On Super Edge Magic and Bimagic Labeling for Duplication Graphs

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Abstract

An edge magic total labeling of a graph \(G(V, E)\) with \(p\) vertices and \(q\) edges is a bijection \(f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p+q\}\) such that for every edge \(uv\) in \(E\), \(f(u) + f(uv) + f(v)\) is a constant \(k\). If there exist two constants \(k_1\) and \(k_2\) such that the above sum is either \(k_1\) or \(k_2\), it is said to be an edge bimagic total labeling. In this paper we study and investigate super edge magic and bimagic labeling for duplication graphs of cycles and paths.

Keywords: Graph, labeling, magic labeling, bimagic labeling, bijective function.

1 Introduction:

A labeling of a graph \(G\) is an assignment \(f\) of labels to either the vertices or the edges or both subject to certain conditions. Labeled graphs are becoming an
increasingly useful family of Mathematical Models from a broad range of applications. Graph labeling was first introduced in the late 1960’s. A useful survey on graph labeling by J.A. Gallian (2012) can be found in [3]. All graphs considered here are finite, simple and undirected.

A (p, q)-graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is called total edge magic if there is a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\} \) such that there exists a constant \( k \) for any edge \( uv \) in \( E \), \( f(u) + f(uv) + f(v) = k \). The original concept of total edge-magic graph is due to Kotzig and Rosa [4]. They called it magic graph. A total edge-magic graph is called a super edge-magic if \( f(V(G)) = \{1, 2, \ldots, p\} \). Wallis [5] called super edge-magic as strongly edge-magic.

It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say \( k_1 \) or \( k_2 \). Edge bimagic total labeling was introduced by J. Baskar Babujee [1] and studied in [2] as \((1, 1)\) edge bimagic labeling.

**Definition 1.1:** A graph \( G(V, E) \) with \( p \) vertices and \( q \) edges is called total edge magic if there is a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\} \) such that there exists a constant \( k \) for any edge \( uv \) in \( E \), \( f(u) + f(uv) + f(v) = k \). A total edge magic graph is called super edge magic if \( f(V(G)) = \{1, 2, \ldots, p\} \).

**Definition 1.2:** A graph \( G(V, E) \) with \( p \) vertices and \( q \) edges is called total edge bimagic if there is a bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\} \) such that for any edge \( uv \) in \( E \), we have two constants \( k_1 \) and \( k_2 \) with \( f(u) + f(v) + f(uv) = k_1 \) or \( k_2 \). A total edge bimagic graph is called super edge bimagic if \( f(V(G)) = \{1, 2, \ldots, p\} \).

**Definition 1.3[6]:** Duplication of an edge \( e = v_i v_{i+1} \) by a new vertex \( v^1 \) in a graph \( G \) produces a new graph \( G' \) such that the neighborhood of \( v^1 \) that is \( N(v^1) = \{v_i, v_{i+1}\} \).

**Definition 1.4[6]:** Duplication of a vertex \( v_k \) by a new edge \( e = v^1 v^{11} \) in a graph \( G \) produces a new graph \( G' \) such that the neighborhood of \( v^1 \) and \( v^{11} \) are respectively \( N(v^1) = \{v_k, v^{11}\} \) and \( N(v^{11}) = \{v_k, v^1\} \).

In this paper we prove that super edge magic and bimagic labeling for some cycles and paths related duplication graphs.

## 2 Main Result

**Theorem 2.1:** The graph \( G \) obtained by duplication of a vertex by an edge in \( C_n \): \( n \equiv 1 \pmod{2} \) has super edge bimagic total labeling.

**Proof:** Let \( u_1, u_2, \ldots, u_n \) be vertices and \( e_1, e_2, \ldots, e_n \) be edges of cycle \( C_n \). Without loss of generality we duplicate the vertex \( u_{n-1} \) by an edge \( e_{n+1} \) with end vertices as
Let $v_1$ and $v_2$. Let the graph so obtained by $(V, E)$. Then vertex set $V = \{v_1, v_2, u_i; 1 \leq i \leq n\}$ and edge set

$E = E_1 \cup E_2$ where $E_1 = \{u_1u_n; u_iu_{i+1}; 1 \leq i \leq n-1\}, E_2 = \{u_{n-1}v_1; u_{n-1}v_2; v_1v_2\}$.

