On a Class of \(\gamma\)-b-Open Sets in a Topological Space

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Abstract

In this paper, we introduce some weak separation axioms by utilizing the notions of \(\gamma\)-b-open sets and the \(\gamma\)-b-closure operator.

Keywords: \(\gamma\)-b-open, \(\gamma\)-b-closure, \(\gamma\)\(D_b\)-set, \(\gamma\)-b-\(T_0\), \(\gamma\)-b-\(T_1\), \(\gamma\)-b-\(T_2\), \(\gamma\)-b-\(R_0\), \(\gamma\)-b-\(R_1\), \(\gamma\)-b-continuous.

1 Introduction

In [1] Andrijevi introduced b-open sets, Kasahara [3] defined an operation \(\alpha\) on a topological space to introduce \(\alpha\)-closed graphs. Following the same technique, Ogata [6] defined an operation \(\gamma\) on a topological space and introduced \(\gamma\)-open sets.

In this paper, we introduce the notion of \(\gamma\)-b-open sets, and \(\gamma\)-b-irresolute in topological spaces. By utilizing these notions we introduce some weak separation axioms. Also we show that some basic properties of \(\gamma\)-b-\(T_i\), \(\gamma\)-b-\(D_i\) for \(i = 0, 1, 2\) spaces and we offer a new class of functions called \(\gamma\)-b-continuous functions and a new notion of the graph of a function called a \(\gamma\)-b-closed graph and investigate some of their fundamental properties.
2 Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$, respectively. A subset $A$ is said to be b-open \cite{1} if $A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$. The complement of a b-open set is said to be b-closed.

An operation $\gamma$ \cite{3} on a topology $\tau$ is a mapping from $\tau$ into the power set $P(X)$ of $X$ such that $V \subset \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of $\gamma$ at $V$. A subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is called $\gamma$-open \cite{6} if for each $x \in A$, there exists an open set $U$ such that $x \in U$ and $\gamma(U) \subset A$. Then, $\tau_{\gamma}$ denotes the set of all $\gamma$-open set in $X$. Clearly $\tau_{\gamma} \subset \tau$. Complements of $\gamma$-open sets are called $\gamma$-closed. The $\gamma$-closure \cite{6} of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is denoted by $\tau_{\gamma} \text{-cl}(A)$ and is defined to be the intersection of all $\gamma$-closed sets containing $A$, and the $\tau_{\gamma}$-interior \cite{4} of $A$ is denoted by $\tau_{\gamma} \text{-int}(A)$ and defined to be the union of all $\gamma$-open sets of $X$ contained in $A$.

A subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is called be $\gamma$-preopen set \cite{5} if and only if $A \subseteq \tau_{\gamma} \text{-int}(\tau_{\gamma} \text{-cl}(A))$. A subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is called be $\gamma$-\beta-open set \cite{2} if $A \subseteq \tau_{\gamma} \text{-cl}(\tau_{\gamma} \text{-int}(\tau_{\gamma} \text{-cl}(A))).$ A topological $X$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-regular \cite{6} if for each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $\gamma(U)$ contained in $V$. It is also to be noted that $\tau = \tau_{\gamma}$ if and only if $X$ is a $\gamma$-regular space \cite{6}.

3 $\gamma$-b-Open Sets

Definition 3.1 A subset $A$ of a topological space $(X, \tau)$ is said to be $\gamma$-b-open if $A \subset \tau_{\gamma} \text{-int}(\tau_{\gamma} \text{-cl}(A)) \cup \tau_{\gamma} \text{-cl}(\tau_{\gamma} \text{-int}(A))$.

The complement of a $\gamma$-b-open set is said to be $\gamma$-b-closed. The family of all $\gamma$-b-open (resp. $\gamma$-b-closed) sets in a topological space $(X, \tau)$ is denoted by $\gamma bO(X, \tau)$ (resp. $\gamma bC(X, \tau)$).

Definition 3.2 Let $A$ be a subset of a topological space $(X, \tau)$. The intersection of all $\gamma$-b-closed sets containing $A$ is called the $\gamma$-b-closure of $A$ and is denoted by $\gamma cl_b(A)$.

Definition 3.3 Let $(X, \tau)$ be a topological space. A subset $U$ of $X$ is called a $\gamma$-b-neighbourhood of a point $x \in X$ if there exists a $\gamma$-b-open set $V$ such that $x \in V \subset U$.

Theorem 3.4 For the $\gamma$-b-closure of subsets $A, B$ in a topological space $(X, \tau)$, the following properties hold:
1. A is $\gamma$-b-closed in $(X, \tau)$ if and only if $A = \gamma \text{cl}_b(A)$.

2. If $A \subset B$ then $\gamma \text{cl}_b(A) \subset \gamma \text{cl}_b(B)$.

3. $\gamma \text{cl}_b(A)$ is $\gamma$-b-closed, that is $\gamma \text{cl}_b(A) = \gamma \text{cl}(\gamma \text{cl}_b(A))$.

4. $x \in \gamma \text{cl}_b(A)$ if and only if $A \cap V \neq \phi$ for every $\gamma$-b-open set $V$ of $X$ containing $x$.

Proof. It is obvious.

**Theorem 3.5** For a family $\{A_\alpha : \alpha \in \Delta\}$ of subsets a topological space $(X, \tau)$, the following properties hold:

1. $\gamma \text{cl}_b(\bigcap_{\alpha \in \Delta} A_\alpha) \subset \bigcap_{\alpha \in \Delta} \gamma \text{cl}_b(A_\alpha)$.

2. $\gamma \text{cl}_b(\bigcup_{\alpha \in \Delta} A_\alpha) \supset \bigcup_{\alpha \in \Delta} \gamma \text{cl}_b(A_\alpha)$.

Proof.

1. Since $\bigcap_{\alpha \in \Delta} A_\alpha \subset A_\alpha$ for each $\alpha \in \Delta$, by Theorem 3.4 we have $\gamma \text{cl}_b(\bigcap_{\alpha \in \Delta} A_\alpha) \subset \gamma \text{cl}_b(A_\alpha)$ for each $\alpha \in \Delta$ and hence $\gamma \text{cl}_b(\bigcap_{\alpha \in \Delta} A_\alpha) \subset \bigcap_{\alpha \in \Delta} \gamma \text{cl}_b(A_\alpha)$.

2. Since $A_\alpha \subset \bigcup_{\alpha \in \Delta} A_\alpha$ for each $\alpha \in \Delta$, by Theorem 3.4 we have $\gamma \text{cl}_b(A_\alpha) \subset \gamma \text{cl}_b(\bigcup_{\alpha \in \Delta} A_\alpha)$ for each $\alpha \in \Delta$ and hence $\bigcup_{\alpha \in \Delta} \gamma \text{cl}_b(A_\alpha) \subset \gamma \text{cl}_b(\bigcup_{\alpha \in \Delta} A_\alpha)$.

**Theorem 3.6** Every $\gamma$-preopen set is $\gamma$-b-open.

Proof. It follows from the Definitions.

The converse of the above Theorem need not be true by the following Example.

**Example 3.7** Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\gamma(A) = A$ for all $A \in \tau$. Here $\{a, b\}$ is not $\gamma$-preopen however it is $\gamma$-b-open.

**Corollary 3.8** Every $\gamma$-open set is $\gamma$-b-open.

Proof. It follows from Theorem 3.6.

**Theorem 3.9** Every $\gamma$-b-open set is $\gamma$-\(\beta\)-open.

Proof. It follows from the Definitions.

