Intuitionistic Fuzzy Sets in Ordered $\Gamma$-Semigroups

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(Received: 18-3-11/Accepted:28-6-11)

Abstract

We consider the intuitionistic fuzzification of the concept of several ideal in an ordered $\Gamma$-semigroup, and investigate some properties of such ideals.

Keywords: Ordered $\Gamma$-semigroup, intuitionistic fuzzy $\Gamma$-subsemigroup, intuitionistic left (resp. right) ideal, intuitionistic fuzzy interior ideal, intuitionistic fuzzy left (resp. right) simple.

1 Introduction

The concept of a fuzzy set given by L.A. Zadeh in his classic paper of 1965 [11] has been used by many authors to generalize some of the basic notions of algebra. Fuzzy semigroups have been first considered by N. Kuroki [5], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu and Tsingelis [7]. The notion of a $\Gamma$-semigroup was introduced by Sen [9]. Many classical notions of semigroups have been extended to $\Gamma$-semigroups. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [10]. In [4], N. Kuroki gave some properties of fuzzy ideals and fuzzy semiprime ideals in semigroups [6]. In [1], K. H. Kim gave some properties of several ideals in an ordered semigroup. In this paper, we consider the intuitionistic fuzzification of the concept of several ideals in an ordered $\Gamma$-semigroup, and investigate some properties of such ideal.
2 Preliminaries

We include some elementary aspects of ordered $\Gamma$-semigroups that are necessary for this paper.

**Definition 2.1** Let $S$ and $\Gamma$ be two non-empty sets. Then $S$ is called a $\Gamma$-semigroup if it satisfies

(i) $x\gamma y \in S$,

(ii) $(x\beta y)\gamma z = x\beta (y\gamma z)$,

for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

**Definition 2.2** Let $S$ be a $\Gamma$-semigroup and $(S, \leq)$ a partially ordered set. Then $S$ is called an ordered $\Gamma$-semigroup if $x \leq y$ implies $a\gamma z \leq b\gamma z$ and $z\gamma a \leq z\gamma b$, for all $x, y, z \in S$ and $\gamma \in \Gamma$.

**Definition 2.3** Let $S$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of an ordered $\Gamma$-semigroup $S$ is said to be a $\Gamma$-subsemigroup of $S$ if $A \Gamma A \subseteq A$.

Let $S$ be an ordered $\Gamma$-semigroup. For $A \subseteq S$, we denote

$$(A] := \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$  

For $A, B \subseteq S$, we denote

$$(A \Gamma B) := \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$  

**Definition 2.4** Let $S$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is called a left ideal of $S$ if it satisfies

(i) $S \Gamma A \subseteq A$.

(ii) For any $b \in S$ and $a \in A$ such that $b \leq a$ implies $b \in A$.

**Definition 2.5** Let $S$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is called a right ideal of $S$ if it satisfies

(i) $A \Gamma S \subseteq A$.

(ii) For any $b \in S$ and $a \in A$ such that $b \leq a$ implies $b \in A$.

**Definition 2.6** Let $S$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is called an ideal of $S$ if it satisfies

(i) $S \Gamma A \subseteq A$.
(ii) $A \Gamma S \subseteq A$.

(iii) For any $b \in S$ and $a \in A$ such that $b \leq a$ implies $b \in A$.

**Definition 2.7** Let $S$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is called a bi-ideal of $S$ if it satisfies

(i) $A \Gamma S \subseteq A$.

(ii) For any $b \in S$ and $a \in A$ such that $b \leq a$ implies $b \in A$.

**Definition 2.8** Let $S$ be an ordered $\Gamma$-semigroup. A $\Gamma$-subsemigroup $A$ of $S$ is called an interior ideal of $S$ if it satisfies

(i) $S \Gamma A \subseteq A$.

(ii) For any $b \in S$ and $a \in A$ such that $b \leq a$ implies $b \in A$.

