A Common Fixed Point Theorem for Sub Compatibility and Occasionally Weak Compatibility in Intuitionistic Fuzzy Metric Spaces

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Abstract

In this paper, we establish a common fixed point theorem for six maps using concept of subcompatibility and occasionally weak compatibility in Intuitionistic Fuzzy metric space. S. kutukcu [10] obtained a fixed point theorem for Menger spaces; we obtain its Intuitionistic Fuzzy metric space version with more generalized conditions relaxing completeness criteria. We also justify our findings with an example.

Keywords: Intuitionistic fuzzy metric space, Subcompatible mapping, Occasionally weakly compatible mapping, Common fixed point.
1 Introduction

The concept of fuzzy sets was introduced initially by Zadeh [17] in 1965. Since, then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Atanassov [4] introduced and studied the concept of intuitionistic fuzzy sets. Intuitionistic fuzzy sets as a generalization of fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems too rough. Coker [6] introduced the concept of intuitionistic fuzzy topological spaces. Alaca et al. [2] proved the well-known fixed point theorems of Banach [5] in the setting of intuitionistic fuzzy metric spaces. Later on, Turkoglu et al. [16] proved Jungck’s [8] common fixed point theorem in the setting of intuitionistic fuzzy metric space. Turkoglu et al. [16] further formulated the notions of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of pants theorem [12]. Gregori et al. [7], Saadati and Park [13] studied the concept of intuitionistic fuzzy metric space and its applications.

Recently, Saurabh Manro et al. [11] introduced the notion of subcompatibility and subsequential continuity in Intuitionistic Fuzzy metric space and proved some result for four self maps. Inspired by the result of Saurabh Manro et al. [11], in this paper we prove a common fixed point theorem for six self maps which is a generalization of [10].

2 Preliminaries

Definition 2.1[14] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm, if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a$ for all $a \in [0,1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0,1]$.

Definition 2.2[14] A binary operation $\diamondsuit: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-conorm if $\diamondsuit$ it satisfies the following conditions:

(i) $\diamondsuit$ is commutative and associative
(ii) $\diamondsuit$ is continuous
(iii) $a \diamondsuit 0 = a$ for all $a \in [0, 1]$
(iv) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0,1]$.

Definition 2.3[2] A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if $X$ is an arbitrary set, $*$ is a continuous $t$-
norm. $\phi$ is a continuous $t$-conorm and $M$, $N$ are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:

For all $x, y, z \in X$ and $s, t > 0$,

1. (IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$
2. (IFM-2) $M(x, y, 0) = 0$
3. (IFM-3) $M(x, y, t) = 1$ if and only if $x = y$
4. (IFM-4) $M(x, y, t) = M(y, x, t)$
5. (IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
6. (IFM-6) $M(x, y, \cdot): [0, \infty) \to [0,1]$ is left continuous
7. (IFM-7) $\lim_{t \to \infty} M(x, y, t) = 1$
8. (IFM-8) $N(x, y, 0) = 1$
9. (IFM-9) $N(x, y, t) = 0$ if and only if $x = y$
10. (IFM-10) $N(x, y, t) = N(y, x, t)$
11. (IFM-11) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$
12. (IFM-12) $N(x, y, \cdot): [0, \infty) \to [0,1]$ is right continuous.
13. (IFM-13) $\lim_{t \to \infty} N(x, y, t) = 0$

Then $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Remark 2.1** ([1], [3]) Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space if $X$ is of the form $(X, M, 1 - M, \ast, \diamond)$ such that $t$-norm $\ast$ and $t$-conorm $\diamond$ are associated, that is, $x \diamond y = 1 - ((1 - x) \ast (1 - y))$ for any $x, y \in X$. But the converse is not true.

