Differential Sandwich Theorems for Integral Operator of Certain Analytic Functions

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Abstract

In the present paper, we obtain some subordination and superordination results involving the integral operator \( \mathcal{I}^{\alpha}_{\beta} \) for certain normalized analytic functions in the open unit disk. These results are applied to obtain sandwich results.

Keywords: Analytic functions, Differential subordination, Superordination, Sandwich theorems, Dominant, Subordinant, Integral operator.

1 Introduction

Let \( H = H(U) \) be the class of analytic functions in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( n \) a positive integer and \( a \in \mathbb{C} \). Let \( H[a, n] \) be the subclass of \( H \) consisting of functions of the form:

\[
f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}).
\]  

(1.1)

Also, let \( T \) be the subclass of \( H \) consisting of functions of the form:
Let \( f, g \in H \). The function \( f \) is said to be subordinate to \( g \), or \( g \) is said to be superordinate to \( f \), if there exists a Schwarz function \( w \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)) such that \( f(z) = g(w(z)) \). In such a case we write \( f \prec g \) or \( f(z) < g(z) \) (\( z \in U \)). If \( g \) is univalent in \( U \), then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).

Let \( p, h \in H \) and \( \psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C} \). If \( p \) and \( \psi(p(z), zp'(z), z^2p''(z); z) \) are univalent functions in \( U \) and if \( p \) satisfies the second-order differential superordination

\[
h(z) < \psi(p(z), zp'(z), z^2p''(z); z),
\]

then \( p \) is called a solution of the differential superordination (1.3). (If \( f \) is subordinate to \( g \), then \( g \) is superordinate to \( f \).) An analytic function \( q \) is called a subordinant of (1.3), if \( q < p \) for all the functions \( p \) satisfying (1.3). An univalent subordinant \( \tilde{q} \) that satisfies \( q < \tilde{q} \) for all the subordinants \( q \) of (1.3) is called the best subordinant. Miller and Mocanu [6] have obtained conditions on the functions \( h, q \) and \( \psi \) for which the following implication holds:

\[
h(z) < \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z).
\]

Komatu [4] introduced and investigated a family of integral operator \( \mathcal{I}_\mu^\lambda : T \to T \), which is defined as follows:

\[
\mathcal{I}_\mu^\lambda f(z) = \frac{\mu^\lambda}{\Gamma(\lambda)z^{\mu-1}} \int_0^z \left( \log \frac{z}{\varepsilon} \right)^{\lambda-1} f(\varepsilon) \, d\varepsilon
\]

\[
= z + \sum_{n=2}^{\infty} \left( \frac{\mu}{\mu + n - 1} \right)^\lambda a_n z^n \quad (z \in U, \mu > 0, \lambda \geq 0).
\]

We note from (1.5) that, we have

\[
z \left( \mathcal{I}_\mu^{\lambda+1} f(z) \right)' = \mu \mathcal{I}_\mu^\lambda f(z) - (\mu - 1) \mathcal{I}_\mu^{\lambda+1} f(z).
\]

Ali et al. [1] obtained sufficient conditions for certain normalized analytic functions to satisfy

\[
q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

Also, Tuneski [9] obtained a sufficient conditions for star likeness of \( f \) in terms of
the quantity $\frac{f'(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [7,8], Goyal et al. [3] also obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) < \left(\mathcal{F}^{a+1}f(z)\right) < q_2(z),$$

and

$$q_1(z) < \left(\frac{t\mathcal{F}^{a+1}f(z) + (1-t)\mathcal{F}^{a}f(z)}{z}\right)^r < q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$.

2 Preliminaries

In order to prove our subordination and superordination results, we need the following definition and lemmas.

**Definition 2.1 [5]:** Denote by $E(f)$ the set of all functions $f$ that are analytic and injective on $U \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U: \lim_{z \to \zeta} f(z) = \infty\} \quad (2.1)$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

**Lemma 2.1 [5]:** Let $q$ be univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in $U$,

(ii) $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$.

If $p$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)), \quad (2.2)$$

then $p < q$ and $q$ is the best dominant of (2.2).

**Lemma 2.2 [6]:** Let $q$ be a convex univalent function in $U$ and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ with
If \( p \) is analytic in \( U \) and
\[
\alpha p(z) + \beta z p'(z) < \alpha q(z) + \beta z q'(z),
\]
then \( p < q \) and \( q \) is the best dominant of (2.3).

