The n- Dimensional Generalized Weyl Fractional Calculus Containing to
n- Dimensional $H$-Transforms

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Abstract
The object of this paper is to establish a relation between the n-dimensional $H$-transform involving the Weyl type n-dimensional Saigo operator of fractional integration.

Keywords: Fractional Integral, Riemann-Liouville Operator, Gauss Hypergeometric function, $H$-function, Fox's $H$-function, G-function.

1 Introduction

Our purpose of this paper is to establish a theorem on n-dimensional $H$-transforms involving with Weyl type n-dimensional Saigo operators. Further, a few interesting and elegant results as special cases of our main results has also been recorded.
2 Fractional Integrals and Derivative

An interesting and useful generalization of both the Riemann-Liouville and Erdélyi-Kober
fractional integration operators are introduced by Saigo [9], [10] in terms of Gauss’s
hypergeometric function as given below.

Let $\alpha, \beta$ and $\eta$ are complex numbers and let $y \in \mathbb{R}_+ = (0, \infty)$. Following [9], [10] the
fractional integral $(\text{Re}(\alpha) > 0)$ and derivative $(\text{Re}(\alpha) < 0)$ of the first kind of a function
$f(y)$ on $\mathbb{R}_+$ are defined respectively in the following forms

\[ I_{0,y}^{\alpha,\beta,\eta} f = \frac{y^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \, _2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{y}\right) f(t) \, dt, \quad R(\alpha) > 0 \]  
\[ = \frac{d^n}{dy^n} I_{0,y}^{\alpha\beta\eta} f, \quad 0 < R(\alpha)+n \leq 1, \quad (n = 1, 2, \ldots), \]  

where $_2F_1(\alpha, \beta; \cdot; \gamma)$ is Gauss’s hypergeometric function. The fractional integral
$(\text{Re}(\alpha) > 0)$ and derivative $(\text{Re}(\alpha) < 0)$ of the second kind are given by

\[ J_{y,\infty}^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_y^{\infty} (t-y)^{\alpha-1} t^{-\alpha-\beta} \, _2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{y}\right) f(t) \, dt, \quad R(\alpha) > 0 \]  
\[ = (-1)^n \frac{d^n}{dy^n} J_{y,\infty}^{\alpha\beta\eta} f, \quad 0 < R(\alpha)+n \leq 1 \quad (n = 1, 2, \ldots). \]  

The Riemann-Liouville, Weyl and Erdélyi-Kober fractional calculus operators follow as
special cases of the operators I and J as given below

\[ R_{0,y} \alpha f = I_{0,y}^{\alpha,-\alpha\eta} f = \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} f(t) \, dt, \quad R(\alpha) > 0 \]  
\[ = \frac{d^n}{dy^n} R_{0,y}^{\alpha+n} f, \quad 0 < R(\alpha) + n \leq 1, \quad (n = 1, 2, \ldots) \]  

\[ W_{y,\infty} \alpha f = J_{y,\infty}^{\alpha,-\alpha\eta} f = \frac{1}{\Gamma(\alpha)} \int_y^{\infty} (t-y)^{\alpha-1} f(t) \, dt, \quad R(\alpha) > 0 \]  
\[ = (-1)^n \frac{d^n}{dy^n} W_{y,\infty}^{\alpha+n} f, \quad 0 < R(\alpha) + n \leq 1 \quad (n = 1, 2, \ldots) \]  

\[ E_{0,y}^{\alpha,\eta} f = I_{0,y}^{\alpha,0\eta} f = \frac{y^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} t^\eta f(t) \, dt, \quad R(\alpha) > 0, \]
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\[ K_{\eta \triangleright \gamma}^{\alpha,\eta} f = \int_{y,\infty}^{y,\infty} f = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \eta)} \int_{0}^{\infty} (t - y)^{\alpha - 1 - \eta} f(t) \, dt, \quad R(\alpha) > 0. \]  

Following Miller [8, p.82], we denote by \( u_1 \) the class of functions \( f(x_1) \) on \( \mathbb{R}_+ \) which are infinitely differentiable with partial derivatives of any order behaving as \( O(\frac{1}{x_i}) \) when \( x_i \to \infty \) for all \( \xi_1 \). Similarly by \( u_2 \), we denote the class of functions \( f(x_1, x_2) \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \), which are infinitely differentiable with partial derivatives of any order behaving as \( O(\frac{1}{x_1}) \) when \( x_1 \to \infty, x_2 \to \infty \) for all \( \xi_2(i=1,2) \).