**Example 2.1** Let $(X, d)$ be a metric space. Define $a \ast b = \min \{a, b\}$ and $t$-conorm $a \diamond b = \max \{a, b\}$ for all $x, y \in X$ and $t > 0$, $M_d(x, y, t) = \frac{t}{t + d(x, y)}$ and $N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$. 
Then \((X, M, N, \ast, \bowtie)\) is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric \((M, N)\) induced by the metric \(d\) the standard intuitionistic fuzzy metric.

**Definition 2.4**[2] Let \((X, M, N, \ast, \bowtie)\) be an Intuitionistic Fuzzy metric space, then

(a) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to \(x\) in \(X\) if for all \(t > 0\),
\[
\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_n, x, t) = 0.
\]

(b) A sequence \(\{x_n\}\) in \(X\) is said to be Cauchy if for all \(t > 0\) and \(p > 0\),
\[
\lim_{n \to \infty} M(x_n + p, x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_n + p, x_n, t) = 0.
\]

(c) An Intuitionistic Fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.5**[15] Self mappings \(\varphi\) and \(\psi\) of an intuitionistic fuzzy metric space \((X, M, N, \ast, \bowtie)\) are said to be compatible if for all \(\epsilon > 0\),
\[
\lim_{n \to \infty} M(\varphi x_n, \psi x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(\varphi x_n, \psi x_n, t) = 0
\]
whenever \(\{x_n\}\) is a sequence in \(X\) such that...

**Definition 2.6**[9] Self mappings \(\varphi\) and \(\psi\) of an intuitionistic fuzzy metric space \((X, M, N, \ast, \bowtie)\) are said to be weakly compatible if
\[
\varphi x = \psi x \quad \text{when} \quad \varphi x = \psi x
\]
for some \(x \in X\).

**Definition 2.7**[11] Self mappings \(\varphi\) and \(\psi\) of an intuitionistic fuzzy metric space \((X, M, N, \ast, \bowtie)\) are said to be occasionally weakly compatible (owc) iff there is a point \(x \in X\) which is a coincidence point of \(A\) and \(B\) at which \(A\) and \(B\) commute.

**Definition 2.8**[11] Self mappings \(\varphi\) and \(\psi\) of an intuitionistic fuzzy metric space \((X, M, N, \ast, \bowtie)\) are said to be sub compatible if there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} M(\varphi x_n, \psi x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(\varphi x_n, \psi x_n, t) = 0
\]
for all \(t > 0\).

**Lemma 2.1**[1] Let \((X, M, N, \ast, \bowtie)\) be an intuitionistic fuzzy metric space and for all \(x, y \in X\) and \(t > 0\) if for a number \(k \in (0, 1)\),
\[
M(x, y, kt) \geq M(x, y, t) \quad \text{and} \quad N(x, y, kt) \leq N(x, y, t).
\]
Then \(x = y\).

**Lemma 2.2**[1] Let \((X, M, N, \ast, \bowtie)\) be an intuitionistic fuzzy metric space and \(\{y_n\}\) be a sequence in \(X\). If there exists a number \(k \in [0, 1]\) such that
\[
M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t), \quad N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t)
\]
for all \(t > 0\) and \(n \in N\), then \(\{y_n\}\) is a Cauchy sequence in \(X\).
3 Main Result

**Theorem 3.1** Let $A, B, S, T, P$ and $Q$ be self maps on an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ with $t * t \geq t$ and $(1 - t) \Diamond (1 - t) \leq (1 - t)$ for some $t \in [0,1]$ such that

(i) There exists a number $k \in (0,1)$ such that

$$M^2(Px, Qy, kt) \ast [M(ABx, Px, kt), M(STy, Qy, kt)] \geq [pM(ABx, Px, t) + qM(ABx, STy, t)]. M(ABx, Qy, 2kt)$$

and

$$N^2(Px, Qy, kt) \Diamond [N(ABx, Px, kt), N(STy, Qy, kt)] \leq [pN(ABx, Px, t) + qN(ABx, STy, t)]. N(ABx, Qy, 2kt)$$

for all $x, y \in X$ and $t > 0$, where $0 < p, q < 1$ such that $p+q=1$.