We define a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, ..., 2n+5\}$ as follows.

For $i = 1$ to $n; i \equiv 1 \text{ (mod2)}, f(u_i) = \frac{i+1}{2}$. For $i = 2$ to $n-1; i \equiv 0 \text{ (mod2)},$

$$f(u_i) = \frac{n+1}{2} + \frac{i}{2}.$$  

For $i = 1$ to $n-1; f(u_1u_{i+1}) = 2n+5-i \cdot f(u_i) = 1 \cdot f(u_n) = \frac{n+1}{2}, f(v_i) = n+1,$

$$f(v_1) = n+2, f(u_{n+1}) = n, f(u_iu_{i+1}) = 2n+5, f(v_1v_2) = n+3,$$  

$$f(v_1u_{n+1}) = n+5, f(v_2u_{n+1}) = n+4.$$  

**Case (i):** For every edge $u_iu_{i+1} \in E_1$

**Subcase (i):** $i \equiv 1 \text{ (mod2)}$

$$f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = \frac{i+1}{2} + \frac{n+1}{2} + \frac{i+1}{2} + 2n+5 - i = \frac{5n+13}{2} = k_i$$  

**Subcase (ii):** $i \equiv 0 \text{ (mod2)}$

$$f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = \frac{n+1}{2} + \frac{i}{2} + \frac{i+2}{2} + 2n+5 - i = \frac{5n+13}{2} = k_i$$  

**Subcase (iii):** For an edge $u_1u_n \in E_1$

$$f(u_1) + f(u_n) + f(u_1u_n) = 1+\frac{n+1}{2} + 2n+5 = \frac{5n+13}{2} = k_i$$  

**Case (ii):** For an edge $u_{n-1}v_1 \in E_2$

$$f(v_1) + f(u_{n-1}) + f(v_1u_{n-1}) = n+1+n+n+5 = 3n+6 = k_2$$

For an edge $u_{n-1}v_2 \in E_2$

$$f(v_2) + f(u_{n-1}) + f(v_2u_{n-1}) = n+2+n+n+4 = 3n+6 = k_2$$

For an edge $v_1v_2 \in E_2$
\[ f(v_1) + f(v_2) + f(v_1v_2) = n+1+n+2+n+3 = 3n+6 = k_2 \]

For all the above two cases the edge counts are \( k_1 = \frac{5n+13}{2} \) and \( k_2 = 3n+6 \).
Hence, the graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_n; n \equiv 1(\text{mod}2) \) is super edge bimagic total labeling.

**Theorem 2.2:** The graph \( G \) obtained by duplication of an edge by a vertex in \( C_n; n \equiv 1(\text{mod}2) \) admits super edge bimagic total labeling.

**Proof:** Let \( u_1, u_2, \ldots, u_n \) be vertices and \( e_1, e_2, \ldots, e_n \) be edges of cycle \( C_n; n \equiv 1(\text{mod}2) \). Without loss of generality we duplicate the edge \( u_iu_{i+1} \) by a vertex \( v_1 \).

Let the graph so obtained by \((V, E)\). Then the vertex set \( V = \{v_1, u_i; 1 \leq i \leq n\} \) and edge set \( E = E_1 \cup E_2 \) where \( E_1 = \{u_iu_{i+1}; 1 \leq i \leq n-1\} \), \( E_2 = \{u_1u_n; v_1u_n; u_{n-1}v_1\} \).

We define a bijective function \( f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 2n+3\} \) as follows.

For \( i = 1 \) to \( n; i \equiv 1(\text{mod}2) \), \( f(u_i) = \frac{i+1}{2} \).

For \( i = 2 \) to \( n-1; i \equiv 0(\text{mod}2) \), \( f(u_i) = \frac{n+1}{2} + \frac{i}{2} \).