On the same way, we denote the class of functions \( f(x_1, x_2, \ldots, x_n) \) on \( \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \), which are infinitely differentiable with partial derivatives of any order behaving as \( O(\frac{1}{x_i}) \) when \( x_i \to \infty, \) where \( i=1,2,\ldots, n \) for all \( \xi_n(i=1,2,\ldots,n) \) by \( u_n \).

The n-dimensional operator of Weyl type fractional integration of orders \( Re(\alpha_i) > 0, \) where \( i=1,2,\ldots, n \) is defined in the class \( u_n \) by,

\[
\prod_{i=1}^{n} \left[ \frac{\Gamma(\alpha_i, \beta_i, \gamma_i)}{p_i, \infty} \right] p_{i=1}^{\frac{\beta_i}{\Gamma(\alpha_i)}} f(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} \left[ \frac{p_i^{\beta_i}}{\Gamma(\alpha_i)} \right] \int_{0}^{\infty} \prod_{i=1}^{n} \left( t_i - p_i \right)^{\alpha_i - 1 - \beta_i} \Gamma(\alpha_i - \beta_i, \gamma_i; \alpha_i; 1 - \frac{p_i}{t_i}) f(t_1, t_2, \ldots, t_n) \, dt_1, dt_2, \ldots dt_n, \tag{11}
\]

where \( \beta_i and \gamma_i, i=1,2,\ldots,n \) are real numbers.

More generally, the operator (11) of Weyl type fractional calculus in \( n \)-variables is defined by the differ-integral expression as,

\[
\prod_{i=1}^{n} \left[ \frac{\Gamma(\alpha_i, \beta_i, \gamma_i)}{p_i, \infty} \right] p_{i=1}^{\frac{\beta_i}{\Gamma(\alpha_i)}} f(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} \left[ \frac{p_i^{\beta_i}}{\Gamma(\alpha_i + r_i)} \right] \left( -1 \right)^{r_i} \sum_{i=1}^{n} \frac{\partial^{r_i + r_{i+1} + \cdots + r_n}}{\partial p_1^{r_1} \partial p_2^{r_2} \cdots \partial p_n^{r_n}} \left\{ \prod_{i=1}^{n} \left( t_i - p_i \right)^{\alpha_i - 1 - \beta_i} \Gamma(\alpha_i - \beta_i, \gamma_i; \alpha_i; 1 - \frac{p_i}{t_i}) f(t_1, t_2, \ldots, t_n) \, dt_1, dt_2, \ldots dt_n, \tag{12}
\]

for arbitrary real (complex) \( \alpha_i and r_1, r_2, \ldots, r_n = 0,1,2,\ldots \) .
In particular, if \( R(\alpha_i) < 0 \) and \( r_i > 0 \), where \( i = 1, 2, \ldots, n \), then (12) yields the partial fractional derivative of \( f(p_1, p_2, \ldots, p_n) \).

On the other hand if we set \( \beta = 0 \), (12) yields the Weyl type Erdélyi-Kober operators in \( n \)-dimensions

\[
\prod_{i=1}^{n} \left[ K_{\nu_i, \infty}^{\alpha_i, \gamma_i} \right] [f(p_1, p_2, \ldots, p_n)] = \prod_{i=1}^{n} \left[ J_{\nu_i, \infty}^{\alpha_i, 0} \right] [f(p_1, p_2, \ldots, p_n)]
\]

\[
= \prod_{i=1}^{n} \left[ \frac{p_i^{\beta_i}}{\Gamma(\alpha_i + \nu_i)} \right] \sum_{i=1}^{n} r_i \partial_{p_i}^{\alpha_i + \nu_i + \ldots + \nu_n} \prod_{i=1}^{n} \left[ (t_i - p_i)^{-\alpha_i + r_i - 1} t_i^{-\nu_i - \gamma_i} \right] f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n \right].
\]