(ii) $AB=BA$, $ST=TS$, $PB=BP$, $QT=TQ$

(iii) $AB$ is continuous.

(iv) The pair $(P, AB)$ is subcompatible and $(Q, ST)$ is occasionally weakly compatible (owc).

Then $A, B, S, T, P$ and $Q$ have a unique common fixed point in $X$.

**Proof:** Since the pair $(P, AB)$ is subcompatible, then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Px_n = \lim_{n \to \infty} ABx_n = z$ for some $z \in X$ and satisfy $\lim_{n \to \infty} P(AB)x_n = \lim_{n \to \infty} AB(P)x_n$.

Since $AB$ is continuous, $AB(AB)x_n \to ABz$ and $(AB)Px_n \to ABz$.

Since $(P, AB)$ is subcompatible, $P(AB)x_n \to ABz$.

Since $(Q, ST)$ is occasionally weakly compatible, then there exists a point $v \in X$ such that $Qv = STv$ and $QSTv = STQv$.

**Step-1:** By taking $x = x_n$ and $y = v$ in (i), we have

$$M^2(Px_n, Qv, kt) \ast [M(ABx_n, Px_n, kt), M(STv, Qv, kt)] \geq [pM(ABx_n, Px_n, t) + qM(ABx_n, STv, t)]. M(ABx_n, Qv, 2kt)$$

and

$$N^2(Px_n, Qv, kt) \Diamond [N(ABx_n, Px_n, kt), N(STv, Qv, kt)] \leq [pN(ABx_n, Px_n, t) + qN(ABx_n, STv, t)]. N(ABx_n, Qv, 2kt)$$

Taking limit as $n \to \infty$ and using $Qv = STv$, we have

$$M^2(z, Qv, kt) \ast [M(z, z, kt), M(Qv, Qv, kt)] \geq [pM(z, z, t) + qM(z, Qv, t)]. M(z, Qv, 2kt)$$
\[ M^{2}(z, Qv, kt) \geq [p + qM(z, Qv, t)]. M(z, Qv, 2kt) \]

\[ M^{2}(z, Qv, kt) \geq [p + qM(z, Qv, t)]. M(z, Qv, kt) \]

\[ \Rightarrow M(z, Qv, kt) \geq \frac{p}{1 - q} = 1. \]

and \[ N^{2}(z, Qv, kt) \geq [N(z, z, kt). N(Qv, Qv, kt)] \]
\[ \geq [pN(z, z, t) + qN(z, Qv, t)]. N(z, Qv, 2kt) \]

\[ \Rightarrow N^{2}(z, Qv, kt) \leq qN(z, Qv, t). N(z, Qv, 2kt) \]
\[ \leq qN(z, Qv, t). N(z, Qv, kt) \]

\[ \Rightarrow N(z, Qv, kt) \leq 0 \text{ for } k \in (0,1) \text{and all } t > 0. \]

Therefore, we have \( z = Qv \) and so \( z = Qv = STv \), then we get \( Qz = STz \).

**Step-2:** By taking \( x = ABx_{n} \) and \( y = v \) in (i), we have

\[ M^{2}(P(AB)x_{n}, Qv, kt) \geq [M(AB(AB)x_{n}, P(AB)x_{n}, kt). M(STv, Qv, kt)] \]
\[ \geq [pM(AB(AB)x_{n}, P(AB)x_{n}, t) + qM(AB(AB)x_{n}, STv, t)]. M(AB(AB)x_{n}, Qv, 2kt) \]

and \[ N^{2}(P(AB)x_{n}, Qv, kt) \geq [N(AB(AB)x_{n}, P(AB)x_{n}, kt). N(STv, Qv, kt)] \]
\[ \leq [pN(AB(AB)x_{n}, P(AB)x_{n}, t) + qN(AB(AB)x_{n}, STv, t)]. N(AB(AB)x_{n}, Qv, 2kt) \]