For \( i = 1 \) to \( n-1; f(u_iu_{i+1}) = 2n+3-i \). \( f(u_1) = 1 \), \( f(u_n) = \frac{n+1}{2} \), \( f(v_1) = n+1 \),

\( f(u_{n-1}) = n \), \( f(v_1u_{n-1}) = n+2 \), \( f(v_1u_n) = n+3 \), \( f(u_1u_n) = 2n+3 \).

**Case (i):** For any edge \( u_iu_{i+1} \in E_1 \)

**Subcase (i):** \( i \equiv 1 \) (mod2)

\[ f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = \frac{i+1}{2} + \frac{n+1}{2} + \frac{i+1}{2} + 2n+3 - i = \frac{5n+9}{2} = k_1 \]

**Subcase (ii):** \( i \equiv 0 \) (mod2)

\[ f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = \frac{n+1}{2} + \frac{i+2}{2} + 2n+3 - i = \frac{5n+9}{2} = k_1 \]

**Case (ii):** For an edge \( u_1u_n \in E_2 \)

\[ f(u_i) + f(u_n) + f(u_1u_n) = 1 + \frac{n+1}{2} + 2n+3 = \frac{5n+9}{2} = k_1 \]
For an edge $v_1u_n \in E_2$

$$f(v_1) + f(u_n) + f(v_1u_n) = n+1+\frac{n+1}{2}+n+3 = \frac{5n+9}{2} = k_1$$

For an edge $u_{n-1}v_1 \in E_2$

$$f(v_1) + f(u_{n-1}) + f(v_1u_{n-1}) = n+1+n+n+2 = 3n+3 = k_2$$

For all the above two cases the edge counts are $k_1 = \frac{5n+9}{2}$ and $k_2 = 3n+3$.

Hence the graph obtained by duplication of an arbitrary edge by a new vertex in cycle $C_n$: $n \equiv 1(\mod 2)$ is super edge bimagic total labeling.

**Theorem 2.3:** The graph $G$ obtained by duplicating all the vertices by edges in $C_n$: $n \equiv 1(\mod 2)$ admits super edge magic total labeling.

**Proof:** Let $u_1, u_2, \ldots, u_n$ be vertices and $e_1, e_2, \ldots, e_n$ be edges of cycle $C_n$. Let the graph obtained by duplicating all the vertices by edges in cycle $C_n$. Then vertex set $V = \{v_i, u_i, w_i; 1 \leq i \leq n\}$ and edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where $E_1 = \{u_iu_{i+1}; 1 \leq i \leq n-1\}$, $E_2 = \{v_1w_i; 1 \leq i \leq n\}$, $E_3 = \{u_iw_i; 1 \leq i \leq n-1\}$, $E_4 = \{v_iu_i; 1 \leq i \leq n\}$ and $E_5 = \{u_1u_n; w_nu_n; v_nw_n; u_nv_n\}$. We define a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 7n\}$ as follows.

For $i = 1$ to $n$; $i \equiv 1(\mod 2)$, $f(u_i) = \frac{i+1}{2}$. For $i = 2$ to $n-1$; $i \equiv 0(\mod 2)$, $f(u_i) = \frac{n+1}{2} + \frac{i}{2}$.

For $i = 1$ to $n-1$; $f(u_1u_{i+1}) = 7n - i$. For $i = 1$ to $n$; $f(v_i) = 2n - i$.

For $i = 1$ to $n$; $i \equiv 1(\mod 2)$, $f(w_i) = 2n+\frac{n+1}{2} + \frac{i+1}{2}$.

For $i = 2$ to $n-1$; $i \equiv 0(\mod 2)$, $f(w_i) = 2n+1+\frac{i}{2}$.

For $i = 1$ to $n$; $i \equiv 1(\mod 2)$, $f(v_1w_i) = 3n+\frac{i+1}{2}$.