An ordered $\Gamma$-semigroup $S$ is called left-zero (resp. right-zero) if $x \leq x \alpha y$ (resp. $y \leq x \alpha y$) for all $x, y \in S$ and $\alpha \in \Gamma$. An ordered $\Gamma$-semigroup $S$ is said to be left (resp. right) simple if for every left (resp. right) ideal $A$ of $S$, we have $A = S$. An ordered $\Gamma$-semigroup $S$ is said to be regular if for every $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq \alpha a x \beta a$. $L[x]$ denote the principal left ideal of a $\Gamma$-semigroup $S$ generated by $x$ in $S$, that is, $L[x] = (x \cup S \Gamma x)$. By a fuzzy set $\mu$ in a non-empty set $X$, we mean a function $\mu : X \rightarrow [0, 1]$ and the complement of $\mu$, denoted by $\mu'$, is the fuzzy set in $X$ given by $\mu'(x) := 1 - \mu(x)$ for all $x \in X$. For any fuzzy subset $\mu$ in $S$ and $t \in [0, 1]$, we define

$$U(\mu; f) := \{x \in S \mid \mu(x) \geq t\},$$

which is called an upper $t$-level cut of $\mu$ and can be used to the characterization of $\mu$.

An intuitionistic fuzzy set (briefly, $IFS$) $A$ in a non-empty set $X$ is an object having the form

$$A := \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the function $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. For the sake of simplicity, we shall use the symbol $A := (\mu_A, \gamma_A)$ for the $IFS$ $A := \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

Let $\chi_U$ denote the characteristic function of a non-empty subset $U$ of an ordered $\Gamma$-semigroup.
Definition 2.9 Let $S$ be an ordered $\Gamma$-semigroup. A fuzzy set $\mu$ is called a fuzzy $\Gamma$-subsemigroup of $S$ if

$$\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 2.10 Let $S$ be an ordered $\Gamma$-semigroup. A fuzzy $\Gamma$-subsemigroup $\mu$ of $S$ is called a fuzzy bi-ideal of $S$, if the following axioms are satisfied:

1. If $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in S$,
2. $\mu(x\alpha a\beta y) \geq \min\{\mu(x), \mu(y)\}$, for all $a, x, y \in S$ and $\alpha, \beta \in \Gamma$.

3 Main Results

In what follows, we use $S$ to denote an ordered $\Gamma$-semigroup unless otherwise specified.

Definition 3.1 For an IFS $A = (\mu_A, \gamma_A)$ in $S$, consider the following axioms:

1. $\text{(IFS}_1) \mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
2. $\text{(IFS}_2) \gamma_A(x\alpha y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$, for all $x, y \in S$ and $\alpha \in \Gamma$.

Then $A = (\mu_A, \gamma_A)$ is called a first (resp. second) intuitionistic fuzzy $\Gamma$-subsemigroup (briefly, IFTSS$_1$ (resp. IFTSS$_2$)) of $S$ if satisfies (IFS$_1$) (resp. (IFS$_2$)). Also, $A = (\mu_A, \gamma_A)$ is said to be an intuitionistic fuzzy $\Gamma$-semigroup (briefly, IFTSS) of $S$ if it is both a first and a second intuitionistic fuzzy $\Gamma$-semigroup.

Theorem 3.2 If $U$ is a $\Gamma$-subsemigroup of ordered $\Gamma$-semigroup $S$, then $U' = (\chi_U, \chi'_U)$ is an IFTSS of $S$.

Let $x, y \in S$ and $\alpha \in \Gamma$. From the hypothesis, $x\alpha y \in U$ if $x, y \in U$. In this case,

$$\chi_U(x\alpha y) = 1 \geq \min\{\chi_U(x), \chi_U(y)\}$$

and

$$\chi'_U(x\alpha y) = 1 - \chi_U(x\alpha y)$$

$$\leq 1 - \min\{\chi_U(x), \chi_U(y)\}$$

$$= \max\{1 - \mu_U(x), 1 - \mu_U(y)\}$$

$$= \max\{\mu'_U(x), \mu'_U(y)\}.$$ 

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Thus $\min\{\chi_U(x), \chi_U(y)\} = 0$, which is implies that

$$\chi_U(x\alpha y) \geq 0 = \min\{\chi_U(x), \chi_U(y)\}$$
and
\[ \chi_U(x\alpha y) \leq 1 \]
\[ = 1 - \min\{\chi_U(x), \chi_U(y)\} \]
\[ = \max\{1 - \chi_U(x), 1 - \chi_U(y)\} \]
\[ = \max\{\chi_U(x), \chi_U(y)\}. \]

This completes the proof.