Taking limit as \( n \rightarrow \infty \) and using \( z = Qv = STv \), we have

\[ M^{2}(ABz, z, kt) \geq [M(ABz, ABz, kt). M(z, z, kt)] \]
\[ \geq [pM(ABz, ABz, t) + qM(ABz, z, t)]. M(ABz, z, 2kt) \]

\[ \Rightarrow M^{2}(ABz, z, kt) \geq [p + qM(ABz, z, t)]. M(ABz, z, 2kt) \]
\[ \geq [p + qM(ABz, z, t)]. M(ABz, z, kt) \]
\[ \Rightarrow M(ABz, z, kt) \geq [p + qM(ABz, z, t)] \]
\[ \geq [p + qM(ABz, z, kt)] \]
\[ \Rightarrow M(ABz, z, kt) \geq \frac{p}{1 - q} = 1. \]

and \[ N^{2}(ABz, z, kt) \geq [N(ABz, ABz, kt). N(z, z, kt)] \]
\[ \leq [pN(ABz, ABz, t) + qN(ABz, z, t)]. N(ABz, z, 2kt) \]
\[ N^2(ABz, z, kt) \leq qN(ABz, z, t) \cdot N(ABz, z, 2kt) \]
\[ \leq qN(ABz, z, t) \cdot N(ABz, z, kt) \]
\[ N(ABz, z, kt) \leq qN(ABz, z, t) \leq qN(ABz, z, kt) \]
\[ N(ABz, z, kt) \leq 0 \text{ for } k \in (0,1) \text{ and all } t > 0. \]

Thus, we have \( z = ABz \).

**Step-3:** By taking \( x = z \) and \( y = v \) in (i), we have

\[
M^2(Pz, Qv, kt) \ast [M(ABz, Pz, kt) \cdot M(STv, Qv, kt)] \\
\geq [pM(ABz, Pz, t) + qM(ABz, STv, t)]. M(ABz, Qv, 2kt)
\]

and \( N^2(Pz, Qv, kt) \ast [N(ABz, Pz, kt) \cdot N(STv, Qv, kt)] \\
\leq [pN(ABz, Pz, t) + qN(ABz, STv, t)]. N(ABz, Qv, 2kt) \)

Using \( z = Qv = STv = ABz \); we have

\[
M^2(Pz, z, kt) \ast [M(z, Pz, kt) \cdot M(z, z, kt)] \\
\geq [pM(z, Pz, t) + qM(z, z, t)]. M(z, z, 2kt)
\]

\[ \Rightarrow M^2(z, Pz, kt) \ast M(z, Pz, kt) \geq [pM(z, Pz, t) + q] \]

Since \( M^2(Pz, z, kt) \leq 1 \) and using (iii) in definition 2.1, we have

\[ M(z, Pz, kt) \geq [pM(z, Pz, t) + q] \geq pM(z, Pz, kt) + q \]

\[ \Rightarrow M(z, Pz, kt) \geq \frac{q}{1 - p} = 1. \]

and \( N^2(Pz, z, kt) \ast [N(z, Pz, kt) \cdot N(z, z, kt)] \\
\leq [pN(z, Pz, t) + qN(z, z, t)]. N(z, z, 2kt) \)

\[ \Rightarrow N^2(Pz, z, kt) \leq 0 \]

\[ \Rightarrow N(Pz, z, kt) \leq 0 \text{ for } k \in (0,1) \text{ and all } t > 0. \]

Thus, we have \( z = Pz = ABz \).