For $i = 2$ to $n-1$; $i \equiv 0(\mod 2)$, $f(v_1w_i) = 3n+\frac{n+1}{2} + \frac{i}{2}$.
For $i = 1$ to $n-2; \ i \equiv 1 (\text{mod} 2)$, $f(u_i v_i) = 5n + \frac{i+1}{2} + \frac{n+1}{2}$.

For $i = 2$ to $n-1; \ i \equiv 0 (\text{mod} 2)$, $f(u_i v_i) = 5n + 1 + \frac{i}{2}$

$f(u_n) = \frac{n+1}{2}, \ f(u_{i} u_{n}) = 7n, \ f(w_n) = 2n+1, \ f(u_{n} w_{n}) = 5n, \ f(v_n w_n) = 3n + \frac{n+1}{2},$

$f(u_n v_n) = 5n+1, \ f(v_n) = 2n$.

Case (i): For any edge $u_i u_{i+1} \in E_1$

Subcase (i): $i \equiv 1 \ (\text{mod} 2)$

$f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) = \frac{i+1}{2} + \frac{n+1}{2} + \frac{i+1}{2} + 7n - i = \frac{15n+3}{2} = k_i$

Subcase (ii): $i \equiv 0 \ (\text{mod} 2)$

$f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) = \frac{n+1}{2} + \frac{i}{2} + \frac{i+2}{2} + 7n - i = \frac{15n+3}{2} = k_i$

Case (ii): For any edge $v_i w_i \in E_2$

Subcase (i): $i \equiv 1 \ (\text{mod} 2)$

$f(v_i) + f(w_i) + f(v_i w_i) = 2n - i + 2n + \frac{n+1}{2} + \frac{i+1}{2} + 3n + \frac{i+1}{2} = \frac{15n+3}{2} = k_i$

Subcase (ii): $i \equiv 0 \ (\text{mod} 2)$

$f(v_i) + f(w_i) + f(v_i w_i) = 2n - i + 2n + \frac{i}{2} + 1 + 3n + \frac{n+1}{2} + \frac{i}{2} = \frac{15n+3}{2} = k_i$

Case (iii): For any edge $u_i w_i \in E_3$

Subcase (i): $i \equiv 1 \ (\text{mod} 2)$

$f(u_i) + f(w_i) + f(u_i w_i) = \frac{i+1}{2} + 2n + \frac{n+1}{2} + \frac{i+1}{2} + 5n - i = \frac{15n+3}{2} = k_i$
Subcase (ii): $i \equiv 0 \pmod{2}$

\[
\begin{align*}
f(u_i) + f(w_i) + f(u_iw_i) &= \frac{i}{2} + \frac{n+1}{2} + 2n + \frac{i}{2} + 1 + 5n - i = \frac{15n+3}{2} = k_i
\end{align*}
\]

Case (iv): For any edge $u_iv_i \in E_4$

Subcase (i): $i \equiv 1 \pmod{2}$

\[
\begin{align*}
f(u_i) + f(v_i) + f(u_iv_i) &= \frac{i+1}{2} + 2n - i + 5n + \frac{n+1}{2} + \frac{i+1}{2} = \frac{15n+3}{2} = k_i
\end{align*}
\]

Subcase (ii): $i \equiv 0 \pmod{2}$

\[
\begin{align*}
f(u_i) + f(v_i) + f(u_iv_i) &= \frac{i}{2} + \frac{n+1}{2} + 2n - i + 5n + 1 + \frac{i}{2} = \frac{15n+3}{2} = k_i
\end{align*}
\]

Case (v): For an edge $u_5u_n \in E_5$

\[
\begin{align*}
f(u_n) + f(u_n) + f(u_nu_n) &= 1 + \frac{n+1}{2} + 7n = \frac{15n+3}{2} = k_i
\end{align*}
\]

For an edge $u_nw_n \in E_5$

\[
\begin{align*}
f(u_n) + f(w_n) + f(u_nw_n) &= \frac{n+1}{2} + 2n + 1 + 5n = \frac{15n+3}{2} = k_i
\end{align*}
\]

For an edge $v_nw_n \in E_5$

\[
\begin{align*}
f(v_n) + f(w_n) + f(v_nw_n) &= 2n + 2n + 1 + 3n + \frac{n+1}{2} = \frac{15n+3}{2} = k_i
\end{align*}
\]

For edge $u_nv_n \in E_5$

\[
\begin{align*}
f(u_n) + f(v_n) + f(u_nv_n) &= \frac{n+1}{2} + 2n + 5n + 1 = \frac{15n+3}{2} = k_i.
\end{align*}
\]

For all the above five cases the edge count is a constant $k_1 = \frac{15n+3}{2}$.