**Lemma 2.3 [6]:** Let \( q \) be convex univalent in \( U \) and let \( \beta \in \mathbb{C} \). Further assume that \( \Re(\beta) > 0 \). If \( p \in H \{ q(0),1 \} \cap Q \) and \( p(z) + \beta z p'(z) \) is univalent in \( U \), then
\[
q(z) + \beta z q'(z) < p(z) + \beta z p'(z),
\]
which implies that \( q < p \) and \( q \) is the best subordinant of (2.4).

**Lemma 2.4 [2]:** Let \( q \) be convex univalent in the unit disk \( U \) and let \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
\begin{enumerate}
  \item \( \Re \left( \frac{\theta(q(z))}{\phi(q(z))} \right) > 0 \) for \( z \in U \),
  \item \( Q(z) = z q'(z) \phi(q(z)) \) is starlike univalent in \( U \).
\end{enumerate}
If \( p \in H \{ q(0),1 \} \cap Q \), with \( p(U) \subset D \), \( \theta(p(z)) + z p'(z) \phi(p(z)) \) is univalent in \( U \) and
\[
\theta(q(z)) + z q'(z) \phi(q(z)) < \theta(p(z)) + z p'(z) \phi(p(z)),
\]
then \( q < p \) and \( q \) is the best subordinant of (2.5).

### 3 Subordination Results

**Theorem 3.1:** Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \), \( 0 \neq \eta \in \mathbb{C}, \gamma > 0 \) and suppose that \( q \) satisfies
\[
\Re \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\gamma}{\eta} \right) \right\}.
\]  
(3.1)

If \( f \in T \) satisfies the subordination
\[
(1 - \mu \eta) \left( \frac{\Im^{\lambda+1} f(z)}{z} \right) + \mu \eta \left( \frac{\Im^{\lambda+1} f(z)}{z} \right)^{\gamma} \left( \frac{\Im^{\lambda} f(z)}{\Im^{\lambda+1} f(z)} \right) < q(z) + \frac{\eta}{\gamma} z q'(z),
\]  
(3.2)
then
\[
\left( \frac{3_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} < q(z)
\]  
(3.3)
and \( q \) is the best dominant of (3.2).

**Proof:** Define the function \( p \) by
\[
p(z) = \left( \frac{3_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma}.
\]  
(3.4)
Differentiating (3.4) with respect to \( z \) logarithmically, we get
\[
\frac{zp'(z)}{p(z)} = \gamma \left( \frac{z\left(3_{\mu}^{\lambda+1} f(z)\right)'}{3_{\mu}^{\lambda+1} f(z)} - 1 \right).
\]  
(3.5)
Now, in view of (1.6), we obtain the following subordination
\[
\frac{zp'(z)}{p(z)} = \gamma \mu \left( \frac{3_{\mu}^{\lambda} f(z)}{3_{\mu}^{\lambda+1} f(z)} - 1 \right).
\]
Therefore,
\[
\frac{zp'(z)}{\gamma} = \mu \left( \frac{3_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} \left( \frac{3_{\mu}^{\lambda} f(z)}{3_{\mu}^{\lambda+1} f(z)} - 1 \right).
\]
The subordination (3.2) from the hypothesis becomes
\[
p(z) + \frac{\eta}{\gamma}zp'(z) < q(z) + \frac{\eta}{\gamma}zq'(z).
\]
An application of Lemma 2.2 with \( \beta = \frac{\eta}{\gamma} \) and \( \alpha = 1 \), we obtain (3.3).
Putting \( q(z) = \left( \frac{1+z}{1-z} \right)^{\sigma} \) \((0 < \sigma \leq 1)\) in Theorem 3.1, we obtain the following corollary:

**Corollary 3.1:** Let \( 0 < \sigma \leq 1, 0 \neq \eta \in \mathbb{C}, \gamma > 0 \) and
\[
\text{Re} \left\{ \frac{1 + 2\sigma z + z^2}{1 - z^2} \right\} > \max \left\{ 0, -\text{Re} \left( \frac{\gamma}{\eta} \right) \right\}.
\]
If \( f \in T \) satisfies the subordination
\[(1 - \mu\eta) \left( \frac{\mathfrak{I}^{\lambda+1}_\mu f(z)}{z} \right)^\gamma + \mu\eta \left( \frac{\mathfrak{I}^{\lambda+1}_\mu f(z)}{z} \right)^\gamma \left( \frac{\mathfrak{I}^{\lambda}_\mu f(z)}{\mathfrak{I}^{\lambda+1}_\mu f(z)} \right) < \left( 1 + \frac{2\eta \sigma z}{\gamma (1 - z^2)} \right) \left( \frac{1 + z}{1 - z} \right)^\sigma, \]

then
\[\left( \frac{\mathfrak{I}^{\lambda+1}_\mu f(z)}{z} \right)^\gamma < \left( \frac{1 + z}{1 - z} \right)^\sigma\]

and \(q(z) = \left( \frac{1 + z}{1 - z} \right)^\sigma\) is the best dominant.