**Step-4:** By taking \( x = x_n \) and \( y = z \) in (i), we have

\[
M^2(Px_n, Qz, kt) \ast [M(ABx_n, Px_n, kt) \cdot M(STz, Qz, kt)] \\
\geq [pM(ABx_n, Px_n, t) + qM(ABx_n, STz, t)]. M(ABx_n, Qz, 2kt)
\]
and $N^2(P_{x_n}, Qz, kt) \diamond [N(ABx_{n}, P_{x_n}, kt). N(STz, Qz, kt)]$
\[ \leq [pN(ABx_{n}, P_{x_n}, t) + qN(ABx_{n}, STz, t)]. N(ABx_{n}, Qz, 2kt) \]

Taking limit as $n \to \infty$ and using $Qz = STz$, we have

\[
M^2(z, Qz, kt) \geq [pM(z, z, t) + qM(z, Qz, t)]. M(z, Qz, 2kt) \geq [p + qM(z, Qz, t)]. M(z, Qz, kt) \]

\[ \Rightarrow M(z, Qz, kt) \geq [p + qM(z, Qz, t)]. M(z, Qz, 2kt) \geq [p + qM(z, Qz, kt)]. \]

\[ \Rightarrow M(z, Qz, kt) \geq \frac{p}{1-q} = 1. \]

and $N^2(z, Qz, kt) \diamond [N(z, z, kt). N(Qz, Qz, kt)]$
\[ \geq [pN(z, z, t) + qN(z, Qz, t)]. N(z, Qz, 2kt) \]

\[ \Rightarrow N^2(z, Qz, kt) \leq qN(z, Qz, t). N(z, Qz, 2kt) \leq qN(z, Qz, kt) \]

\[ \Rightarrow N(z, Qz, kt) \leq 0 \text{ for } k \in (0,1) \text{ and all } t > 0. \]

Thus, we have $z = Qz$ and therefore $z = ABz = Pz = Qz = STz$.

**Step-5:** By taking $x = Bz$ and $y = z$ in (i), we have

\[
M^2(P(B)z, Qz, kt) \geq [pM(AB(B)z, P(B)z, t) + qM(AB(B)z, STz, t)]. M(AB(B)z, Qz, 2kt) \]

and $N^2(P(B)z, Qz, kt) \diamond [N(AB(B)z, P(B)z, kt). N(STz, Qz, kt)]$
\[ \leq [pN(AB(B)z, P(B)z, t) + qN(AB(B)z, STz, t)]. N(AB(B)z, Qz, 2kt) \]

Since $AB = BA$ and $PB = BP$, we have $P(B)z = B(P)z = Bz$ and $AB(B)z = B(AB)z = Bz$ and using $Qz = STz = z$, we have

\[
M^2(Bz, z, kt) \geq [pM(Bz, Bz, t) + qM(Bz, z, t)]. M(Bz, z, 2kt) \]
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⇒ $M^2(Bz, z, kt) \geq [p + qM(Bz, z, t)] \cdot M(Bz, z, 2kt)$

≥ $[p + qM(Bz, z, t)] \cdot M(Bz, z, kt)$

$M(Bz, z, kt) \geq [p + qM(Bz, z, t)] \geq [p + qM(Bz, z, kt)]$

⇒ $M(Bz, z, kt) \geq \frac{p}{1-q} = 1.$

and $N^2(Bz, z, kt) \leq [N(Bz, Bz, kt) \cdot N(z, z, kt)]$

$\leq [pN(Bz, Bz, t) + qN(Bz, z, t)] \cdot N(Bz, z, 2kt)$

⇒ $N^2(Bz, z, kt) \leq qN(Bz, z, t) \cdot N(Bz, z, 2kt)$

≤ $qN(Bz, z, t) \cdot N(Bz, z, kt)$

⇒ $N(Bz, z, kt) \leq 0$ for $k \in (0,1)$ and all $t > 0.$

Thus, we have $z = Bz.$ Since $z = ABz,$ we also have $z = Az,$ therefore $z = Az = Bz = Pz = Qz = STz.$