Hence the graph obtained by duplication of all the vertices by edges in cycle $C_5$: $n \equiv 1 \pmod{2}$ is super edge magic total labeling.

Illustration 2.4: Super edge magic labeling of a graph obtained by duplicating all the vertices by edges in $C_5$ is shown in figure 1.
Theorem 2.5: The graph $G$ obtained by duplicating all the vertices by edges in path $P_n$ admits super edge magic total labeling for $n \equiv 1 \pmod{2}$.

Proof: Let $u_1, u_2, \ldots, u_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. Let the graph obtained by duplicating all the vertices by edges in path $P_n$ in $G$. Then the vertex set $V = \{v_i, u_i, w_i; 1 \leq i \leq n\}$ and edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4$ where

$E_1 = \{u_iu_{i+1}; 1 \leq i \leq n-1\}$,

$E_2 = \{u_iv_i; 1 \leq i \leq n\}$, $E_3 = \{v_iw_i; 1 \leq i \leq n\}$, $E_4 = \{u_iw_i; 1 \leq i \leq n\}$.

We define a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 7n-1\}$ as follows.

Case (i): For any edge $u_iu_{i+1} \in E_1$

Subcase (i): $i \equiv 1 \pmod{2}$

$f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = 3n + 1 - \frac{i+1}{2} + 2n + \frac{n+1}{2} - \frac{i+1}{2} + 3n + i = \frac{17n+1}{2} = k_i$

Subcase (ii): $i \equiv 0 \pmod{2}$

$f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) = 2n + \frac{n+1}{2} - \frac{i}{2} + 3n - \frac{i+2}{2} + 3n + i = \frac{17n+1}{2} = k_i$
Case (ii): For any edge $u_iv_i \in E_2$

**Subcase (i):** $i \equiv 1 \pmod{2}$

$$f(u_i) + f(v_i) + f(u_iv_i) = 3n - \frac{i+1}{2} + 1 + 5n + \frac{n+1}{2} - \frac{i+1}{2} = \frac{17n+1}{2} = k_i$$

**Subcase (ii):** $i \equiv 0 \pmod{2}$

$$f(u_i) + f(v_i) + f(u_iv_i) = 2n + \frac{n+1}{2} - \frac{i}{2} + 6n - \frac{i}{2} = \frac{17n+1}{2} = k_i$$

Case (iii): For any edge $v_iw_i \in E_3$

**Subcase (i):** $i \equiv 1 \pmod{2}$

$$f(w_i) + f(v_i) + f(w_iv_i) = i + n + \frac{n+1}{2} + 1 - \frac{i+1}{2} + 7n - \frac{i+1}{2} = \frac{17n+1}{2} = k_i$$

**Subcase (ii):** $i \equiv 0 \pmod{2}$

$$f(w_i) + f(v_i) + f(w_iv_i) = i + 2n + 1 - \frac{i}{2} + 6n + \frac{n-1}{2} - \frac{i}{2} = \frac{17n+1}{2} = k_i$$

Case (iv): For every edge $u_iw_i \in E_4$

**Subcase (i):** $i \equiv 1 \pmod{2}$

$$f(w_i) + f(u_i) + f(w_iu_i) = 3n + 1 - \frac{i+1}{2} + n + \frac{n+1}{2} + 1 - \frac{i+1}{2} + 4n - 1 + i = \frac{17n+1}{2} = k_i$$