**Remark 3.10** The concepts of b-open and $\gamma$-b-open sets are independent, while in a $\gamma$-regular space these concepts are equivalent.
Example 3.11 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Define an operation $\gamma$ on $\tau$ by

$$\gamma(A) = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, \{a\}, X\}$. Then $\{b\}$ is $b$-open but not $\gamma$-b-open. Again, if we define $\gamma$ on $\tau$ by $\gamma(A) = X$, then $\{c\}$ is $\gamma$-b-open but not $b$-open.

**Theorem 3.12** An arbitrary union of $\gamma$-b-open sets is $\gamma$-b-open.

**Proof.** Let $\{A_k : k \in \Delta\}$ be a family of $\gamma$-b-open sets. Then for each $k$, $A_k \subseteq \tau_\gamma$-int$(\tau_\gamma$-cl$(A_k)) \cup \tau_\gamma$-cl$(\tau_\gamma$-int$(A_k))$ and so

$$\bigcup_{k \in \Delta} A_k \subseteq \bigcup_{k \in \Delta} [\tau_\gamma$-int$(\tau_\gamma$-cl$(A_k)) \cup \tau_\gamma$-cl$(\tau_\gamma$-int$(A_k)))]$$

$$\subseteq [\bigcup_{k \in \Delta} \tau_\gamma$-int$(\tau_\gamma$-cl$(A_k))] \cup [\bigcup_{k \in \Delta} \tau_\gamma$-cl$(\tau_\gamma$-int$(A_k))]$$

$$\subseteq [\tau_\gamma$-int$(\bigcup_{k \in \Delta} \tau_\gamma$-cl$(A_k))] \cup [\tau_\gamma$-cl$(\bigcup_{k \in \Delta} \tau_\gamma$-int$(A_k))]$$

$$\subseteq \tau_\gamma$-int$(\bigcup_{k \in \Delta} A_k) \cup \tau_\gamma$-cl$(\bigcup_{k \in \Delta} A_k))].$$

Therefore, $\bigcup_{k \in \Delta} A_k$ is $\gamma$-b-open.

**Remark 3.13**

1. An arbitrary intersection of $\gamma$-b-closed sets is $\gamma$-b-closed.

2. The intersection of even two $\gamma$-b-open sets may not be $\gamma$-b-open.

Example 3.14 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Define an operation $\gamma$ on $\tau$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \\ X & \text{otherwise} \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, \{a, b\}, X\}$, take $A = \{a, c\}$ and $B = \{b, c\}$ are $\gamma$-b-open. Then $A \cap B = \{c\}$, which is not a $\gamma$-b-open set.

**Definition 3.15** A subset $A$ of a topological space $(X, \tau)$ is called a $\gamma D_b$-set if there are two $U, V \in \gamma bO(X, \tau)$ such that $U \neq X$ and $A = U \setminus V$.

It is true that every $\gamma$-b-open set $U$ different from $X$ is a $\gamma D_b$-set if $A = U$ and $V = \emptyset$. So, we can observe the following.

**Remark 3.16** Every proper $\gamma$-b-open set is a $\gamma D_b$-set.
Definition 3.17  A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be

1. $\gamma$-$b$-$D_0$ if for any pair of distinct points $x$ and $y$ of $X$ there exists a $\gamma D_b$-set of $X$ containing $x$ but not $y$ or a $\gamma D_b$-set of $X$ containing $y$ but not $x$.

2. $\gamma$-$b$-$D_1$ if for any pair of distinct points $x$ and $y$ of $X$ there exists a $\gamma D_b$-set of $X$ containing $x$ but not $y$ and a $\gamma D_b$-set of $X$ containing $y$ but not $x$.

3. $\gamma$-$b$-$D_2$ if for any pair of distinct points $x$ and $y$ of $X$ there exist disjoint $\gamma D_b$-sets $G$ and $E$ of $X$ containing $x$ and $y$, respectively.

Definition 3.18  A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be

1. $\gamma$-$b$-$T_0$ (resp. $\gamma$-pre $T_0$ [5] and $\gamma$-$\beta$ $T_0$ [2]) if for any pair of distinct points $x$ and $y$ of $X$ there exists a $\gamma$-b-open (resp. $\gamma$-preopen and $\gamma$-$\beta$-open) set $U$ in $X$ containing $x$ but not $y$ or a $\gamma$-b-open (resp. $\gamma$-preopen and $\gamma$-$\beta$-open) set $V$ in $X$ containing $y$ but not $x$.

2. $\gamma$-$b$-$T_1$ (resp. $\gamma$-pre $T_1$ [5] and $\gamma$-$\beta$ $T_1$ [2]) if for any pair of distinct points $x$ and $y$ of $X$ there exists a $\gamma$-b-open (resp. $\gamma$-preopen and $\gamma$-$\beta$-open) set $U$ in $X$ containing $x$ but not $y$ and a $\gamma$-b-open (resp. $\gamma$-preopen and $\gamma$-$\beta$-open) set $V$ in $X$ containing $y$ but not $x$.

3. $\gamma$-$b$-$T_2$ (resp. $\gamma$-pre $T_2$ [5] and $\gamma$-$\beta$ $T_2$ [2]) if for any pair of distinct points $x$ and $y$ of $X$ there exist disjoint $\gamma$-b-open (resp. $\gamma$-preopen and $\gamma$-$\beta$-open) sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively.

Remark 3.19  For a topological space $(X, \tau)$, the following properties hold:

1. If $(X, \tau)$ is $\gamma$-$b$-$T_i$, then it is $\gamma$-$b$-$T_{i-1}$, for $i = 1, 2$.

2. If $(X, \tau)$ is $\gamma$-$b$-$T_i$, then it is $\gamma$-$b$-$D_i$, for $i = 0, 1, 2$.

3. If $(X, \tau)$ is $\gamma$-$b$-$D_i$, then it is $\gamma$-$b$-$D_{i-1}$, for $i = 1, 2$.

4. If $(X, \tau)$ is $\gamma$-pre $T_i$, then it is $\gamma$-$b$-$T_i$, for $i = 0, 1, 2$.

5. If $(X, \tau)$ is $\gamma$-$b$-$T_i$, then it is $\gamma$-$\beta$ $T_i$, for $i = 0, 1, 2$. 
By Remark 3.19 we have the following diagram.

\[ \begin{array}{c}
\gamma\text{-pre } T_2 & \longrightarrow & \gamma\text{-pre } T_1 & \longrightarrow & \gamma\text{-pre } T_0 \\
\downarrow & & \downarrow & & \downarrow \\
\gamma\text{-b-} T_2 & \longrightarrow & \gamma\text{-b-} T_1 & \longrightarrow & \gamma\text{-b-} T_0 \\
\downarrow & & \downarrow & & \downarrow \\
\gamma\text{-} \beta \text{ } T_2 & \longrightarrow & \gamma\text{-} \beta \text{ } T_1 & \longrightarrow & \gamma\text{-} \beta \text{ } T_0 \\
\end{array} \]

**Theorem 3.20** A topological space \((X, \tau)\) is \(\gamma\text{-}b\text{-}D_1\) if and only if it is \(\gamma\text{-}b\text{-}D_2\).

**Proof.** sufficiency. Follows from Remark 3.19.
Necessity. Let \(x, y \in X\), \(x \neq y\). Then there exist \(\gamma D_1\)-sets \(G_1, G_2\) in \(X\) such that \(x \in G_1\), \(y \notin G_1\) and \(y \in G_2\), \(x \notin G_2\). Let \(G_1 = U_1 \setminus U_2\) and \(G_2 = U_3 \setminus U_4\), where \(U_1, U_2, U_3\) and \(U_4\) are \(\gamma\text{-}b\text{-}open\) sets in \(X\). From \(x \notin G_2\), it follows that either \(x \notin U_3\) or \(x \in U_3\) and \(x \in U_4\). We discuss the two cases separately.