(13)

3 n-Dimensional Laplace and \( \overline{H} \)-Transactions

The Laplace transform \( \zeta(p_i) \) of a function \( f(x_i) \in U_n \) is defined as

\[
\zeta(p_1, p_2, \ldots, p_n) = L[f(x_1, x_2, \ldots, x_n); p_1, p_2, \ldots, p_n] = \int_{0}^{\infty} \int_{0}^{\infty} \ldots \int_{0}^{\infty} e^{-\sum_{i=1}^{n} p_i x_i} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n
\]

(14)

where \( R(p_i) > 0 \), where \( i = 1, 2, \ldots, n \). Similarly, the Laplace transform of

\[
f[u_1 \sqrt{x_1^2 - \lambda_1^2} H(x_1 - \lambda_1), u_2 \sqrt{x_2^2 - \lambda_2^2} H(x_2 - \lambda_2), \ldots, u_n \sqrt{x_n^2 - \lambda_n^2} H(x_n - \lambda_n)],
\]

is defined by the Laplace transform of \( F(x_1, x_2, \ldots, x_n) \) where

\[
F(x_1, x_2, \ldots, x_n) = f[u_1 \sqrt{x_1^2 - \lambda_1^2} H(x_1 - \lambda_1), u_2 \sqrt{x_2^2 - \lambda_2^2} H(x_2 - \lambda_2), \ldots, u_n \sqrt{x_n^2 - \lambda_n^2} H(x_n - \lambda_n)],
\]

\[
x_i > \lambda_i > 0, \text{ where } i = 1, 2, \ldots, n;
\]

(15)

and \( H(t) \) denotes Heaviside’s unit step function.

**Definition:** By \( n \)-dimensional \( \overline{H} \)-transform \( \varphi(p_1, p_2, \ldots, p_n) \) of a function \( F(x_1, x_2, \ldots, x_n) \), we mean the following repeated integral involving \( n \)-different \( \overline{H} \)-functions.
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\[ \varphi(p_1, p_2, \ldots, p_n) = \frac{M_1N_1;M_2N_2;\ldots;M_nN_n}{P_1Q_1;P_2Q_2;\ldots;P_nQ_n} [F(x_1, x_2, \ldots, x_n) ; a_1, a_2, \ldots, a_n ; p_1, p_2, \ldots, p_n] \]

\[ = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \ldots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} (p x_i)^{a_i-1} \frac{M_iN_i}{P_iQ_i} \left( p x_i \right)^{k_i} \left( a_{ij}, A_{ij} \right)^{-1/N_i} \left( b_{ij}, B_{ij} \right)^{1/N_i+1/P_i} \right] \]

\[ \times \mathcal{F}(x_1, x_2, \ldots, x_n) \, dx_1 \ldots dx_n, \quad (16) \]

here we suppose that $\lambda_i > 0, k_i > 0, \text{where} i = 1, 2, \ldots, n; \varphi(p_1, p_2, \ldots, p_n)$, exists and belongs to $u_n$.

Further suppose that,

\[ |\arg p_i| < \frac{1}{2} T_i \pi, \quad (17) \]

where,

\[ T_i = \sum_{j=1}^{M_i} |\beta_{ij}| - \sum_{j=1}^{N_i} |\alpha_{ij}| - \sum_{j=M_i+1}^{Q_i} |B_{ij}|, \quad (18) \]

where $i=1,2,\ldots,n$.

The $\overline{H}$-function appearing in (16), introduced by Inayat-Hussain (\cite{1}, see also \cite{14}) in terms of Mellin-Barnes type contour integral, is defined by,

\[ \overline{H}_{M,N}^{P,Q} \left[ z \begin{bmatrix} (\alpha_j, \beta_j; A_j, B_j)_{M+1, Q} \\ (\alpha_j, \beta_j; A_j, B_j)_{N+1, P} \end{bmatrix} \right] = \frac{1}{2 \pi i} \int_{-i\infty}^{i\infty} \psi(\xi) z^\xi d\xi, \quad (19) \]