**Step-6:** By taking $x = x_n$ and $y = Tz$ in (i), we have

\[
M^2(Px_n, Q(Tz), kt) \ast [M(ABx_n, Px_n, kt) \cdot M(ST(Tz), Q(Tz), kt)] \\
\geq [pM(ABx_n, Px_n, t) + qM(ABx_n, ST(Tz), t)] \cdot M(ABx_n, Q(Tz), 2kt)
\]

and $N^2(Px_n, Q(Tz), kt) \cdot [N(ABx_n, Px_n, kt) \cdot N(ST(Tz), Q(Tz), kt)]$

\[
\leq [pN(ABx_n, Px_n, t) + qN(ABx_n, ST(Tz), t)] \cdot N(ABx_n, Q(Tz), 2kt)
\]

Since $QT = TQ$ and $ST = TS,$ we have $QTz = TQz = Tz$ and $ST(Tz) = T(STz) = Tz.$

Letting $n \to \infty,$ we have

\[
M^2(z, Tz, kt) \ast [M(z, z, kt) \cdot M(Tz, Tz, kt)] \\
\geq [pM(z, z, t) + qM(z, Tz, t)] \cdot M(z, Tz, 2kt)
\]

⇒ $M^2(z, Tz, kt) \geq [p + qM(z, Tz, t)] \cdot M(z, Tz, 2kt)$

≥ $[p + qM(z, Tz, t)]M(z, Tz, kt)$

⇒ $M(z, Tz, kt) \geq [p + qM(z, Tz, t)] \geq [p + qM(z, Tz, kt)].$
\[ \Rightarrow M(z, Tz, kt) \geq \frac{p}{1 - q} = 1. \]

and \[ N^2(z, Tz, kt) \leq [N(z, z, kt) \cdot N(Tz, Tz, kt)] \]
\[ \geq [pN(z, z, t) + qN(z, Tz, t)] \cdot N(z, Tz, 2kt) \]
\[ \Rightarrow N^2(z, Tz, kt) \leq qN(z, Tz, t) \cdot N(z, Tz, 2kt) \]
\[ \leq qN(z, Tz, t) \cdot N(z, Tz, kt) \]
\[ \Rightarrow N(z, Tz, kt) \leq qN(z, Tz, t) \leq qN(z, Tz, kt) \]
\[ \Rightarrow N(z, Tz, kt) \leq 0 \text{ for } k \in (0,1) \text{ and all } t > 0. \]

Thus, we have \( z = Tz \). Since \( Tz = STz \), we also have \( z = Sz \). Therefore \( z = Az = Bz = Pz = Qz = Sz = Tz \), that is, \( z \) is the common fixed point of the six maps.

**Step-7:** For uniqueness, let \( w, (w \neq z) \) be another common fixed point of \( A, B, S, T, P \) and \( Q \).

By taking \( x = z \) and \( y = w \) in (i), we have

\[ M^2(Pz, Qw, kt) * [M(ABz, Pz, kt) \cdot M(STw, Qw, kt)] \]
\[ \geq [pM(ABz, Pz, t) + qM(ABz, STw, t)] \cdot M(ABz, Qw, 2kt) \]

and \[ N^2(Pz, Qw, kt) \triangleq [N(ABz, Pz, kt) \cdot N(STw, Qw, kt)] \]
\[ \leq [pN(ABz, Pz, t) + qN(ABz, STw, t)] \cdot N(ABz, Qw, 2kt) \]