**Theorem 3.2:** Let \(q\) be convex univalent in \(U\) with \(q(0) = 1, q(z) \neq 0 (z \in U)\) and assume that \(q\) satisfies
\[
\Re \left\{ 1 + \frac{um}{\eta} + \frac{v(m + 1)}{\eta} q(z) + (m - 1) \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q(z)} \right\} > 0,
\]
where \(u, v, m \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\}\) and \(z \in U\).

Suppose that \(z(q(z))^{m-1} q'(z)\) is starlike univalent in \(U\). If \(f \in T\) satisfies
\[
\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z) < (u + vq(z))(q(z))^m + \eta z(q(z))^{m-1} q'(z),
\]
where
\[
\varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z) = u \left( \frac{t\mathfrak{I}^{\lambda+1}_\mu f(z) + (1 - t)\mathfrak{I}^{\lambda}_\mu f(z)}{z} \right)^y + v \left( \frac{t\mathfrak{I}^{\lambda+1}_\mu f(z) + (1 - t)\mathfrak{I}^{\lambda}_\mu f(z)}{z} \right)^y (t\mathfrak{I}^{\lambda}_\mu f(z) + (1 - t)\mathfrak{I}^{\lambda-1}_\mu f(z) - 1),
\]
\[(0 \leq t \leq 1, \gamma > 0, z \in U),\]
then
\[
\left( \frac{t\mathfrak{I}^{\lambda+1}_\mu f(z) + (1 - t)\mathfrak{I}^{\lambda}_\mu f(z)}{z} \right)^\gamma < q(z)
\]
and \(q\) is the best dominant of (3.7).

**Proof:** Define the function \(p\) by
\[
p(z) = \left( \frac{t\mathfrak{I}^{\lambda+1}_\mu f(z) + (1 - t)\mathfrak{I}^{\lambda}_\mu f(z)}{z} \right)^\gamma.
\]
By setting
\[ \theta(w) = (u + vw)w^m \quad \text{and} \quad \phi(w) = \eta w^{m-1}, w \neq 0, \]
we see that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \phi(w) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\} \). Also, we get
\[ Q(z) = zq'(z)\phi(q(z)) = \eta z(q(z))^{m-1}q'(z) \]
and
\[ h(z) = \theta(q(z)) + Q(z) = (u + vq(z))(q(z))^{m} + \eta z(q(z))^{m-1}q'(z). \]
It is clear that \( Q(z) \) is starlike univalent in \( U \),
\[ \text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ 1 + \frac{um}{\eta} + \frac{v(m+1)}{\eta}q(z) + (m-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q(z)} \right\} > 0. \]
By a straightforward computation, we obtain
\[ (u + vp(z))(p(z))^m + \eta z(p(z))^{m-1}p'(z) = \varphi(u,v,\lambda,t,m,\mu,\eta;z), \quad (3.11) \]
where \( \varphi(u,v,\lambda,t,m,\mu,\eta;z) \) is given by (3.8).

From (3.7) and (3.11), we have
\[ (u + vp(z))(p(z))^m + \eta z(p(z))^{m-1}p'(z) < (u + vq(z))(q(z))^m + \eta z(q(z))^{m-1}q'(z). \quad (3.12) \]
Therefore, by Lemma 2.1, we get \( p(z) < q(z) \). By using (3.10), we obtain the result.

Putting \( q(z) = \frac{1+Az}{1+Bz} \) \((-1 \leq B < A \leq 1) \) in Theorem 3.2, we obtain the following corollary:

**Corollary 3.2:** Let \(-1 \leq B < A \leq 1\) and
\[ \text{Re} \left\{ \frac{um}{\eta} + \frac{v(m+1)(1+Az)}{\eta(1+Bz)} + \frac{1 + m(A-B)z - ABz^2}{(1+Az)(1+Bz)} \right\} > 0, \]
where \( u, v, m \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\} \) and \( z \in U \). If \( f \in T \) satisfies
\[ \varphi(u,v,\lambda,t,m,\mu,\eta;z) \]
\[ < \left( u + v \left( \frac{1+Az}{1+Bz} \right) \left( \frac{1+Az}{1+Bz} \right)^m + \frac{\eta(A-B)(1+Az)^{m-1}z}{(1+Bz)^{m+1}}, \]
where \( \varphi(u,v,\lambda,t,m,\mu,\eta;z) \) is given by (3.8),
then
\[
\left( t^{3\mu+1} f(z) + (1 - t) 3^{3\mu} f(z) \right) < \frac{1 + Az}{1 + Bz}
\]
and \( q(z) = \frac{1 + Az}{1 + Bz} \) is the best dominant.