where

\[ \psi(\xi) = \prod_{j=1}^{M} \Gamma(\beta_j - \beta_{ij} + \alpha_j \xi)^{-1} \prod_{j=1}^{N} \{\Gamma(1-a_j + \alpha_j \xi)^{\alpha_j} \}
\]

\[ \prod_{j=M+1}^{Q} \{\Gamma(1-b_j + \beta_{ij} \xi)^{\beta_j} \} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)^{-1}, \quad (20) \]

which contains fractional powers of some of the $\Gamma$-functions. Here and throughout the paper $a_j (j = 1, \ldots, P)$ and $b_j (j = 1, \ldots, Q)$ are complex parameters.

$\alpha_j \geq 0 \; (j=1,\ldots,P), \beta_j \geq 0 \; (j=1,\ldots,Q),(\text{not all zero simultaneously})$ and the exponents $A_j (j = 1, \ldots, N)$ and $B_j (j = M+1, \ldots, Q)$ can take on non-integer values. The contour in (19) is imaginary axis $R(\xi) = 0$. It is suitably indented in order to avoid the singularities of the $\Gamma$-functions and to keep these singularities on appropriate sides. Again, for $A_j (j = 1, \ldots, N)$ not an integer, the poles of the $\Gamma$-functions of the numerator in (16) are converted to branch points. However, as long as there is no coincidence of poles from any
\( \Gamma(b_j - \beta_j \zeta_j), \quad (j = 1, \ldots, M) \) and \( \Gamma(1-a_j + \alpha_j \zeta_j), \quad (j = 1, \ldots, N) \) pair the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

For the sake of brevity

\[
T = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0. \tag{21}
\]

4 Relationship between n-Dimensional \( \mathcal{H} \)-Transforms in Terms of n-Dimensional Saigo Operator of Weyl Type

To prove the theorem in this section, we need the following n-dimensional \( \mathcal{H} \)-transform \( \phi_i(p_1, p_2, \ldots, p_n) \) of \( F(x_1, x_2, \ldots, x_n) \) defined by,

\[
\phi_i(p_1, p_2, \ldots, p_n) = \mathcal{H}\left[ F(x_1, x_2, \ldots, x_n) \right] \alpha_1, \alpha_2, \ldots, \alpha_n; p_1, p_2, \ldots, p_n \]

\[
= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} (p_i x_i)^{\alpha_i - 1} \phi_i^M + 2N_1 \left[ (\alpha_{ij}, \delta_{ij} : A_{ij} N_{ij} (\alpha_{ij} - 1, k_{ij}), (\alpha_1 - 1, k_{ij}), (\beta_{ij} + \gamma_{ij} - \alpha_{ij} - 1, k_{ij}) \right]
\]

\[
\mathcal{H}_{1+2Q+2} \left[ f(x_1, x_2, \ldots, x_n) \right] dx_1 dx_2 \ldots dx_n, \tag{22}
\]

where it is assumed that \( \phi_i(p_1, p_2, \ldots, p_n) \) exists and belongs to \( u_n \) as well as \( k_i > 0 \), where \( i = 1, 2, \ldots, n \) and other conditions on the parameters, in which additional parameters \( \alpha_i, \beta_i, \gamma_i \) where \( i = 1, 2, \ldots, n \) included correspond to those in (11).

**Theorem 1** Let \( \phi_i(p_1, p_2, \ldots, p_n) \) be given by definition (14) then for \( R(\alpha_i) > 0, \lambda_i > 0, k_i > 0 \), where \( i = 1, 2, \ldots, n \) there holds the formula

\[
\prod_{i=1}^{n} \left\{ \gamma_i \right\}_{p_i, \infty} [\phi(p_1, p_2, \ldots, p_n)] = \phi_i(p_1, p_2, \ldots, p_n) \quad \tag{23}
\]

provided that \( \phi_i(p_1, p_2, \ldots, p_n) \) exists and belong to \( u_n \).