**Theorem 3.3** Let \( U \) be a non-empty subset of ordered \( \Gamma \)-semigroup \( S \). If \( U' = (\chi_U, \chi_U') \) is an IFTSS1 or IFTSS2 of \( S \), then \( U \) is a \( \Gamma \)-subsemigroup of \( S \).

Suppose that \( U' = (\chi_U, \chi_U') \) is an IFTSS1 of \( S \) and \( x \in UTU \). In this case \( x = u\alpha v \) for some \( u, v \in U \) and \( \alpha \in \Gamma \). It follows from \((\Gamma IS_1)\) that
\[ \chi_U(x) = \chi_U(u\alpha v) \geq \min\{\chi_U(u), \chi_U(v)\} = 1. \]

Hence \( \chi_U(x) = 1 \), that is, \( x \in U \). Thus \( U \) is a \( \Gamma \)-subsemigroup of \( S \). Now, assume that \( U' = (\chi_U, \chi_U') \) is an IFTSS2 of \( S \) and \( x' \in UTU \). Then \( x' = u'\alpha' v' \) for some \( u', v' \in U \) and \( \alpha' \in \Gamma \). Using \((\Gamma IS_2)\), we get that
\[ \chi_U(x') = \chi_U(u'\alpha' v') \]
\[ \leq \max\{\chi_U(u'), \chi_U(v')\} \]
\[ = \max\{1 - \chi_U(u'), 1 - \chi_U(v')\} \]
\[ = 0, \]
and so \( 1 - \chi_U(x') = \chi_U(x') = 0 \), which implies that \( \chi_U(x') = 1 \), i.e. \( x' \in U \). Thus \( U \) is a \( \Gamma \)-subsemigroup of \( S \). This completes the proof.

**Definition 3.4** For an IFS \( A = (\mu_A, \gamma_A) \) in \( S \), consider the following axioms:

\((\Gamma IL1)\) \( x \leq y \) implies \( \mu_A(x) \geq \mu_A(y) \) and \( \mu_A(x\alpha y) \geq \mu_A(y) \),
\((\Gamma IL2)\) \( x \leq y \) implies \( \gamma_A(x) \leq \gamma_A(y) \) and \( \gamma_A(x\alpha y) \leq \gamma_A(y) \), for all \( x, y \in S \) and \( \alpha \in \Gamma \).

Then \( A = (\mu_A, \gamma_A) \) is called a first (resp. second) intuitionistic fuzzy left ideal (briefly, IFTL1 (resp. IFTL2)) of \( S \) if it satisfies \((\Gamma IL1)\) (resp. \((\Gamma IL2)\)). Also, \( A = (\mu_A, \gamma_A) \) is said to be an intuitionistic fuzzy left ideal (briefly, IFTL1) of \( S \) if it is both a first and a second intuitionistic fuzzy left ideal.

**Definition 3.5** For an IFS \( A = (\mu_A, \gamma_A) \) in \( S \), consider the following axioms:

\((\Gamma IR1)\) \( x \leq y \) implies \( \mu_A(x) \geq \mu_A(y) \) and \( \mu_A(x\alpha y) \geq \mu_A(x) \),
\((\Gamma IR2)\) \( x \leq y \) implies \( \gamma_A(x) \leq \gamma_A(y) \) and \( \gamma_A(x\alpha y) \leq \gamma_A(x) \), for all \( x, y \in S \) and \( \alpha \in \Gamma \).

Then \( A = (\mu_A, \gamma_A) \) is called a first (resp. second) intuitionistic fuzzy right ideal (briefly, IFTRI1 (resp. IFTRI2)) of \( S \) if it satisfies \((\Gamma IR1)\) (resp. \((\Gamma IR2)\)). Also, \( A = (\mu_A, \gamma_A) \) is said to be an intuitionistic fuzzy right ideal (briefly, IFTRI) of \( S \) if it is both a first and a second intuitionistic fuzzy right ideal.
Definition 3.6 Let $A = (\mu_A, \gamma_A)$ be an IFS in $S$. Then $A$ is called an intuitionistic fuzzy ideal of $S$ if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal.