Which implies that

\[ M^2(z, w, kt) * [M(z, z, kt) \cdot M(w, w, kt)] \]
\[ \geq [pM(z, z, t) + qM(z, w, t)] \cdot M(z, w, 2kt) \]
\[ \Rightarrow M^2(z, w, kt) \geq [p + qM(z, w, t)] \cdot M(z, w, 2kt) \]
\[ \geq [p + qM(z, w, t)] \cdot M(z, w, kt) \]
\[ \Rightarrow M(z, w, kt) \geq p + qM(z, w, t) \geq p + qM(z, w, kt) \]
\[ \Rightarrow M(z, w, kt) \geq \frac{p}{1 - q} = 1. \]

and \[ N^2(z, w, kt) \triangleq [N(z, z, kt) \cdot N(w, w, kt)] \]
\[ \geq [pN(z, z, t) + qN(z, w, t)] \cdot N(z, w, 2kt) \]
\[ \Rightarrow N^2(z, w, kt) \leq qN(z, w, t) \cdot N(z, w, 2kt) \]
\[ \leq qN(z, w, t).N(z, w, kt) \]
\[ \Rightarrow N(z, w, kt) \leq qN(z, w, t)q \leq N(z, w, kt) \]
\[ \Rightarrow N(z, w, kt) \leq 0 \text{ for } k \in (0,1) \text{ and all } t > 0. \]

Thus, we have \( z = w \). This completes the proof of the theorem.

If we take \( B=T=I_X \text{ (the identity map on } X) \) in the main theorem, we have the following:

**Corollary 3.2:** Let \( A, S, P \) and \( Q \) be self maps on an intuitionistic fuzzy metric space \( (X, M, N, *, \bowtie) \) with \( t \ast t \geq t \) and \( (1 - t) \bowtie (1 - t) \leq (1 - t) \) for some \( t \in [0,1] \) such that

(i) There exists a number \( k \in (0,1) \) such that
\[
M^2(Px, Qy, kt) \ast [M(Ax, Px, kt).M(Sy, Qy, kt)] \\
\geq [pM(Ax, Px, t) + qM(Ax, Sy, t)].M(Ax, Qy, 2kt)
\]
\[
N^2(Px, Qy, kt) \bowtie [N(Ax, Px, kt).N(Sy, Qy, kt)] \\
\leq [pN(Ax, Px, t) + qN(Ax, Sy, t)].N(Ax, Qy, 2kt)
\]

for all \( x, y \in X \) and \( t>0 \), where \( 0<p, q<1 \) such that \( p+q=1 \).

(ii) \( A \) is continuous.

(iii) The pair \( (P, A) \) is subcompatible and \( (Q, S) \) is occasionally weakly compatible (owc).

Then \( A, S, P \) and \( Q \) have a unique common fixed point in \( X \).

**Example 3.3** Let \( X = \{ \frac{1}{n} : n \in \mathbb{N} \cup \{0\} \} \) with metric \( d \) defined by \( d(x, y) = |x - y| \).

For all \( x, y \in X \) and \( t \in (0, \infty) \), define
\[
M(x, y, t) = \frac{t}{t+|x-y|}, \quad N(x, y, t) = \frac{|x-y|}{t+|x-y|}, \quad M(x, y, 0) = 0, \quad N(x, y, 0) = 1.
\]

Clearly \( (X, M, N, *, \bowtie) \) is an Intuitionistic Fuzzy metric space, where \( * \) and \( \bowtie \) are defined by \( a \ast b = \min\{a, b\} \) and \( a \bowtie b = \min\{1, a + b\} \).

Let \( A, B, S, T, P \) and \( Q \) be maps from \( X \) into itself defined as \( Ax = x, Bx = \frac{x}{2}, Sx = \frac{x}{5}, Tx = \frac{x}{3}, Px = 0, Qx = \frac{x}{6} \) for all \( x \in X \).
Clearly AB=BA, ST=TS, PB=BP, QT=TQ and AB is continuous. If we take $k = 0.5$ and $t = 1$, we see that the condition (i) of the main theorem is also satisfied. Moreover, the maps P and AB are subcompatible if $\lim_{n \to \infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Px_n = \lim_{n \to \infty} ABx_n = 0$ for $0 \in X$. The maps Q and ST are occasionally weakly compatible at 0. Thus, all conditions of the main theorem are satisfied and 0 is the unique common fixed point of A, B, S, T, P and Q.

References