**Proof:** Let \( R(\alpha_i) > 0 \), where \( i = 1, 2, \ldots, n \) then in view of (11) and (18), we find that
\[ \prod_{i=1}^{n} \left[ p_{i}^{\beta_{i}} \pi \right]^{\frac{\gamma_{i}}{\pi}} = \prod_{i=1}^{n} \left( \frac{\beta_{i}}{\Gamma(\alpha_{i})} \right) \]

\[ \int_{p_{1}}^{\infty} \int_{p_{2}}^{\infty} \cdots \int_{p_{n}}^{\infty} \left( t_{1} - p_{1} \right)^{\alpha_{1} - 1} \left( t_{2} - p_{2} \right)^{\alpha_{2} - 1} \cdots \left( t_{n} - p_{n} \right)^{\alpha_{n} - 1} \frac{2I}{t_{1} - \alpha_{1} - \beta_{1}} \cdots \frac{2I}{t_{n} - \alpha_{n} - \beta_{n}} \phi(t_{1}, t_{2}, \ldots, t_{n}) dt_{1} dt_{2} \cdots dt_{n}, \]

or

\[ \prod_{i=1}^{n} \left[ \frac{p_{i}^{\beta_{i}}}{\Gamma(\alpha_{i})} \right] \int_{p_{1}}^{\infty} \int_{p_{2}}^{\infty} \cdots \int_{p_{n}}^{\infty} \left( t_{1} - p_{1} \right)^{\alpha_{1} - 1} \left( t_{2} - p_{2} \right)^{\alpha_{2} - 1} \cdots \left( t_{n} - p_{n} \right)^{\alpha_{n} - 1} \frac{2I}{t_{1} - \alpha_{1} - \beta_{1}} \cdots \frac{2I}{t_{n} - \alpha_{n} - \beta_{n}} \phi(t_{1}, t_{2}, \ldots, t_{n}) dt_{1} dt_{2} \cdots dt_{n}, \]

On interchanging the order of integration which is permissible and on evaluating the integrals through the integral formula

\[ \int_{x}^{\infty} u^{-\mu-v} (u-x)^{\nu-1} 2F_{1}(\tau, \alpha, \nu; 1; \frac{x}{u}) H_{1}^{(M, N)} P_{Q} \left[ 
\begin{array}{c}
(a, \alpha; J)_{1}, N, (a, \alpha; J)_{N+1} P_{1} \\
(b, \beta; J)_{1}, M, (b, \beta; J)_{M+1} Q_{1}
\end{array} \right] du \]

\[ = \frac{\Gamma(\nu)}{x^{\mu}} H_{P+2, Q+2} \left[ 
\begin{array}{c}
(a, \alpha; J)_{1}, N, (a, \alpha; J)_{N+1} P_{1} \\
(b, \beta; J)_{1}, M, (b, \beta; J)_{M+1} Q_{1}
\end{array} \right], \]

where, \( R(\nu) > 0, \quad R\left( \mu + \nu + \frac{k (1-a)_{j}}{\alpha_{j}} \right) > 0 \)

\[ \left( \mu + \nu - \tau - \omega + \frac{k (1-a)_{j}}{\alpha_{j}} \right) > 0, |\arg z| < \frac{1}{2} T \pi \quad (T \text{ is given in } (21)) \]

(23) can be established by means of the following formula [2, p.399].

\[ \int_{0}^{1} x^{\gamma-1} (1-x)^{\rho-1} 2F_{1}(\alpha, \beta; \gamma, x) dx = \frac{\Gamma(\gamma) \Gamma(\rho)}{\Gamma(\gamma + \rho - \alpha - \beta)} \frac{\Gamma(\gamma + \rho - \alpha) \Gamma(\gamma + \rho - \beta)}{\Gamma(\gamma + \rho - \alpha - \beta)} \]

(26)
for \( R(\gamma) > 0, R(\rho) > 0, R(\gamma + \rho - \alpha - \beta) > 0. \)

by using the formula, left hand side of (24) becomes

\[
\int_1^\infty \cdots \int_1^\infty H_{M+2,N_1 \cdots M+2,N_n} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i-1} (t,x_i) \]

\[
\int_1^\infty \cdots \int_1^\infty H_{P+2,Q_1 \cdots P+2,Q_n+2} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i} \]

\[
\int_1^\infty \cdots \int_1^\infty H_{M+2,N_1 \cdots M+2,N_n} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i-1} (t,x_i) \]

\[
\int_1^\infty \cdots \int_1^\infty H_{P+2,Q_1 \cdots P+2,Q_n+2} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i} \]

\[
\int_1^\infty \cdots \int_1^\infty H_{M+2,N_1 \cdots M+2,N_n} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i-1} (t,x_i) \]

\[
\int_1^\infty \cdots \int_1^\infty H_{P+2,Q_1 \cdots P+2,Q_n+2} \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i} \]