Let $U$ be a left-zero $\Gamma$-subsemigroup of $S$. If $A = (\mu_A, \gamma_A)$ is an IFTLI of $S$. Then the restriction of $A$ to $U$ is constant, that is, $A(x) = A(y)$ for all $x, y \in S$.

Let $x, y \in S$ and $\alpha \in \Gamma$. Since $U$ is a left-zero of $\Gamma$-subsemigroup of $S$, we have $x \leq x\alpha y$ and $y \leq y\alpha x$. In this case, from the hypothesis, we have

$$
\mu_A(x) \geq \mu_A(x\alpha y) \geq \mu_A(y), \quad \mu_A(y) \geq \mu_A(y\alpha x) \geq \mu_A(x)
$$

and

$$
\gamma_A(x) \leq \gamma_A(x\alpha y) \leq \gamma_A(y), \quad \gamma_A(y) \leq \gamma_A(y\alpha x) \leq \gamma_A(x).
$$

Thus we obtain $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$ for all $x, y \in U$. Hence $A(x) = A(y)$.

Lemma 3.7 If $U$ is a left ideal of $S$, then $U' = (\chi_U, \chi_U')$ is an IFTLI of $S$.

Let $x, y \in S$ and $\alpha \in \Gamma$ be such that $x \leq y$. Since $U$ is a left ideal of $S$, we have $x \in U$ and $x\alpha y \in U$ if $y \in U$. It follows that $x \leq y$ implies

$$
\chi_U(x) = 1 = \chi_U(y)
$$

and

$$
\chi_U'(x) = 1 - \chi_U(x) = 0 = 1 - \chi_U(y) = \chi_U'(y).
$$

Also, we have

$$
\chi_U(x\alpha y) = 1 = \chi_U(y)
$$

and

$$
\chi_U'(x\alpha y) = 1 - \chi_U(x\alpha y) = 0 = 1 - \chi_U(y) = \chi_U'(y).
$$

If $y \notin U$, then $\chi_U(y) = 0$. In this case, $x \leq y$ implies $\chi_U(x) \geq 0 = \chi_U(y)$ and $\chi_U'(x) \leq \chi_U'(y) = 1 - \chi_U(y) = 1$. Also, we obtain $\chi_U(x\alpha y) \geq 0 = \chi_U(y)$ and $\chi_U'(y) = 1 - \chi_U(y) = 1 \geq \chi_U'(x\alpha y)$. Consequently, $U' = (\chi, \chi_U')$ is an IFTLI of $S$.

An element $e$ in an ordered $\Gamma$-semigroup $S$ is called an idempotent if $eae \geq e$, for all $\alpha \in \Gamma$. Let $E_S$ denote the set of all idempotents in an ordered $\Gamma$-semigroup $S$.

Theorem 3.8 Let $A = (\mu_A, \gamma_A)$ be an IFTLI of $S$. If $E_S$ is a left-zero $\Gamma$-subsemigroup of $S$, then $A(e) = A(e')$ for all $e, e' \in E_S$. 

Let \( e, e' \in E_S \). From the hypothesis, \( eae' \geq e \) and \( e' \beta e \geq e' \) for all \( \alpha, \beta \in \Gamma \). Thus, since \( A = (\mu_A, \gamma_A) \) is an IFTLI of \( S \), we get that
\[
\mu_A(e) \geq \mu_A(eae') \geq \mu_A(e'), \quad \mu_A(e' \beta e) \geq \mu_A(e')
\]
and
\[
\gamma_A(e) \leq \gamma_A(eae') \leq \gamma_A(e'), \quad \gamma_A(e' \beta e) \leq \gamma_A(e).
\]
Hence we have \( \mu_A(e) = \mu_A(e') \) and \( \gamma_A(e) = \gamma_A(e') \) for all \( e, e' \in E_S \). This completes the proof.