(i) \(x \notin U_3\). By \(y \notin G_1\) we have two subcases:
(a) \(y \notin U_1\). From \(x \in U_1 \setminus U_2\), it follows that \(x \in U_1 \setminus (U_2 \cup U_3)\), and by \(y \in U_3 \setminus U_4\) we have \(y \in U_3 \setminus (U_1 \cup U_3)\). Therefore \((U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_3)) = \emptyset\).
(b) \(y \in U_1\) and \(y \in U_2\). We have \(x \in U_1 \setminus U_2\) and \(y \in U_2\). Therefore \((U_1 \setminus U_2) \cap U_2 = \emptyset\).

(ii) \(x \in U_3\) and \(x \in U_4\). We have \(y \in U_3 \setminus U_4\) and \(x \in U_4\). Hence \((U_3 \setminus U_4) \cap U_4 = \emptyset\). Therefore \(X\) is \(\gamma\text{-}b\text{-}D_2\).

**Definition 3.21** A point \(x \in X\) which has only \(X\) as the \(\gamma\text{-}b\text{-}neighborhood\) is called a \(\gamma\text{-}b\text{-}neat point.

**Theorem 3.22** If a topological space \((X, \tau)\) is \(\gamma\text{-}b\text{-}D_1\), then it has no \(\gamma\text{-}b\text{-}neat point.

**Proof.** Since \((X, \tau)\) is \(\gamma\text{-}b\text{-}D_1\), so each point \(x\) of \(X\) is contained in a \(\gamma D_1\)-set \(A = U \setminus V\) and thus in \(U\). By definition \(U \neq X\). This implies that \(x\) is not a \(\gamma\text{-}b\text{-}neat point.

**Theorem 3.23** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\text{-}b\text{-}T_0\) if and only if for each pair of distinct points \(x, y\) of \(X\), \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\).

**Theorem 3.24** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\text{-}b\text{-}T_1\) if and only if the singletons are \(\gamma\text{-}b\text{-}closed\) sets.
**Proof.** Let \((X, \tau)\) be \(\gamma\)-b-\(T_1\) and \(x\) any point of \(X\). Suppose \(y \in X \setminus \{x\}\), then \(x \neq y\) and so there exists a \(\gamma\)-b-open set \(U\) such that \(y \in U\) but \(x \notin U\). Consequently \(y \in U \subset X \setminus \{x\}\ i.e., X \setminus \{x\} = \bigcup\{ U : y \in X \setminus \{x\}\}\) which is \(\gamma\)-b-open.

Conversely, suppose \(\{p\}\) is \(\gamma\)-b-closed for every \(p \in X\). Let \(x, y \in X\) with \(x \neq y\). Now \(x \neq y\) implies \(y \in X \setminus \{x\}\). Hence \(X \setminus \{x\}\) is a \(\gamma\)-b-open set contains \(y\) but not \(x\). Similarly \(X \setminus \{x\}\) is a \(\gamma\)-b-open set contains \(x\) but not \(y\). Accordingly \(X\) is a \(\gamma\)-b-\(T_1\) space.

**Definition 3.25** A topological space \((X, \tau)\) is \(\gamma\)-b-symmetric if for \(x\) and \(y\) in \(X\), \(x \in \gamma cl_b(\{y\})\) implies \(y \in \gamma cl_b(\{x\})\).

**Theorem 3.26** If a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is a \(\gamma\)-b-\(T_1\) space, then it is \(\gamma\)-b-symmetric.

**Proof.** Suppose that \(y \notin \gamma cl_b(\{x\})\). Then, since \(x \neq y\), there exists a \(\gamma\)-b-open set \(U\) containing \(x\) such that \(y \notin U\) and hence \(x \notin \gamma cl_b(\{y\})\). This shows that \(x \in \gamma cl_b(\{y\})\) implies \(y \in \gamma cl_b(\{x\})\). Therefore, \((X, \tau)\) is \(\gamma\)-b-symmetric.

**Definition 3.27** Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \(\gamma, \beta\) operations on \(\tau, \sigma\), respectively. A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be \(\gamma\)-b-irresolute if for each \(x \in X\) and each \(\beta\)-b-open set \(V\) containing \(f(x)\), there is a \(\gamma\)-b-open set \(U\) in \(X\) containing \(x\) such that \(f(U)\subset V\).

**Theorem 3.28** If \(f : (X, \tau) \to (Y, \sigma)\) is a \(\gamma\)-b-irresolute surjective function and \(E\) is a \(\beta D_b\)-set in \(Y\), then the inverse image of \(E\) is a \(\gamma D_b\)-set in \(X\).

**Proof.** Let \(E\) be a \(\beta D_b\)-set in \(Y\). Then there are \(\beta\)-b-open sets \(U_1\) and \(U_2\) in \(Y\) such that \(E = U_1 \setminus U_2\) and \(U_1 \neq Y\). By the \(\gamma\)-b-irresolute of \(f\), \(f^{-1}(U_1)\) and \(f^{-1}(U_2)\) are \(\gamma\)-b-open in \(X\). Since \(U_1 \neq Y\) and \(f\) is surjective, we have \(f^{-1}(U_1) \neq X\). Hence, \(f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)\) is a \(\gamma D_b\)-set.

**Theorem 3.29** If \((Y, \sigma)\) is \(\beta\)-b-\(D_1\) and \(f : (X, \tau) \to (Y, \sigma)\) is \(\gamma\)-b-irresolute bijective, then \((X, \tau)\) is \(\gamma\)-b-\(D_1\).

**Proof.** Suppose that \(Y\) is a \(\beta\)-b-\(D_1\) space. Let \(x\) and \(y\) be any pair of distinct points in \(X\). Since \(f\) is injective and \(Y\) is \(\beta\)-b-\(D_1\), there exist \(\beta D_b\)-set \(G_x\) and \(G_y\) of \(Y\) containing \(f(x)\) and \(f(y)\) respectively, such that \(f(x) \notin G_y\) and \(f(y) \notin G_x\). By Theorem 3.28, \(f^{-1}(G_x)\) and \(f^{-1}(G_y)\) are \(\gamma D_b\)-set in \(X\) containing \(x\) and \(y\), respectively, such that \(x \notin f^{-1}(G_y)\) and \(y \notin f^{-1}(G_x)\). This implies that \(X\) is a \(\gamma\)-b-\(D_1\) space.

**Theorem 3.30** A topological space \((X, \tau)\) is \(\gamma\)-b-\(D_1\) if for each pair of distinct points \(x, y \in X\), there exists a \(\gamma\)-b-irresolute surjective function \(f : (X, \tau) \to (Y, \sigma)\), where \(Y\) is a \(\beta\)-b-\(D_1\) space such that \(f(x)\) and \(f(y)\) are distinct.
Proof. Let \( x \) and \( y \) be any pair of distinct points in \( X \). By hypothesis, there exists a \( \gamma \)-b-irresolute, surjective function \( f \) of a space \( X \) onto a \( \beta \)-b-\( D_1 \) space \( Y \) such that \( f(x) \neq f(y) \). By Theorem 3.20, there exist disjoint \( \beta D_b \)-set \( G_x \) and \( G_y \) in \( Y \) such that \( f(x) \in G_x \) and \( f(y) \in G_y \). Since \( f \) is \( \gamma \)-b-irresolute and surjective, by Theorem 3.28, \( f^{-1}(G_x) \) and \( f^{-1}(G_y) \) are disjoint \( \gamma D_b \)-sets in \( X \) containing \( x \) and \( y \), respectively. Hence by Theorem 3.20, \( X \) is \( \gamma \)-b-\( D_1 \) space.