As far as the \( n \)-dimensional Weyl type operators \( \prod_{i=1}^n \left[ J_{\alpha_i,\beta_i,\gamma_i} \right] \) preserves the class \( u_n \), it follows that \( \varphi_1(p_1,p_2,\ldots,p_n) \) also belongs to \( u_n \).

It is interesting to note that the statement of Theorem 1 can easily be extended for arbitrary real \( \alpha_i \) where \( i = 1, 2, \ldots, n \) by using the definition (12) for the generalized Weyl type fractional calculus operators and differentiating under the signs of the integrals.

## 5 Interesting Special Cases

Putting \( \gamma_i = 0 \) where \( i = 1, 2, \ldots, n \) in theorem 1, we can easily prove Theorem 1(a).

**Theorem 1(a).** For \( R(\alpha_i) > 0, \beta_i > 0, r_i > 0; \) where \( i = 1, 2, \ldots, n \) and also let \( \varphi(p_1,p_2,\ldots,p_n) \) be given by (14) then there holds the following formula,

\[
\prod_{i=1}^n \left[ J_{\alpha_i,\beta_i,0} \right] \left[ \varphi(p_1,p_2,\ldots,p_n) \right] = \varphi_2(p_1,p_2,\ldots,p_n) \]  

(27)

provided that \( \varphi_2(p_1,p_2,\ldots,p_n) \) exists and belongs to \( u_n \) where \( \varphi_2 \) is represented by the repeated integral,

\[
\varphi_2(p_1,p_2,\ldots,p_n) = \int_1^\infty \cdots \int_1^\infty \prod_{i=1}^n \left[ \left( \lambda_i \right) \right]^{a_i-1} (p_i) \]
For $A_i=1$, the $H$-function reduces to Fox's $H$-function [5], [6] and then Theorem 1 (a) reduces to,

$$\prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right] = \varphi_3(p_1, p_2, \ldots, p_n),$$

provided that $\varphi_3(p_1, p_2, \ldots, p_n)$ exists and belongs to $u_n$ where $\varphi_3$ is represented by the repeated integral,

$$\varphi_3(p_1, p_2, \ldots, p_n) = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right]$$

$$= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right]$$

On employing the identity

$$H_{M, N}^{P, Q} \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_Q \end{array} \right] = G_{M, N}^{P, Q} \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_Q \end{array} \right],$$

we see that the $n$-dimensional $H$-transform reduces to the corresponding $n$-dimensional $G$-transform $\theta(p_1, p_2, \ldots, p_n)$ defined by

$$\theta(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right]$$

$$= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right]$$

provided that $\theta(p_1, p_2, \ldots, p_n)$ exists and belongs to class $u_n$, where $k_i$ are positive integers, $\lambda_i > 0, P_i \leq Q_i$. 

\begin{align*}
H_{M_1+N_1}^{P_1+Q_1} \left[ \begin{array}{c} \alpha_i, \beta_i, \gamma_i \\ x_1, x_2, \ldots, x_n \end{array} \right] & \cdot F(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right] \\
& \cdot F(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.
\end{align*}

\begin{align*}
\prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right] & = \varphi_3(p_1, p_2, \ldots, p_n),
\end{align*}

\begin{align*}
\varphi_3(p_1, p_2, \ldots, p_n) & = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right],
\end{align*}

\begin{align*}
H_{M_1+N_1}^{P_1+Q_1} \left[ \begin{array}{c} \alpha_i, \beta_i, \gamma_i \\ x_1, x_2, \ldots, x_n \end{array} \right] & \cdot F(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} J_{p_1, p_2, \ldots, p_n} \left[ \varphi(p_1, p_2, \ldots, p_n) \right] \\
& \cdot F(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.
\end{align*}
(33) \[ \left| \arg p^1 \right| < \frac{1}{2} T_1^*, \]

with \[ T_1^* = 2N_i + 2M_i - P_i - Q_i, \] (34)

where \( i = 1, 2, \ldots, n \), \( G_{M,N}^{P,Q} \), appealing in (31) and (32) represents Meijer’s G-function whose detailed account is available from the monograph of Mathai and Saxena [4].