**Subcase (ii):** $i \equiv 0 \pmod{2}$

$$f(w_i) + f(u_i) + f(w_iu_i) = 2n + \frac{n+1}{2} - \frac{i}{2} + 2n + 1 - \frac{i}{2} + 4n - 1 + i = \frac{17n+1}{2} = k_i$$

For all the above four cases the edge count is a constant $k_i = \frac{17n+1}{2}$. Hence the graph obtained by duplication of all the vertices by edges in path $P_n$: $n \equiv 1 \pmod{2}$

**Theorem 2.6:** The graph $G$ obtained by duplicating all the vertices by edges in path $P_n$ admits super edge bimagic total labeling for $n \equiv 0 \pmod{2}$. 
Proof: Let $u_1, u_2, \ldots, u_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. Let the graph obtained by duplicating all the vertices by edges in path $P_n$ in $G$. Then the vertex set $V = \{v_i, u_i, w_i; 1 \leq i \leq n-1\} \cup \{u^1, v^1, w^1\}$ and edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where $E_1 = \{u_iu_{i+1}; 1 \leq i \leq n-2\}$, $E_2 = \{u_iv_i; 1 \leq i \leq n-1\}$, $E_3 = \{v_iw_i; 1 \leq i \leq n-1\}$, $E_4 = \{u_iw_i; 1 \leq i \leq n-1\}$ and $E_5 = \{u^1u_1; u^1v^1; u^1w^1; v^1w^1\}$. We define a bijective function $f: V(G) \cup (E) \rightarrow \{1, 2, \ldots, 7n-1\}$ as follows.

For $i = 1$ to $n-1$; $i \equiv 1(\text{mod}2)$, $f(u_i) = 3n - 2 - \frac{i+1}{2}$.

For $i = 2$ to $n-2$; $i \equiv 0(\text{mod}2)$, $f(u_i) = 2n + \frac{n}{2} - \frac{i}{2}$.

For $i = 1$ to $n-2$; $f(u_{i+1}) = 3n+i$. For $i = 1$ to $n-1$; $f(v_i) = i$.

For $i = 1$ to $n-1$; $i \equiv 1(\text{mod}2)$, $f(u_{i+1}) = 3n+i$.

For $i = 2$ to $n-2$; $i \equiv 0(\text{mod}2)$, $f(u_{i+1}) = 3n+1$.

For $i = 1$ to $n-1$; $i \equiv 1(\text{mod}2)$, $f(v_i) = 5n+2 - \frac{i+1}{2}$.

For $i = 2$ to $n-2$; $i \equiv 0(\text{mod}2)$, $f(v_i) = 6n+1 - \frac{i}{2}$.

For $i = 1$ to $n-1$; $i \equiv 1(\text{mod}2)$, $f(u_{i+1}) = 7n - \frac{i+1}{2}$.

For $i = 2$ to $n-2$; $i \equiv 0(\text{mod}2)$, $f(u_{i+1}) = 6n+2 - \frac{i}{2}$.

For $i = 1$ to $n-1$; $f(u_{i+1}) = 4n+2 + i$.

$f(u^1) = 3n-1$, $f(v^1) = 3n$, $f(w^1) = 3n-2$, $f(u_1u^1) = 3n+4$, $f(u^1v^1) = 3n+1$, $f(v^1w^1) = 3n+2$, $f(u^1w^1) = 3n+3$.

Case (i): For any edge $u_iu_{i+1} \in E_i$

Subcase (i): $i \equiv 1 \ (\text{mod}2)$
f(u_i) + f(u_{i+1}) = 3n - 2 - \frac{i+1}{2} + 2n + \frac{n}{2} - 2 - \frac{i+1}{2} + 3n + 4 + i = \frac{17n - 2}{2} = k_i

Subcase (ii): i \equiv 0 \pmod{2}

f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) = 2n + \frac{n}{2} - 2 - \frac{i}{2} + 3n - 2 - \frac{i+2}{2} + 3n + 4 + i = \frac{17n - 2}{2} = k_i

Case (ii): For any edge \( u_i v_i \in E_2 \)