4. \( \gamma \)-b-\( R_0 \) and \( \gamma \)-b-\( R_1 \) Spaces

Definition 4.1 Let \( A \) be a subset of a topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \). The \( \gamma \)-b-kernel of \( A \), denoted by \( \gamma ker_b(A) \), is defined to be the set

\[
\gamma ker_b(A) = \cap\{U \in \gamma bO(X): A \subset U\}.
\]

Theorem 4.2 Let \((X, \tau)\) be a topological space with an operation \( \gamma \) on \( \tau \) and \( x \in X \). Then \( y \in \gamma ker_b(\{x\}) \) if and only if \( x \in \gamma cl_b(\{y\}) \).

Proof. Suppose that \( y \notin \gamma ker_b(\{x\}) \). Then there exists a \( \gamma \)-b-open set \( V \) containing \( x \) such that \( y \notin V \). Therefore, we have \( x \notin \gamma cl_b(\{y\}) \). The proof of the converse case can be done similarly.

Lemma 4.3 Let \((X, \tau)\) be a topological space and \( A \) be a subset of \( X \). Then, 
\[
\gamma ker_b(A) = \{x \in X: \gamma cl_b(\{x\}) \cap A \neq \phi\}.
\]

Proof. Let \( x \in \gamma ker_b(A) \) and suppose \( \gamma cl_b(\{x\}) \cap A = \phi \). Hence \( x \notin X \setminus \gamma cl_b(\{x\}) \) which is a \( \gamma \)-b-open set containing \( A \). This is impossible, since \( x \in \gamma ker_b(A) \). Consequently, \( \gamma cl_b(\{x\}) \cap A \neq \phi \). Next, let \( x \in X \) such that \( \gamma cl_b(\{x\}) \cap A \neq \phi \) and suppose that \( x \notin \gamma ker_b(A) \). Then, there exists a \( \gamma \)-b-open set \( V \) containing \( A \) and \( x \notin V \). Let \( y \in \gamma cl_b(\{x\}) \cap A \). Hence, \( V \) is a \( \gamma \)-b-neighborhood of \( y \) which does not contain \( x \). By this contradiction \( x \in \gamma ker_b(A) \) and the claim.

Remark 4.4 The following properties hold for the subsets \( A, B \) of a topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \):

1. \( A \subset \gamma ker_b(A) \), if \( A \) is \( \gamma \)-b-open in \((X, \tau)\), then \( A = \gamma ker_b(A) \).

2. If \( A \subset B \), then \( \gamma ker_b(A) \subset \gamma ker_b(B) \).

Definition 4.5 A topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \) is said to be \( \gamma \)-b-\( R_0 \) if every \( \gamma \)-b-open set \( U \) and \( x \in U \) implies \( \gamma cl_b(\{x\}) \subset U \).

Theorem 4.6 For a topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \), the following properties are equivalent:
1. \((X, \tau)\) is \(\gamma\)\(-b\)-R\(_{0}\).

2. For any \(F \in \gamma bC(X)\), \(x \notin F\) implies \(F \subset U\) and \(x \notin U\) for some \(U \in \gamma bO(X)\).

3. For any \(F \in \gamma bC(X)\), \(x \notin F\) implies \(F \cap \gamma cl_b\{x\} = \phi\).

4. For any distinct points \(x\) and \(y\) of \(X\), either \(\gamma cl_b\{x\} = \gamma cl_b\{y\}\) or \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\).

**Proof.** (1) \(\Rightarrow\) (2). Let \(F \in \gamma bC(X)\) and \(x \notin F\). Then by (1) \(\gamma cl_b\{x\} \subset X \setminus F\). Set \(U = X \setminus \gamma cl_b\{x\}\), then \(U\) is \(\gamma\)\(-b\)-open set such that \(F \subset U\) and \(x \notin U\).

(2) \(\Rightarrow\) (3). Let \(F \in \gamma bC(X)\) and \(x \notin F\). There exists \(U \in \gamma bO(X)\) such that \(F \subset U\) and \(x \notin U\). Since \(U \in \gamma bO(X)\), \(U \cap \gamma cl_b\{x\} = \phi\) and \(F \cap \gamma cl_b\{x\} = \phi\).

(3) \(\Rightarrow\) (4). Suppose that \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\) for distinct points \(x, y \in X\). There exists \(z \in \gamma cl_b\{x\}\) such that \(z \notin \gamma cl_b\{y\}\) (or \(z \in \gamma cl_b\{y\}\) such that \(z \notin \gamma cl_b\{x\}\)). There exists \(V \in \gamma bO(X)\) such that \(y \notin V\) and \(z \notin V\); hence \(x \in V\). Therefore, we have \(x \notin \gamma cl_b\{y\}\). By (3), we obtain \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\). The proof for otherwise is similar.

(4) \(\Rightarrow\) (1). Let \(V \in \gamma bO(X)\) and \(x \in V\). For each \(y \notin V\), \(x \neq y\) and \(x \notin \gamma cl_b\{y\}\). This shows that \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\). By (4), \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\) for each \(y \in X \setminus V\) and hence \(\gamma cl_b\{x\} \cap (\bigcup_{y \in X \setminus V} \gamma cl_b\{y\}) = \phi\).

On other hand, since \(V \in \gamma bO(X)\) and \(y \in X \setminus V\), we have \(\gamma cl_b\{y\} \subset X \setminus V\) and hence \(X \setminus V = \bigcup_{y \in X \setminus V} \gamma cl_b\{y\}\). Therefore, we obtain \((X \setminus V) \cap \gamma cl_b\{x\} = \phi\) and \(\gamma cl_b\{x\} \subset V\). This shows that \((X, \tau)\) is a \(\gamma\)\(-b\)-R\(_{0}\) space.

**Theorem 4.7** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\)\(-b\)-\(T_1\) if and only if \((X, \tau)\) is \(\gamma\)\(-b\)-\(T_0\) and \(\gamma\)\(-b\)-\(R_0\) space.

**Proof.** Necessity. Let \(U\) be any \(\gamma\)\(-b\)-open set of \((X, \tau)\) and \(x \in U\). Then by Theorem 3.24, we have \(\gamma cl_b\{x\} \subset U\) and so by Remark 3.19, it is clear that \(X\) is \(\gamma\)\(-b\)-\(T_0\) and \(\gamma\)\(-b\)-\(R_0\) space.

Sufficiency. Let \(x\) and \(y\) be any distinct points of \(X\). Since \(X\) is \(\gamma\)\(-b\)-\(T_0\), there exists a \(\gamma\)\(-b\)-open set \(U\) such that \(x \in U\) and \(y \notin U\). As \(x \in U\) implies that \(\gamma cl_b\{x\} \subset U\). Since \(y \notin U\), \(y \notin \gamma cl_b\{x\}\). Hence \(y \in V = X \setminus \gamma cl_b\{x\}\) and it is clear that \(x \notin V\). Hence it follows that there exist \(\gamma\)\(-b\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively, such that \(y \notin U\) and \(x \notin V\). This implies that \(X\) is \(\gamma\)\(-b\)-\(T_1\).

**Theorem 4.8** For a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\), the following properties are equivalent:

1. \((X, \tau)\) is \(\gamma\)\(-b\)-\(R_0\).
2. \( x \in \gamma cl_b\{\{y\}\} \) if and only if \( y \in \gamma cl_b\{\{x\}\} \), for any points \( x \) and \( y \) in \( X \).