### 4 Superordination Results

**Theorem 4.1:** Let \( q \) be convex univalent in \( U \) with \( q(0) = 1, \gamma > 0 \) and \( \Re[\eta] > 0 \). Let \( f \in T \) satisfies

\[
\left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} \in H[q(0), 1] \cap Q
\]

and

\[
(1 - \mu\eta) \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} + \mu\eta \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} \left( \frac{3^{3\mu} f(z)}{3^{3\mu+1} f(z)} \right)
\]

be univalent in \( U \). If

\[
q(z) + \frac{\eta}{\gamma} z q'(z) < (1 - \mu\eta) \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} + \mu\eta \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} \left( \frac{3^{3\mu} f(z)}{3^{3\mu+1} f(z)} \right),
\]

then

\[
q(z) < \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma}
\]

and \( q \) is the best subordinant of (4.1).

**Proof:** Define the function \( p \) by

\[
p(z) = \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma}.
\]

Differentiating (4.3) with respect to \( z \) logarithmically, we get

\[
\frac{zp'(z)}{p(z)} = \gamma \left( \frac{z \left( \frac{3^{3\mu+1} f(z)}{z} \right)'}{3^{3\mu+1} f(z)} - 1 \right).
\]

After some computations and using (1.6), from (4.4), we obtain

\[
(1 - \mu\eta) \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} + \mu\eta \left( \frac{3^{3\mu+1} f(z)}{z} \right)^{\gamma} \left( \frac{3^{3\mu} f(z)}{3^{3\mu+1} f(z)} \right) = p(z) + \frac{\eta}{\gamma} z p'(z),
\]

and now, by using Lemma 2.3, we get the desired result.
Putting \( q(z) = \left(\frac{1+z}{1-z}\right)^\sigma \) (0 < \( \sigma \leq 1 \)) in Theorem 4.1, we obtain the following corollary:

**Corollary 4.1:** Let 0 < \( \sigma \leq 1, \gamma > 0 \) and Re\{\( \eta \)\} > 0. If \( f \in T \) satisfies

\[
\left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma \in H[q(0),1] \cap Q
\]

and

\[
(1 - \mu\eta)\left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma + \mu\eta\left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma \left( \frac{3^{\lambda}f(z)}{3^{\lambda+1}f(z)} \right)
\]

be univalent in \( U \). If

\[
\left( 1 + \frac{2\gamma\sigma z}{\gamma(1-z^2)} \right) \left( \frac{1+z}{1-z} \right)^\sigma < (1 - \mu\eta)\left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma + \mu\eta\left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma \left( \frac{3^{\lambda}f(z)}{3^{\lambda+1}f(z)} \right)
\]

then

\[
\left( \frac{1+z}{1-z} \right)^\sigma < \left( \frac{3^{\lambda+1}f(z)}{z} \right)^\gamma
\]

and \( q(z) = \left(\frac{1+z}{1-z}\right)^\sigma \) is the best subordinant.

**Theorem 4.2:** Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \), and assume that \( q \) satisfies

\[
\text{Re}\left\{ \frac{um}{\eta}q'(z) + \frac{v(m+1)}{\eta}q(z)q'(z) \right\} > 0,
\]

where \( u, v, m \in \mathbb{C}, \eta \in \mathbb{C} \setminus \{0\} \) and \( z \in U \).

Suppose that \( z(q(z))^{m-1} q'(z) \) is starlike univalent in \( U \). Let \( f \in T \) satisfies

\[
\left( \frac{t3^{\lambda+1}f(z) + (1-t)3^{\lambda}f(z)}{z} \right)^\gamma \in H[q(0),1] \cap Q
\]

and \( \phi(u,v,\gamma,\lambda,t,m,\mu,\eta;z) \) is univalent in \( U \), where \( \phi(u,v,\gamma,\lambda,t,m,\mu,\eta;z) \) is given by (3.8). If

\[
(u + vq(z))(q(z))^m + \eta z(q(z))^{m-1} q'(z) < \phi(u,v,\gamma,\lambda,t,m,\mu,\eta;z),
\]

then

\[
q(z) < \left( \frac{t3^{\lambda+1}f(z) + (1-t)3^{\lambda}f(z)}{z} \right)^\gamma
\]
and \( q \) is the best subordinant of (4.6).