Subcase (i): i \equiv 1 \pmod{2}

f(u_i) + f(v_i) + f(u_i v_i) = 3n - 2 - \frac{i+1}{2} + i + 5n + \frac{n}{2} + 2 - \frac{i+1}{2} = \frac{17n - 2}{2} = k_i

Subcase (ii): i \equiv 0 \pmod{2}

f(u_i) + f(v_i) + f(u_i v_i) = 2n + \frac{n}{2} - 2 - \frac{i}{2} + i + 6n + 1 - \frac{i}{2} = \frac{17n - 2}{2} = k_i

Case (iii): For any edge \( v_i w_i \in E_1 \)

Subcase (i): i \equiv 1 \pmod{2}

f(w_i) + f(v_i) + f(w_i v_i) = i + n + \frac{n}{2} - \frac{i+1}{2} + 7n - \frac{i+1}{2} = \frac{17n - 2}{2} = k_i

Subcase (ii): i \equiv 0 \pmod{2}

f(w_i) + f(v_i) + f(w_i v_i) = i + 2n - 1 - \frac{i}{2} + 6n + \frac{n}{2} - \frac{i}{2} = \frac{17n - 2}{2} = k_i

Case (iv): For any edge \( u_i w_i \in E_4 \)

Subcase (i): i \equiv 1 \pmod{2}

f(w_i) + f(u_i) + f(w_i u_i) = 3n - 2 - \frac{i+1}{2} + n + \frac{n}{2} - \frac{i+1}{2} + 4n + 2 + i = \frac{17n - 2}{2} = k_i

Subcase (ii): i \equiv 0 \pmod{2}

f(w_i) + f(u_i) + f(w_i u_i) = 2n + \frac{n}{2} - 2 - \frac{i}{2} + 2n - 1 - \frac{i}{2} + 4n + 2 + i = \frac{17n - 2}{2} = k_i
Case (v): For an edge $u_1u_1 \in E_5$

$$f(u_1) + f(u_1) + f(u_1u_1) = 3n - 1 + 3n - 3 + 3n + 4 = 9n = k_2$$

For an edge $u_1v_1 \in E_5$

$$f(u_1) + f(v_1) + f(u_1v_1) = 3n - 1 + 3n + 3n + 1 = 9n = k_2$$

For an edge $u_1w_1 \in E_5$

$$f(u_1) + f(w_1) + f(u_1w_1) = 3n - 1 + 3n - 2 + 3n + 3 = 9n = k_2$$

For an edge $w_1v_1 \in E_5$

$$f(w_1) + f(v_1) + f(w_1v_1) = 3n - 2 + 3n + 3n + 2 = 9n = k_2$$

For all the above five cases the edge counts are $k_1 = \frac{17n - 2}{2}$ and $k_2 = 9n$. Hence the graph obtained by duplication of all the vertices by edges in path $P_n$: $n \equiv 0 \text{ (mod 2)}$ is super edge bimagic total labeling.

Illustration 2.7: Super edge magic labeling of a graph obtained by duplicating all the vertices by edges in $P_5$ is shown in figure 2.

![Figure 2: k = 43](image)

Theorem 2.8: The graph $G$ obtained by duplicating all the edges by vertices in path $P_n$: $n \equiv 1 \text{ (mod 2)}$ admits super edge bimagic total labeling.

Proof: Let $u_1, u_2, \ldots, u_n$ be vertices and $e_1, e_2, \ldots, e_{n-1}$ be edges of path $P_n$. Let the graph obtained by duplicating all the edges by vertices in path $P_n$. Then the vertex set $V = \{u_i; 1 \leq i \leq n+1\} \cup \{v_i; 1 \leq i \leq n\}$ and edge set $E = E_1 \cup E_2 \cup E_3$

where $E_1 = \{u_iu_{i+1}; 1 \leq i \leq n\},$
\( E_2 = \{u_i v_i; 1 \leq i \leq n\}, E_3 = \{v_i u_{i+1}; 1 \leq i \leq n\}. \)

We define a bijective function \( f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 5n+1\} \) as follows.