**Proof.** (1) \( \Rightarrow \) (2). Assume that \( X \) is \( \gamma \)-\( b \)-\( R_0 \). Let \( x \in \gamma cl_b\{\{y\}\} \) and \( V \) be any \( \gamma \)-\( b \)-open set such that \( y \in V \). Now by hypothesis, \( x \in V \). Therefore, every \( \gamma \)-\( b \)-open set which contain \( y \) contains \( x \). Hence \( y \in \gamma cl_b\{\{x\}\} \).

(2) \( \Rightarrow \) (1). Let \( U \) be a \( \gamma \)-\( b \)-open set and \( x \in U \). If \( y \notin U \), then \( x \notin \gamma cl_b\{\{y\}\} \) and hence \( y \notin \gamma cl_b\{\{x\}\} \). This implies that \( \gamma cl_b\{\{x\}\} \subset U \). Hence \( (X, \tau) \) is \( \gamma \)-\( b \)-\( R_0 \).

We observed that by Definition 3.25 and Theorem 4.8 the notions of \( \gamma \)-\( b \)-symmetric and \( \gamma \)-\( b \)-\( R_0 \) are equivalent.

**Theorem 4.9** The following statements are equivalent for any points \( x \) and \( y \) in a topological space \( (X, \tau) \) with an operation \( \gamma \) on \( \tau \):

1. \( \gamma ker_b\{\{x\}\} \neq \gamma ker_b\{\{y\}\} \).
2. \( \gamma cl_b\{\{x\}\} \neq \gamma cl_b\{\{y\}\} \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( \gamma ker_b\{\{x\}\} \neq \gamma ker_b\{\{y\}\} \), then there exists a point \( z \) in \( X \) such that \( z \in \gamma ker_b\{\{x\}\} \) and \( z \notin \gamma ker_b\{\{y\}\} \). From \( z \in \gamma ker_b\{\{x\}\} \) it follows that \( \{x\} \cap \gamma cl_b\{\{z\}\} \neq \phi \) which implies \( x \in \gamma cl_b\{\{z\}\} \). By \( z \notin \gamma ker_b\{\{y\}\} \), we have \( \{y\} \cap \gamma cl_b\{\{z\}\} = \phi \). Since \( x \in \gamma cl_b\{\{z\}\} \), \( \gamma cl_b\{\{x\}\} \subset \gamma cl_b\{\{z\}\} \) and \( \{y\} \cap \gamma cl_b\{\{x\}\} = \phi \). Therefore, it follows that \( \gamma cl_b\{\{x\}\} \neq \gamma cl_b\{\{y\}\} \).

(2) \( \Rightarrow \) (1). Suppose that \( \gamma cl_b\{\{x\}\} \neq \gamma cl_b\{\{y\}\} \). Then there exists a point \( z \) in \( X \) such that \( z \in \gamma cl_b\{\{x\}\} \) and \( z \notin \gamma cl_b\{\{y\}\} \). Then, there exists a \( \gamma \)-\( b \)-open set containing \( z \) and therefore \( x \) but not \( y \), namely, \( y \notin \gamma ker_b\{\{x\}\} \) and thus \( \gamma ker_b\{\{x\}\} \neq \gamma ker_b\{\{y\}\} \).

**Theorem 4.10** Let \( (X, \tau) \) be a topological space and \( \gamma \) be an operation on \( \tau \). Then \( \cap\{\gamma cl_b\{\{x\}\} : x \in X\} = \phi \) if and only if \( \gamma ker_b\{\{x\}\} \neq X \) for every \( x \in X \).

**Proof.** Necessity. Suppose that \( \cap\{\gamma cl_b\{\{x\}\} : x \in X\} = \phi \). Assume that there is a point \( y \) in \( X \) such that \( \gamma ker_b\{\{y\}\} = X \). Let \( x \) be any point of \( X \). Then \( x \in V \) for every \( \gamma \)-\( b \)-open set \( V \) containing \( y \) and hence \( y \in \gamma cl_b\{\{x\}\} \) for any \( x \in X \). This implies that \( y \in \cap\{\gamma cl_b\{\{x\}\} : x \in X\} \). But this is a contradiction.

Sufficiency. Assume that \( \gamma ker_b\{\{x\}\} \neq X \) for every \( x \in X \). If there exists a point \( y \) in \( X \) such that \( y \in \cap\{\gamma cl_b\{\{x\}\} : x \in X\} \), then every \( \gamma \)-\( b \)-open set containing \( y \) must contain every point of \( X \). This implies that the space \( X \) is the unique \( \gamma \)-\( b \)-open set containing \( y \). Hence \( \gamma ker_b\{\{y\}\} = X \) which is a contradiction. Therefore, \( \cap\{\gamma cl_b\{\{x\}\} : x \in X\} = \phi \).
Theorem 4.11 A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\)-b-\(R_0\) if and only if for every \(x\) and \(y\) in \(X\), \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\) implies \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\).

Proof. Necessity. Suppose that \((X, \tau)\) is \(\gamma\)-b-\(R_0\) and \(x, y \in X\) such that \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\). Then, there exists \(z \in \gamma cl_b\{x\}\) such that \(z \notin \gamma cl_b\{y\}\) (or \(z \notin \gamma cl_b\{x\}\)). There exists \(V \in \gamma bo(X)\) such that \(y \notin V\) and \(z \in V\), hence \(x \in V\). Therefore, we have \(x \notin \gamma cl_b\{y\}\). Thus \(x \in [X \setminus \gamma cl_b\{y\}] \in \gamma bo(X)\), which implies \(\gamma cl_b\{x\} \subset [X \setminus \gamma cl_b\{y\}]\) and \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\). The proof for otherwise is similar.

Sufficiency. Let \(V \in \gamma bo(X)\) and let \(x \in V\). We still show that \(\gamma cl_b\{x\} \subset V\). Let \(y \notin V\), i.e., \(y \notin X \setminus V\). Then \(x \neq y\) and \(x \notin \gamma cl_b\{y\}\). This shows that \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\). By assumption, \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\). Hence \(y \notin \gamma cl_b\{x\}\) and therefore \(\gamma cl_b\{x\} \subset V\).

Theorem 4.12 A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\)-b-\(R_0\) if and only if for any points \(x\) and \(y\) in \(X\), \(\gamma ker_b\{x\} \neq \gamma ker_b\{y\}\) implies \(\gamma ker_b\{x\} \cap \gamma ker_b\{y\} = \phi\).

Proof. Suppose that \((X, \tau)\) is a \(\gamma\)-b-\(R_0\) space. Thus by Theorem 4.9, for any points \(x\) and \(y\) in \(X\) if \(\gamma ker_b\{x\} \neq \gamma ker_b\{y\}\) then \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\). Now we prove that \(\gamma ker_b\{x\} \cap \gamma ker_b\{y\} \neq \phi\). Assume that \(z \in \gamma ker_b\{x\} \cap \gamma ker_b\{y\}\). By \(z \in \gamma ker_b\{x\}\) and Theorem 4.2, it follows that \(x \in \gamma cl_b\{z\}\) and \(z \in \gamma cl_b\{y\}\) implies \(\gamma cl_b\{x\} \subset \gamma cl_b\{z\}\). Similarly, we have \(\gamma cl_b\{y\} = \gamma cl_b\{z\} = \gamma cl_b\{x\}\). This is a contradiction. Therefore, we have \(\gamma ker_b\{x\} \cap \gamma ker_b\{y\} = \phi\).