**Proof:** Define the function \( p \) by

\[
p(z) = \left( \frac{t \mathcal{S}_{\mu}^{\lambda+1} f(z) + (1-t) \mathcal{S}_{\mu}^{\lambda} f(z)}{z} \right)^{\gamma}.
\]

By setting

\[
\theta(w) = (u + vw)w^m \quad \text{and} \quad \phi(w) = \eta w^{m-1}, w \neq 0,
\]

we see that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \phi(w) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\} \). Also, we get

\[
Q(z) = zq'(z)\phi(q(z)) = \eta z(q(z))^{m-1} q'(z).
\]

It is clear that \( Q(z) \) is starlike univalent in \( U \),

\[
\Re \left( \frac{\theta(q(z))}{\phi(q(z))} \right) = \Re \left( \frac{um}{\eta} q'(z) + \frac{v(m+1)}{\eta} q(z)q'(z) \right) > 0.
\]

By a straightforward computation, we obtain

\[
\varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z) = (u + vp(z))(p(z))^m + \eta z(p(z))^{m-1} p'(z),
\]

where \( \varphi(u,v,\gamma,\lambda,t,m,\mu,\eta;z) \) is given by (3.8).

From (4.6) and (4.9), we have

\[
(u + vq(z))(q(z))^m + \eta z(q(z))^{m-1} q'(z) < (u + vp(z))(p(z))^m + \eta z(p(z))^{m-1} p'(z).
\]

Therefore, by Lemma 2.4, we get \( q(z) < p(z) \). By using (4.8), we obtain the result.

### 5 Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

**Theorem 5.1:** Let \( q_1 \) be convex univalent in \( U \) with \( q_1(0) = 1, \Re\{\eta\} > 0 \) and let \( q_2 \) be univalent in \( U, q_2(0) = 1 \) and satisfies (3.1). Let \( f \in T \) satisfies

\[
\left( \frac{\mathcal{S}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} \in H[1,1] \cap Q
\]

and
\[
(1 - \mu \eta) \left( \frac{\mathcal{H}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} + \mu \eta \left( \frac{\mathcal{H}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} \left( \frac{\mathcal{H}_{\mu}^{\lambda} f(z)}{\mathcal{H}_{\mu}^{\lambda+1} f(z)} \right)
\]
be univalent in \( U \). If
\[
q_1(z) + \frac{\eta}{\gamma} z q_1'(z) < (1 - \mu \eta) \left( \frac{\mathcal{H}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} + \mu \eta \left( \frac{\mathcal{H}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} \left( \frac{\mathcal{H}_{\mu}^{\lambda} f(z)}{\mathcal{H}_{\mu}^{\lambda+1} f(z)} \right)
\]
\[
< q_2(z) + \frac{\eta}{\gamma} z q_2'(z),
\]
then
\[
q_1(z) < \left( \frac{\mathcal{H}_{\mu}^{\lambda+1} f(z)}{z} \right)^{\gamma} < q_2(z)
\]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**Theorem 5.2:** Let \( q_1 \) be convex univalent in \( U \) with \( q_1(0) = 1 \) and satisfies (4.5) and let \( q_2 \) be univalent in \( U \), \( q_2(0) = 1 \) and satisfies (3.6). Let \( f \in T \) satisfies
\[
\left( \frac{t \mathcal{H}_{\mu}^{\lambda+1} f(z) + (1 - t) \mathcal{H}_{\mu}^{\lambda} f(z)}{z} \right)^{\gamma} \in H \ [1,1] \cap Q
\]
and \( \varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z) \) is univalent in \( U \), where \( \varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z) \) is given by (3.8). If
\[
(u + v q_1(z))(q_1(z))^{m+\eta z q_1(z)} < \varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z)
\]
\[
< (u + v q_2(z))(q_2(z))^{m+\eta z q_2(z)} < \varphi(u, v, \gamma, \lambda, t, m, \mu, \eta; z),
\]
then
\[
q_1(z) < \left( \frac{t \mathcal{H}_{\mu}^{\lambda+1} f(z) + (1 - t) \mathcal{H}_{\mu}^{\lambda} f(z)}{z} \right)^{\gamma} < q_2(z)
\]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**References**


