly contained in \(\pi_\varphi\). Since \(K_1 \times K_2/N\) is abelian, \(1_{K_1 \times K_2}\) is weakly contained in \(\pi_\psi\) and the result follows.

Now if \(K_1 \times K_2\) has property (P*), then \(1_{K_1 \times K_2}\) is weakly contained in the regular representation \(\lambda_{K_1 \times K_2}\) when both representations are considered as representations of \(K_1 \times K_2\). This is equivalent to the amenability of \(K_1 \times K_2\) \([4]\).

## 3 Conditionally Exponential Convex Functions on Product Dual Hypergroups

In this section we will give some properties of the class of conditionally exponential convex functions defined on product dual hypergroups.

**Definition 3.1** Let \(K^*\) be the dual of the hypergroup \(K\) the function \(\psi : K^* \rightarrow \mathbb{C}\) is said to be conditionally exponential convex if for all \(n \in \mathbb{N}\) and any \(y_1, y_2, \ldots, y_n \in K^*\) and \(c_1, c_2, \ldots, c_n \in \mathbb{C}\) we have:

\[
\sum_{i,j=1}^{n} [\psi(y_i) + \psi(y_j) - \psi(y_i + y_j)] c_i \overline{c_j} \geq 0
\]

for all \(n \in \mathbb{N}\), \(c_1, c_2, \ldots, c_n \in \mathbb{C}\) and any \(y_1, y_2, \ldots, y_n \in K^*\).
Theorem 3.2 If \( \psi : K_1^* \rightarrow \mathbb{C}, \psi : K_2^* \rightarrow \mathbb{C} \) are conditionally exponential convex functions respectively, then \( \psi : K_1^* \times K_2^* \rightarrow \mathbb{C} \) defined by
\[
\psi(y_1, y_2) = \psi(y_1) + \psi(y_2)
\]
is conditionally exponential convex function.

Proof: Let \( \psi : K_1^* \rightarrow \mathbb{C} \), and \( \psi : K_2^* \rightarrow \mathbb{C} \), then
\[
\begin{align*}
\sum_{i,j=1}^{n}[\psi(y_1)_i + \psi(y_1)_j - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j} &\geq 0 \\
\sum_{i,j=1}^{n}[\psi(y_2)_i + \psi(y_2)_j - \psi((y_2)_i + (y_2)_j)]c_i\overline{c_j} &\geq 0
\end{align*}
\]
then we have
\[
\psi(y_1, y_2) = \sum_{i,j=1}^{n}[\psi(y_1)_i + \psi(y_1)_j - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j}
\]
there for \( \psi(y_1, y_2) \) is conditionally exponential convex function.

Theorem 3.3 A continuous function \( \psi : K_1^* \times K_2^* \rightarrow \mathbb{C} \) is conditionally exponential convex iff the following conditions are satisfied: (i) \( \psi(0,0) \geq 0 \), (ii) \( \psi_t(y_1, y_2) = \exp[-\psi(y_1, y_2)] \) is conditionally exponential convex for all \( t \).

Proof: Suppose that \( \psi \) is continuous conditionally exponential convex function, then (i) is easily satisfied. To establish (ii) we have:
\[
\sum_{i,j=1}^{n}[\psi(y_1)_i + \psi(y_1)_j - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j} \geq 0
\]
which implies that
\[
\sum_{i,j=1}^{n}\exp[\psi(y_1)_i + \psi(y_1)_j - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j} \geq 0
\]
So, we have for \( t = 1 \),
\[
\sum_{i,j=1}^{n}\exp[\psi(y_1)_i + \psi(y_1)_j - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j}
\]

where \( c'_k = c_k \exp[-\psi(y_1, y_2)_k] \). Hence, \( \Psi_1(y_1, y_2) \) is conditionally exponential convex.

Since \( t\psi(t) \) is conditionally exponential convex, then it’s clear that \( \Psi_1(y_1, y_2) \) is conditionally exponential convex all \( t > 0 \).

To prove the converse, let (i) and (ii) be satisfied. By (i) we have \( \exp[-t\psi(0, 0)] \leq 1 \) for all \( t > 0 \). So \( \Psi_t(y_1, y_2) = \frac{1}{t}[1 - \exp(-t\psi(y_1, y_2))] \) is conditionally exponential convex for all \( t > 0 \). Using Fattou’s lemma we can easily get that \( \psi_t(y_1, y_2) = \lim \Psi_t(y_1, y_2) \) is conditionally exponential convex.

**Theorem 3.4** Let \( \psi : K_1^* \times K_2^* \to \mathbb{C} \) be a conditionally exponential convex function and suppose that \( \psi(0, 0) \geq 0 \) then \( \frac{1}{\psi} \) is conditionally exponential convex.

**Proof:** Since \( \psi \) is conditionally exponential convex function, then the function \( \exp[-t\psi(y_1, y_2)] \) is conditionally exponential convex for all \( t > 0 \). The function \( \frac{1}{\psi} \) can be written in the form:

\[
\frac{1}{\psi(y_1, y_2)} = \int_0^\infty \exp[-t\psi(y_1, y_2)]dt
\]

Hence,

\[
\sum_{i,j=1}^n \frac{1}{\psi((y_1, y_2)_i + (y_1, y_2)_j)c_i\overline{c_j}} = \sum_{i,j=1}^n c_i\overline{c_j} \int_0^\infty \exp[-t\psi((y_1, y_2)_i + (y_1, y_2)_j)]dt
\]

\[
= \int_0^\infty \left( \sum_{i,j=1}^n \exp[-t\psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \right) dt \geq 0.
\]

Thus, \( \frac{1}{\psi} \) is conditionally exponential convex.

**References**