Conversely, let \((X, \tau)\) be a topological space such that for any points \(x\) and \(y\) in \(X\), \(\gamma ker_b\{x\} \neq \gamma ker_b\{y\}\) implies \(\gamma ker_b\{x\} \cap \gamma ker_b\{y\} = \phi\). If \(\gamma cl_b\{x\} \neq \gamma cl_b\{y\}\), then by Theorem 4.9, \(\gamma ker_b\{x\} \neq \gamma ker_b\{y\}\). Hence, \(\gamma ker_b\{x\} \cap \gamma ker_b\{y\} = \phi\) which implies \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\). Because \(z \in \gamma cl_b\{x\}\) implies that \(x \in \gamma ker_b\{z\}\) and therefore \(\gamma ker_b\{x\} \cap \gamma ker_b\{z\} \neq \phi\). By hypothesis, we have \(\gamma ker_b\{x\} = \gamma ker_b\{z\}\). Then \(z \in \gamma cl_b\{x\} \cap \gamma cl_b\{y\}\) implies that \(\gamma ker_b\{x\} = \gamma ker_b\{z\} = \gamma ker_b\{y\}\). This is a contradiction. Therefore, \(\gamma cl_b\{x\} \cap \gamma cl_b\{y\} = \phi\) and by Theorem 4.6 \((X, \tau)\) is a \(\gamma\)-b-\(R_0\) space.

Theorem 4.13 For a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\), the following properties are equivalent:

1. \((X, \tau)\) is a \(\gamma\)-b-\(R_0\) space.

2. For any nonempty set \(A\) and \(G \in \gamma bo(X)\) such that \(A \cap G \neq \phi\), there exists \(F \in \gamma bC(X)\) such that \(A \cap F \neq \phi\) and \(F \subset G\).
3. Any \( G \in \gamma bO(X) \), \( G = \bigcup \{ F \in \gamma bC(X) : F \subset G \} \).

4. Any \( F \in \gamma bC(X) \), \( F = \bigcap \{ G \in \gamma bO(X) : F \subset G \} \).

5. For every \( x \in X \), \( \gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\}) \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( A \) be a nonempty subset of \( X \) and \( G \in \gamma bO(X) \) such that \( A \cap G \neq \emptyset \). There exists \( x \in A \cap G \). Since \( x \in G \in \gamma bO(X) \), \( \gamma cl_b(\{x\}) \subset G \). Set \( F = \gamma cl_b(\{x\}) \), then \( F \in \gamma bC(X) \), \( F \subset G \) and \( A \cap F \neq \emptyset \).

(2) \( \Rightarrow \) (3). Let \( G \in \gamma bO(X) \), then \( G \supseteq \bigcup \{ F \in \gamma bC(X) : F \subset G \} \). Let \( x \) be any point of \( G \). There exists \( F \in \gamma bC(X) \) such that \( x \in F \) and \( F \subset G \). Therefore, we have \( x \in F \subset \bigcup \{ F \in \gamma bC(X) : F \subset G \} \) and hence \( G = \bigcup \{ F \in \gamma bC(X) : F \subset G \} \).

(3) \( \Rightarrow \) (4). This is obvious.

(4) \( \Rightarrow \) (5). Let \( x \) be any point of \( X \) and \( y \notin \gamma ker_b(\{x\}) \). There exists \( V \in \gamma bO(X) \) such that \( x \in V \) and \( y \notin V \), hence \( \gamma cl_b(\{y\}) \cap V = \emptyset \). By (4) \( \bigcap \{ G \in \gamma bO(X) : \gamma cl_b(\{y\}) \subset G \} \cap V = \emptyset \) and there exists \( G \in \gamma bO(X) \) such that \( x \notin G \) and \( \gamma cl_b(\{y\}) \subset G \). Therefore \( \gamma cl_b(\{x\}) \cap G = \emptyset \) and \( y \notin \gamma cl_b(\{x\}) \). Consequently, we obtain \( \gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\}) \).

(5) \( \Rightarrow \) (1). Let \( G \in \gamma bO(X) \) and \( x \in G \). Let \( y \in \gamma ker_b(\{x\}) \), then \( x \in \gamma cl_b(\{y\}) \) and \( y \in G \). This implies that \( \gamma ker_b(\{x\}) \subset G \). Therefore, we obtain \( x \in \gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\}) \subset G \). This shows that \( (X, \tau) \) is a \( \gamma b-R_0 \) space.

**Corollary 4.14** For a topological space \( (X, \tau) \) with an operation \( \gamma \) on \( \tau \), the following properties are equivalent:

1. \( (X, \tau) \) is a \( \gamma b-R_0 \) space.

2. \( \gamma cl_b(\{x\}) = \gamma ker_b(\{x\}) \) for all \( x \in X \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( (X, \tau) \) is a \( \gamma b-R_0 \) space. By Theorem 4.13, \( \gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\}) \) for each \( x \in X \). Let \( y \in \gamma ker_b(\{x\}) \), then \( x \in \gamma cl_b(\{y\}) \) and by Theorem 4.6 \( \gamma cl_b(\{x\}) = \gamma cl_b(\{y\}) \). Therefore, \( y \in \gamma cl_b(\{x\}) \) and hence \( \gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\}) \). This shows that \( \gamma cl_b(\{x\}) = \gamma ker_b(\{x\}) \).

(2) \( \Rightarrow \) (1). This is obvious by Theorem 4.13.

**Theorem 4.15** For a topological space \( (X, \tau) \) with an operation \( \gamma \) on \( \tau \), the following properties are equivalent:

1. \( (X, \tau) \) is a \( \gamma b-R_0 \) space.

2. If \( F \) is \( \gamma b \)-closed, then \( F = \gamma ker_b(F) \).

3. If \( F \) is \( \gamma b \)-closed and \( x \in F \), then \( \gamma ker_b(\{x\}) \subset F \).

4. If \( x \in X \), then \( \gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\}) \).
Proof. (1) ⇒ (2). Let $F$ be a $\gamma$-b-closed and $x \notin F$. Thus $(X \setminus F)$ is a $\gamma$-b-open set containing $x$. Since $(X, \tau)$ is $\gamma$-b-$R_0$, $\gamma cl_b(\{x\}) \subset (X \setminus F)$. Thus $\gamma cl_b(\{x\}) \cap F = \phi$ and by Lemma 4.3 $x \notin \gamma ker_b(F)$. Therefore $\gamma ker_b(F) = F$.

(2) ⇒ (3). In general, $A \subseteq B$ implies $\gamma ker_b(A) \subset \gamma ker_b(B)$. Therefore, it follows from (2) that $\gamma ker_b(\{x\}) \subset \gamma ker_b(F) = F$.

(3) ⇒ (4). Since $x \in \gamma cl_b(\{x\})$ and $\gamma cl_b(\{x\})$ is $\gamma$-b-closed, by (3), $\gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$.

(4) ⇒ (1). We show the implication by using Theorem 4.8. Let $x \in \gamma cl_b(\{y\})$. Then by Theorem 4.2, $y \in \gamma ker_b(\{x\})$. Since $x \in \gamma cl_b(\{x\})$ and $\gamma cl_b(\{x\})$ is $\gamma$-b-closed, by (4) we obtain $y \in \gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$. Therefore $x \in \gamma cl_b(\{y\})$ implies $y \in \gamma cl_b(\{x\})$. The converse is obvious and $(X, \tau)$ is $\gamma$-b-$R_0$.

Definition 4.16 A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, is said to be $\gamma$-b-$R_1$ if for $x, y \in X$ with $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$, there exist disjoint $\gamma$-b-open sets $U$ and $V$ such that $\gamma cl_b(\{x\}) \subset U$ and $\gamma cl_b(\{y\}) \subset V$.