**Definition 3.9** Let \( S \) be an ordered \( \Gamma \)-semigroup. A fuzzy \( \Gamma \)-subsemigroup \( \mu \) of \( S \) is called a fuzzy interior ideal of \( S \), if the following axioms are satisfied:
\[
(1) \quad \mu(xo\alpha\beta y) \geq \mu(a),
\]
\[
(2) \quad \text{if } x \leq y, \text{ then } \mu(x) \geq \mu(y) \text{ for all } a, x, y \in S \text{ and } \alpha, \beta \in \Gamma.
\]

**Definition 3.10** For an IFS \( A = (\mu_A, \gamma_A) \) in \( S \), consider the following axioms:
\[
(\Gamma I1) \quad x \leq y \implies \mu_A(x) \geq \mu_A(y) \text{ and } \mu_A(xo\alpha\beta y) \geq \mu_A(s),
\]
\[
(\Gamma I2) \quad x \leq y \implies \gamma_A(x) \leq \gamma_A(y) \text{ and } \gamma_A(xo\alpha\beta y) \leq \gamma_A(s) \text{ for all } s, x, y \in S \text{ and } \alpha, \beta \in \Gamma.
\]

Then \( A = (\mu_A, \gamma_A) \) is called a first (resp. second) intuitionistic fuzzy interior ideal (briefly, IFTL1 (resp. IFTL2)) of \( S \) if it is an IFTS1 (resp. IFTS2) satisfying \((\Gamma I1)\) (resp. \((\Gamma I2)\)). Also, \( A = (\mu_A, \gamma_A) \) is said to be an intuitionistic fuzzy interior ideal (briefly, IFTI) of \( S \) if it is both a first and a second intuitionistic fuzzy interior ideal of \( S \).

**Theorem 3.11** If \( S \) is regular, then every IFTI of \( S \) is an IFTI of \( S \).

Let \( A = (\mu_A, \gamma_A) \) be an IFTI of \( S \) and \( x, y \in S \). In this case, because \( S \) is regular, there exist \( s, s' \in S \) and \( \alpha, \beta, \alpha', \beta' \in \Gamma \) such that \( x \leq xo\alpha\beta x \) and \( y \leq yo\alpha' s' \beta' y \). Thus
\[
\mu_A(xy) \geq \mu_A(x\gamma' yo\alpha s' \beta' y) = \mu_A(x\gamma' yo\alpha (s' \beta' y)) \geq \mu_A(y).
\]
and
\[
\gamma_A(xy) \leq \gamma_A(x\gamma' yo\alpha s' \beta' y) = \gamma_A(x\gamma' yo\alpha (s' \beta' y)) \leq \gamma_A(y),
\]
for some \( \gamma' \in \Gamma \). It follows that \( A = (\mu_A, \gamma_A) \) is an IFTLI of \( S \). Similarly, we can show that \( A = (\mu_A, \gamma_A) \) is an IFTRI of \( S \). This completes the proof.

**Theorem 3.12** If \( U \) is an interior ideal of \( S \), then \( U' = (\chi_U, \chi_U') \) is an IFTI of \( S \).
Since $U$ is a $\Gamma$-subsemigroup of $S$, we have $U' = (\chi_U, \chi'_U)$ is an $IFTSS$ of $S$ by Theorem 3.2. Let $x, y \in S$ be such that $x \leq y$. Then we have $x \in U$ if $y \in U$. Thus $x \leq y$ implies $\chi_U(x) = 1 = \chi_U(y)$ and
\[
\chi_U(x) = 1 - \chi_U(x)
\]
\[= 0
\]
\[= 1 - \chi_U(y)
\]
\[= \chi'_U(y).
\]
If $y \notin U$, then $\chi_U(x) \geq 0 = \chi_U(y)$ and $\chi'_U(x) \leq \chi'_U(y) = 1 - \chi_U(y) = 1$.
Now, let $s, x, y \in S$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x\alpha s\beta y \in U$ if $s \in U$.
In this case, $\chi_U(x\alpha s\beta y) = 1 = \chi_U(s)$ and
\[
\chi'_U(x\alpha s\beta y) = 1 - \chi_U(x\alpha s\beta y)
\]
\[= 0
\]
\[= 1 - \chi_U(s) = \chi'_U(s).
\]
If $s \notin U$, then $\chi_U(s) = 0$. Thus $\chi(x\alpha s\beta y) \geq 0 = \chi_U(s)$ and
\[
\chi'_U(s) = 1 - \chi_U(s)
\]
\[= 1
\]
\[\geq \chi'_U(x\alpha s\beta y).
\]
Consequently, $U' = (\chi_U, \chi'_U)$ is an $IFTII$ of $S$.