For \( i = 1 \) to \( n; i \equiv 1(\text{mod} 2), \ f(u_i) = \frac{i+1}{2}. \)

For \( i = 2 \) to \( n+1; i \equiv 0(\text{mod} 2), \ f(u_i) = \frac{n+1}{2} + \frac{i}{2}. \)

For \( i = 1 \) to \( n; f(u_i, u_{i+1}) = 4n+3-i. \) For \( i = 1 \) to \( n; f(v_i) = 2n+2-i. \)

For \( i = 1 \) to \( n; i \equiv 1(\text{mod} 2), \ f(u_i v_i) = 2n+\frac{n+1}{2} + \frac{i+1}{2}. \)

For \( i = 2 \) to \( n-1; i \equiv 0(\text{mod} 2), \ f(u_i v_i) = 4n+2+i. \)

For \( i = 1 \) to \( n; i \equiv 1(\text{mod} 2), \ f(u_{i+1} v_i) = 2n+1+\frac{n+1}{2}. \)

For \( i = 2 \) to \( n-1; i \equiv 0(\text{mod} 2), \ f(u_{i+1} v_i) = 4n+1+\frac{n+1}{2} + \frac{i}{2}. \)

**Case (i):** For any edge \( u_i u_{i+1} \in E_i \)

**Subcase (i):** \( i \equiv 1 \ (\text{mod} \ 2) \)

\[ f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) = \frac{i+1}{2} + \frac{n+1}{2} + \frac{i+1}{2} + 4n+3-i = \frac{9(n+1)}{2} = k_i \]

**Subcase (ii):** \( i \equiv 0 \ (\text{mod} \ 2) \)

\[ f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) = \frac{n+1}{2} + \frac{i}{2} + \frac{i+2}{2} + 4n+3-i = \frac{9(n+1)}{2} = k_i \]

**Case (ii):** For any edge \( u_i v_i \in E_2 \)

**Subcase (i):** \( i \equiv 1 \ (\text{mod} \ 2) \)

\[ f(u_i) + f(v_i) + f(u_i v_i) = \frac{i+1}{2} + 2n+2-i+2n+\frac{i+1}{2} + \frac{i+1}{2} = \frac{9(n+1)}{2} = k_i \]
Subcase (ii): $i \equiv 0 \pmod{2}$

$$f(u_{i+1}) + f(v_i) + f(u_{i+1}v_i) = \frac{n+1}{2} + i + 2n+2 - i + 4n+2 + \frac{i}{2} = \frac{13n+9}{2} = k_2$$

Case (iii): For any edge $u_{i+1}v_i \in E_3$

Subcase (i): $i \equiv 1 \pmod{2}$

$$f(u_{i+1}) + f(v_i) + f(u_{i+1}v_i) = 2n + 2 - i + \frac{n+1}{2} + \frac{i+1}{2} + 2n+1 + \frac{i+1}{2} = \frac{9(n+1)}{2} = k_1$$

Subcase (ii): $i \equiv 0 \pmod{2}$

$$f(u_{i+1}) + f(v_i) + f(u_{i+1}v_i) = 2n + 2 - i + \frac{i+2}{2} + 4n + \frac{n+1}{2} + 1 + \frac{i}{2} = \frac{13n+9}{2} = k_2$$

For all the above three cases the edge counts are $k_1 = \frac{9(n+1)}{2}$ and $k_2 = \frac{13n+9}{2}$.

Hence the graph obtained by duplication of all the edges by vertices in path $P_n$: $n \equiv 1 \pmod{2}$ is super edge bimagic total labeling.

3 Conclusion:

In our present study, we investigated super edge magic and bimagic labeling for duplication graphs of cycles and paths. In this direction, we are interested in establishing the following results. (i) The graph $G$ obtained by duplicating all the edges by vertices in path $P_n$: $n \equiv 0 \pmod{2}$ has super edge bimagic labeling. (ii) The graph $G$ obtained by duplicating all the vertices by edges in cycle $C_n$: $n \equiv 0 \pmod{2}$ has super edge magic labeling.

References