Theorem 4.17 A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is $\gamma$-b-$R_1$ if it is $\gamma$-b-$T_2$.

Proof. Let $x$ and $y$ be any points of $X$ such that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. By Remark 3.19, every $\gamma$-b-$T_2$ space is $\gamma$-b-$T_1$. Therefore, by Theorem 3.24, $\gamma cl_b(\{x\}) = \{x\}$, $\gamma cl_b(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since $(X, \tau)$ is $\gamma$-b-$T_2$, there exist disjoint $\gamma$-b-open sets $U$ and $V$ such that $\gamma cl_b(\{x\}) = \{x\} \subset U$ and $\gamma cl_b(\{y\}) = \{y\} \subset V$. This shows that $(X, \tau)$ is $\gamma$-b-$R_1$.

Theorem 4.18 For a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, the following are equivalent:

1. $(X, \tau)$ is $\gamma$-b-$T_2$.
2. $(X, \tau)$ is $\gamma$-b-$R_1$ and $\gamma$-b-$T_1$.
3. $(X, \tau)$ is $\gamma$-b-$R_1$ and $\gamma$-b-$T_0$.

Proof. Proof is easy and hence omitted.

Theorem 4.19 For a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, the following statements are equivalent:

1. $(X, \tau)$ is $\gamma$-b-$R_1$.
2. If $x, y \in X$ such that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$, then there exist $\gamma$-b-closed sets $F_1$ and $F_2$ such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. Proof is easy and hence omitted.
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Theorem 4.20 If $(X, \tau)$ is $\gamma$-$b$-$R_1$, then $(X, \tau)$ is $\gamma$-$b$-$R_0$.

**Proof.** Let $U$ be $\gamma$-$b$-open such that $x \in U$. If $y \notin U$, since $x \notin \gamma cl_b\{y\}$, we have $\gamma cl_b\{x\} \neq \gamma cl_b\{y\}$. So, there exists a $\gamma$-$b$-open set $V$ such that $\gamma cl_b\{y\} \subset V$ and $x \notin V$, which implies $y \notin \gamma cl_b\{x\}$. Hence $\gamma cl_b\{x\} \subset U$. Therefore, $(X, \tau)$ is $\gamma$-$b$-$R_0$.

The converse of the above Theorem need not be true as shown in the following example.

**Example 4.21** Consider $X = \{a,b,c\}$ with the discrete topology on $X$. Define an operation $\gamma$ on $\tau$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Then $X$ is a $\gamma$-$b$-$R_0$ space but not a $\gamma$-$b$-$R_1$ space.

**Theorem 4.22** A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is $\gamma$-$b$-$R_1$ if and only if for $x, y \in X$, $\gamma ker\{\{x\}\} \neq \gamma ker\{\{y\}\}$, there exist disjoint $\gamma$-$b$-open sets $U$ and $V$ such that $\gamma cl_b\{x\} \subset U$ and $\gamma cl_b\{y\} \subset V$.

**Proof.** It follows from Theorem 4.9.

**Theorem 4.23** A topological space $(X, \tau)$ is $\gamma$-$b$-$R_1$ if and only if the inclusion $x \in X \setminus \gamma cl_b\{y\}$ implies that $x$ and $y$ have disjoint $\gamma$-$b$-open neighborhoods.

**Proof.** Necessity. Let $x \in X \setminus \gamma cl_b\{y\}$. Then $\gamma cl_b\{x\} \neq \gamma cl_b\{y\}$ and $x$ and $y$ have disjoint $\gamma$-$b$-open neighborhoods.

Sufficiency. First, we show that $(X, \tau)$ is $\gamma$-$b$-$R_0$. Let $U$ be a $\gamma$-$b$-open set and $x \in U$. Suppose that $y \notin U$. Then, $\gamma cl_b\{y\} \cap U = \emptyset$ and $x \notin \gamma cl_b\{y\}$. There exist $\gamma$-$b$-open sets $U_x$ and $U_y$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. Hence, $\gamma cl_b\{x\} \subset \gamma cl_b\{U_x\}$ and $\gamma cl_b\{x\} \cap U_y \subset \gamma cl_b\{U_x\} \cap U_y = \emptyset$. Therefore, $y \notin \gamma cl_b\{x\}$. Consequently, $\gamma cl_b\{x\} \subset U$ and $(X, \tau)$ is $\gamma$-$b$-$R_0$. Next, we show that $(X, \tau)$ is $\gamma$-$b$-$R_1$. Suppose that $\gamma cl_b\{x\} \neq \gamma cl_b\{y\}$. Then, we can assume that there exists $z \in \gamma cl_b\{x\}$ such that $z \notin \gamma cl_b\{y\}$. There exist $\gamma$-$b$-open sets $V_z$ and $V_y$ such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \emptyset$. Since $z \in \gamma cl_b\{x\}$, $x \in V_z$. Since $(X, \tau)$ is $\gamma$-$b$-$R_0$, we obtain $\gamma cl_b\{x\} \subset V_z$, $\gamma cl_b\{y\} \subset V_y$ and $V_z \cap V_y = \emptyset$. This shows that $(X, \tau)$ is $\gamma$-$b$-$R_1$.

5 $\gamma$-$b$-Continuous Functions and $\gamma$-$b$-Closed Graphs

**Definition 5.1** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\gamma$-$b$-continuous if for every open set $V$ of $Y$, $f^{-1}(V)$ is $\gamma$-$b$-open in $X$. 
**Theorem 5.2** The following are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \( \gamma \)-b-continuous.

2. The inverse image of every closed set in \( Y \) is \( \gamma \)-b-closed in \( X \).

3. For each subset \( A \) of \( X \), \( f(\gamma \text{cl}_b(A)) \subset \text{cl}(f(A)) \).

4. For each subset \( B \) of \( Y \), \( \gamma \text{cl}_b(f^{-1}(B)) \subset f^{-1}(\text{cl}(B)) \).

**Proof.** (1) \( \Leftrightarrow \) (2). Obvious.

(3) \( \Leftrightarrow \) (4). Let \( B \) be any subset of \( Y \). Then by (3), we have \( f(\gamma \text{cl}_b(f^{-1}(B))) \subset \text{cl}(f(f^{-1}(B))) \subset \text{cl}(B) \). This implies \( \gamma \text{cl}_b(f^{-1}(B)) \subset f^{-1}(\text{cl}(B)) \).

Conversely, let \( B = f(A) \) where \( A \) is a subset of \( X \). Then, by (4), we have, \( \gamma \text{cl}_b(A) \subset \gamma \text{cl}_b(f^{-1}(f(A))) \subset f^{-1}(\text{cl}(f(A))) \). Thus, \( f(\gamma \text{cl}_b(A)) \subset \text{cl}(f(A)) \).

(2) \( \Rightarrow \) (4). Let \( B \subset Y \). Since \( f^{-1}(\text{cl}(B)) \) is \( \gamma \)-b-closed and \( f^{-1}(B) \subset f^{-1}(\text{cl}(B)) \), then \( \gamma \text{cl}_b(f^{-1}(B)) \subset f^{-1}(\text{cl}(B)) \).

(4) \( \Rightarrow \) (2). Let \( K \subset Y \) be a closed set. By (4), \( \gamma \text{cl}_b(f^{-1}(K)) \subset f^{-1}(\text{cl}(K)) = f^{-1}(K) \). Thus, \( f^{-1}(K) \) is \( \gamma \)-b-closed.