**Theorem 3.13** Let $S$ be regular and $U$ a non-empty subset of $S$. If $U' = (\chi_U, \chi'_U)$ is an $IFTII_1$ or $IFTII_2$ of $S$, then $U$ is an interior ideal of $S$.

It is clear that $U$ is a $\Gamma$-subsemigroup of $S$ be Theorem 3.3. Suppose that $U' = (\chi_U, \chi'_U)$ is an $IFTII_1$ of $S$ and $x \in STUTS$. In this case, $x = s\alpha u\beta t$ for some $s, t \in S, u \in U$ and $\alpha, \beta \in \Gamma$. It follows from ($\Gamma II_i$) that
\[
\chi_U(x) = \chi_U(s\alpha u\beta t) \geq \chi_U(u) = 1.
\]
Hence $\chi_U(x) = 1$, i.e. $x \in U$. Let $x \leq y$ and $y \in U$. Then
\[
\chi_U(x) \geq \chi_U(y) = 1.
\]
Hence $\chi_U(x) = 1$, i.e. $x \in U$. Thus $U$ is an interior ideal of $S$. Now, assume that $U' = (\chi_U, \chi'_U)$ is an $IFTII_2$ of $S$ and $x' = s'\alpha' u'\beta' t'$ for some $s, t' \in S, u \in U$ and $\alpha, \beta \in \Gamma$. Using ($\Gamma II_2$), we obtain
\[
\chi'_U(x') = \chi'_U(s'\alpha' u'\beta' t')
\]
\[\leq \chi'_U(u')
\]
\[= 1 - \chi_U(u') = 0,
\]
and so $\chi'_U(x') = 1 - \chi_U(x') = 0$. Therefore, $\chi_U(x') = 1$, i.e. $x' \in U$. Also, let $x, y \in S$ be such that $x \leq y$ and $y \in U$. Then we have $\chi'_U(x) \leq \chi'_U(y)$, i.e.
\[1 - \chi_U(x) \leq 1 - \chi_U(y).
\]
Thus $\chi_U(x) \geq \chi_U(y)$, i.e. $\chi_U(x) = 1$, and so $x \in U$. This completes the proof.
Definition 3.14 S is called first (resp. second) intuitionistic fuzzy left simple if IFTLI₁ (resp. IFTLI₂) of S is constant. Also, S is said to be intuitionistic fuzzy left simple if it is both first and second intuitionistic fuzzy left simple, i.e. every IFTLI of S is constant.

Lemma 3.15 An ordered Γ-semigroup S is left (resp. right) simple if and only if (SΓa) = S (resp. (aΓS) = S) for every a ∈ S.

Theorem 3.16 If S is left simple, then S is intuitionistic fuzzy left simple.

Let A = (µ_A, γ_A) be an IFTLI of S and x, x' ∈ S. In this case, because S is left simple, there exist s, s' ∈ S and α, β ∈ Γ such that x ≤ sax' and x' ≤ s'βx. Thus, since A = (µ_A, γ_A) is an IFTLI of S, we get that

\[ µ_A(x) ≥ µ_A(sax') ≥ µ_A(x'), \quad µ_A(x') ≥ µ_A(s'βx) ≥ µ_A(x) \]

and

\[ γ_A(x) ≤ γ_A(sax') ≤ γ_A(x'), \quad γ_A(x') ≤ γ_A(s'βx) ≤ γ_A(x). \]

Hence we have µ_A(x) = µ_A(x') and γ_A(x) = γ_A(x') for all x, x' ∈ S, that is, A(x) = A(x') for all x, x' ∈ S. Consequently, S is intuitionistic fuzzy left simple. This completes the proof.

Theorem 3.17 If S is first or second intuitionistic fuzzy left simple, then S is left simple.

Let U be a left ideal of S. Suppose that S is first (or second) intuitionistic fuzzy left simple. Because U' = (χ_U, χ'_U) is an IFTLI of S by Lemma 3.8, U' = (χ_U, χ'_U) is an IFTLI₁ (and IFTLI₂) of S. From the hypothesis, χ_U (and χ'_U) is constant. Since U is non-empty, it follows that χ_U = 1 (or χ'_U = 0), where 1 and 0 are fuzzy sets in χ_U defined by 1(x) = 1 and 0(x) = 0 for all x ∈ S, respectively. Thus x ∈ U for all x ∈ S. This completes the proof.