Thus, we obtain the following Theorem 1(b).

**Theorem 1(b).** For \( R(\alpha_i) > 0, \beta_i > 0, k_i > 0; \) where \( i = 1, 2, \ldots, n \) being positive integers and also let \( \theta(p_1, p_2, \ldots, p_n) \) be given by (31) then the following formula

\[
\prod_{i=1}^{n} \left[ \int_{p_i}^{\infty} \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \left( p, x \right)_a a_i^{-1} \right]
\]

\[
\prod_{i=1}^{n} \left( \frac{\Delta(k_1, k_1 - a_i + 1)}{\alpha_1 + \beta_1 + \gamma_i - a_i + 1} \right)
\]

\[
= \theta_1(p_1, p_2, \ldots, p_n)
\]

(35)

holds, provided that \( \theta_1(p_1, p_2, \ldots, p_n) \) exists and belongs to class \( u \) for other conditions on the parameters, in which additional parameters \( \alpha_i, \beta_i \) and \( \gamma_i \) where \( i = 1, 2, \ldots, n \) included correspond to those in (32). Here

\[
\theta_1(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} \left[ \int_{k_i}^{\infty} \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \cdots \int_{\lambda_n}^{\infty} \left( p, x \right)_a a_i^{-1} \right]
\]

\[
\prod_{i=1}^{n} \left( \frac{\Delta(k_1, k_1 - a_i + 1)}{\alpha_1 + \beta_1 + \gamma_i - a_i + 1} \right)
\]

\[
= \theta_1(p_1, p_2, \ldots, p_n)
\]

(36)

and the symbol \( \Delta(n, \alpha) \) represents the sequence of parameters

\[
\frac{\alpha}{n}, \frac{\alpha + 1}{n}, \ldots, \frac{\alpha + n - 1}{n}
\]

On taking \( \gamma_i = 0, \) where \( i = 1, 2, \ldots, n, \) (36) becomes
\[
\prod_{i=1}^{n} \left[ \int_{p_i, \infty}^{\alpha, \beta, 0} \right] \left[ \theta(p_1, p_2, \ldots, p_n) \right] = \theta_2(p_1, p_2, \ldots, p_n)
\]

provided \( \theta_2(p_1, p_2, \ldots, p_n) \) exists and belongs to class \( u_n \), where \( \theta_2 \) is represented by the integral

\[
\theta_2(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} \left[ k_i^{-\alpha_i} \right] \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \prod_{i=1}^{n} \left( p_i x_i \right)^{a_i - 1}
\]

\[
G \left[ \left( \frac{a_1, \ldots, a_p, \Delta(k_1, \alpha_1 + \beta_1 - a_1 + 1)}{\Delta(k_1, \beta_1 - a_1 + 1), b_Q, \ldots, b_Q, \lambda} \right) \right] .
\]

\[
F(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.
\]

6 Special Case

(i) Converting our Theorem 1, 1(a) and 1(b) for \( i=1,2,3 \); we find the known result defined by Chaurasia and Jain [19].

(ii) Converting our Theorem 1, 1(a) and 1(b) for \( i=1,2 \); we find the known result defined by Chaurasia and Shrivastava [18], if we tack \( N = N' = 0 \).

(iii) Taking \( A_j = B_j = 1 \), then Theorem 1, 1(a) and 1(b) for \( i=1,2 \); we find the known result defined by Saigo, Saxena and Ram [13].

References


[19] V.B.L. Chaurasia and Monika Jain, Three dimensional generalized Weyl fractional calculus pertaining to three-dimensional $\mathbf{H}$ - transforms, *Chile, SCIENTIA Series A: Mathematical Sciences*, 20(2009), 37–43.