**Definition 5.3** For a function \( f : (X, \tau) \to (Y, \sigma) \), the graph \( G(f) = \{(x, f(x)) : x \in X\} \) is said to be \( \gamma \)-b-closed if for each \( (x, y) \notin G(f) \), there exist a \( \gamma \)-b-open set \( U \) containing \( x \) and an open set \( V \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \).

**Lemma 5.4** The function \( f : (X, \tau) \to (Y, \sigma) \) has a \( \gamma \)-b-closed graph if and only if for each \( x \in X \) and \( y \in Y \) such that \( y \neq f(x) \), there exist a \( \gamma \)-b-open set \( U \) and an open set \( V \) containing \( x \) and \( y \) respectively, such that \( f(U) \cap V = \emptyset \).

**Proof.** It follows readily from the above definition.

**Theorem 5.5** If \( f : (X, \tau) \to (Y, \sigma) \) is an injective function with the \( \gamma \)-b-closed graph, then \( X \) is \( \gamma \)-b-\( T_1 \).

**Proof.** Let \( x \) and \( y \) be two distinct points of \( X \). Then \( f(x) \neq f(y) \). Thus there exist a \( \gamma \)-b-open set \( U \) and an open set \( V \) containing \( x \) and \( f(y) \), respectively, such that \( f(U) \cap V = \emptyset \). Therefore \( y \notin U \) and it follows that \( X \) is \( \gamma \)-b-\( T_1 \).

**Theorem 5.6** If \( f : (X, \tau) \to (Y, \sigma) \) is an injective \( \gamma \)-b-continuous with a \( \gamma \)-b-closed graph \( G(f) \), then \( X \) is \( \gamma \)-b-\( T_2 \).
Proof. Let $x_1$ and $x_2$ be any distinct points of $X$. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since the graph $G(f)$ is $\gamma$-b-closed, there exist a $\gamma$-b-open set $U$ containing $x_1$ and open set $V$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since $f$ is $\gamma$-b-continuous, $f^{-1}(V)$ is a $\gamma$-b-open set containing $x_2$ such that $U \cap f^{-1}(V) = \emptyset$. Hence $X$ is $\gamma$-b-$T_2$.

Recall that a space $X$ is said to be $T_1$ if for each pair of distinct points $x$ and $y$ of $X$, there exist an open set $U$ containing $x$ but not $y$ and an open set $V$ containing $y$ but not $x$.

**Theorem 5.7** If $f : (X, \tau) \to (Y, \sigma)$ is an surjective function with the $\gamma$-b-closed graph, then $Y$ is $T_1$.

Proof. Let $y_1$ and $y_2$ be two distinct points of $Y$. Since $f$ is surjective, there exists $x$ in $X$ such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By Lemma 5.4, there exist $\gamma$-b-open set $U$ and an open set $V$ containing $x$ and $y_1$ respectively, such that $f(U) \cap V = \emptyset$. We obtain an open set $V$ containing $y_1$ which does not contain $y_2$. It follows that $y_2 \notin V$. Hence, $Y$ is $T_1$.

**Definition 5.8** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\gamma$-b-$W$-continuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-b-open set $U$ in $X$ containing $x$ such that $f(U) \subset cl(V)$.

**Theorem 5.9** If $f : (X, \tau) \to (Y, \sigma)$ is $\gamma$-b-$W$-continuous and $Y$ is Hausdorff, then $G(f)$ is $\gamma$-b-closed.

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that $Y$ is Hausdorff, there exist open sets $W$ and $V$ such that $f(x) \in W$, $y \in V$ and $V \cap W = \emptyset$. It follows that $cl(W) \cap V = \emptyset$. Since $f$ is $\gamma$-b-$W$-continuous, there exists a $\gamma$-b-open set $U$ containing $x$ such that $f(U) \subset cl(W)$. Hence, we have $f(U) \cap V = \emptyset$. This means that $G(f)$ is $\gamma$-b-closed.

**Definition 5.10** A subset $A$ of a space $X$ is said to be $\gamma$-b-compact relative to $X$ if every cover of $A$ by $\gamma$-b-open sets of $X$ has a finite subcover.

**Theorem 5.11** Let $f : (X, \tau) \to (Y, \sigma)$ have a $\gamma$-b-closed graph. If $K$ is $\gamma$-b-compact relative to $X$, then $f(K)$ is closed in $Y$.

Proof. Suppose that $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By lemma 5.4, there exists a $\gamma$-b-open set $U_x$ containing $x$ and an open neighbourhood $V_y$ of $y$ such that $f(U_x) \cap V_y = \emptyset$. The family $\{U_x : x \in K\}$ is a cover of $K$ by $\gamma$-b-open sets of $X$ and there exists a finite subset $K_0$ of $K$ such that $K \subset \bigcup\{U_x : x \in K_0\}$. Put $V = \cap\{V_y : x \in K_0\}$. Then $V$ is an open neighbourhood of $y$ and $f(K) \cap V = \emptyset$. This means that $f(K)$ is closed in $Y$. 
Theorem 5.12 If \( f : (X, \tau) \to (Y, \sigma) \) has a \( \gamma \)-b-closed graph \( G(f) \), then for each \( x \in X \). \( \{f(x)\} \cap \{\text{cl}(f(A)) : A \text{ is } \gamma \text{-b-open set containing } x\} \).

Proof. Suppose that \( y \neq f(x) \) and \( y \in \bigcap \{\text{cl}(f(A)) : A \text{ is } \gamma \text{-b-open set containing } x\} \). Then \( y \in \text{cl}(f(A)) \) for each \( \gamma \)-b-open set \( A \) containing \( x \). This implies that for each open set \( B \) containing \( y \), \( B \cap f(A) \neq \phi \). Since \( (x, y) \notin G(f) \) and \( G(f) \) is a \( \gamma \)-b-closed graph, this is a contradiction.

Definition 5.13 A function \( f : (X, \tau) \to (Y, \sigma) \) is called a \( \gamma \)-b-open if the image of every \( \gamma \)-b-open set in \( X \) is open in \( Y \).

Theorem 5.14 If \( f : (X, \tau) \to (Y, \sigma) \) is a surjective \( \gamma \)-b-open function with a \( \gamma \)-b-closed graph \( G(f) \), then \( Y \) is \( T_2 \).

Proof. Let \( y_1 \) and \( y_2 \) be any two distinct points of \( Y \). Since \( f \) is surjective \( f(x) = y_1 \) for some \( x \in X \) and \( (x, y_2) \in (X \times Y) \setminus G(f) \). This implies that there exist a \( \gamma \)-b-open set \( A \) of \( X \) and an open set \( B \) of \( Y \) such that \( (x, y_2) \in (A \times B) \) and \( (A \times B) \cap G(f) = \phi \). We have \( f(A) \cap B = \phi \). Since \( f \) is \( \gamma \)-b-open, then \( f(A) \) is open such that \( f(x) = y_1 \in f(A) \). Thus, \( Y \) is \( T_2 \).

Theorem 5.15 If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \gamma \)-b-continuous injective function and \( Y \) is \( T_2 \), then \( X \) is \( \gamma \)-b-\( T_2 \).

Proof. Let \( x \) and \( y \) in \( X \) be any pair of distinct points, then there exist disjoint open sets \( A \) and \( B \) in \( Y \) such that \( f(x) \in A \) and \( f(y) \in B \). Since \( f \) is \( \gamma \)-b-continuous, \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( \gamma \)-b-open in \( X \) containing \( x \) and \( y \) respectively, we have \( f^{-1}(A) \cap f^{-1}(B) = \phi \). Thus, \( X \) is \( \gamma \)-b-\( T_2 \).

References