Lemma 3.18 An ordered Γ-semigroup S is simple if and only if for every a ∈ S, we have S = (SΓaS).

Theorem 3.19 If S is simple, then every IFTLI of S is constant.

Let A = (µ_A, γ_A) be an IFTLI of S and x, x' ∈ S. In this case, because S is simple, there exist s, s', t, t' ∈ S and α, β, α', β' ∈ Γ such that x ≤ sax'βt and x' ≤ s'α'x'β't. Thus, since A = (µ_A, γ_A) is an IFTLI of S, we obtain that

\[ µ_A(x) ≥ µ_A(sax'βt) ≥ µ_A(x'), \quad µ_A(x') ≥ µ_A(s'α'x'β't) ≥ µ_A(x) \]

and

\[ γ_A(x) ≤ γ_A(sax'βt) ≤ γ_A(x'), \quad γ_A(x') ≤ γ_A(s'α'x'β't) ≤ γ_A(x). \]

Hence we get µ_A(x) = µ_A(x') and γ_A(x) = γ_A(x') for all x, x' ∈ S. Consequently, A = (µ_A, γ_A) is constant.
Definition 3.20 For an IFS $A = (\mu_A, \gamma_A)$ in $S$, consider the following axioms:

\((TIB_1)\) $x \leq y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(x \alpha s \beta y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

\((TIB_2)\) $x \leq y$ implies $\gamma_A(x) \leq \gamma_A(y)$ and $\gamma_A(x \alpha s \beta y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$

for all $s, x, y \in S$ and $\alpha, \beta \in \Gamma$. Then $A = (\mu_A, \gamma_A)$ is called an intuitionistic fuzzy bi-ideal (briefly, IFTB) of $S$ if it satisfies $(TIB_1)$ and $(TIB_2)$.

Theorem 3.21 If $S$ is left simple, then every IFTB of $S$ is an IFTRI of $S$. Let $A = (\mu_A, \gamma_A)$ be an IFTB of $S$ and $x, y \in S$. In this case, from the hypothesis, there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $y \leq s \alpha x$. Thus, because $A = (\mu_A, \gamma_A)$ is an IFTB of $S$, we have that

$$\mu_A(x \beta y) \geq \mu_A(x \beta s \alpha x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$$

and

$$\gamma_A(x \beta y) \leq \gamma_A(x \beta s \alpha x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x).$$

It follows that $A = (\mu_A, \gamma_A)$ is an IFTRI of $S$.

Theorem 3.22 If $U$ is a bi-ideal of $S$, then $U' = (\chi_U, \chi'_U)$ is an IFTB of $S$.

Since $U$ is a $\Gamma$-subsemigroup of $S$, we obtain that $U' = (\chi_U, \chi'_U)$ is an IFTS of $S$ by Theorem 3.2. Let $x, y \in S$ be such that $x \leq y$ and $y \in U$. Then $x \in U$, and so $\chi_U(x) = 1 = \chi_U(y)$ and $\chi'_U(x) = 1 - \chi_U(x) = 0 = 1 - \chi_U(y) = \chi'_U(y)$. Let $s, x, y \in S$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x \alpha s \beta y \in U$ if $x, y \in U$. In this case,

$$\chi_U(x \alpha s \beta y) = 1 = \min\{\chi_U(x), \chi_U(y)\}$$

and

$$\chi'_U(x \alpha s \beta y) = 1 - \chi_U(x \alpha s \beta y) = 0 = \max\{\chi'_U(x), \chi'_U(y)\}.$$

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Thus

$$\chi_U(x \alpha s \beta y) \geq 0 = \min\{\chi_U(x), \chi_U(y)\}$$

and

$$\max\{\chi'_U(x), \chi'_U(y)\} = \max\{1 - \chi_U(x), 1 - \chi_U(y)\}$$

$$= 1 - \min\{\chi_U(x), \chi_U(y)\}$$

$$= 1$$

$$\geq \chi'_U(x \alpha s \beta y).$$

Consequently, $U' = (\chi_U, \chi'_U)$ is an IFTB of $S$. 
References


