Lacunary Strongly Almost Generalized
Convergence with Respect to Orlicz Function

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Abstract
Kizmaz [5] defined the concept of difference sequence spaces. Later some authors introduced and studied some generalizations of this idea. In this paper, we study some properties of \( \hat{c}, M \) \( (\Delta^m) \)-convergence which was defined by Esi [1].

Keywords: Lacunary sequence, difference sequence, Orlicz function, strongly almost convergence.

1 Definitions and Notations

Let \( l_\infty, c \) and \( c_0 \) be the sequence spaces of bounded, convergent and null sequences \( x = (x_i) \), respectively. A sequence \( x = (x_i) \in l_\infty \) is said to be almost convergent [8] if all Banach limits of \( x = (x_i) \) coincide. In [8], it was shown that

\[
\hat{c} = \left\{ x = (x_i) : \lim_{n \to \infty} \sum_{i=1}^{n} x_{i+s} \text{ exists, uniformly in } s \right\}.
\]

In [9, 10], Maddox defined a sequence \( x = (x_i) \) strongly almost convergent to a number \( L \), if

\[
\lim_{n \to \infty} \sum_{i=1}^{n} |x_{i+s} - L| = 0, \text{ uniformly in } s.
\]
By a lacunary sequence \( \theta = (k_r), r = 0, 1, 2, \ldots \), where \( k_0 = 0 \), we shall mean increasing sequence of non-negative integers \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). The intervals determined by \( \theta \) are denoted by \( I_r = (k_{r-1}, k_r] \) and the ratio \( \frac{k_r}{k_{r-1}} \) will be denoted by \( q_r \). The space of lacunary strongly convergent sequence \( N_\theta \) was defined by Freedman et al. [3] as follows:

\[
N_\theta = \left\{ x = (x_i) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0 \text{ for some } L \right\}.
\]

In [5], Kizmaz defined the sequence spaces \( Z(\Delta) = \{ x = (x_i) : (\Delta x_i) \in Z \} \) for \( Z = l_\infty, c \) and \( c_0 \), where \( \Delta x = (\Delta x_i) = (x_i - x_{i-1}) \). After, Et and Colak [2] defined generalized the difference sequence spaces as follows:

\[
Z(m) = \{ x = (x_i) : (\Delta^m x_i) \in Z \} \text{ for } Z = l_\infty, c \text{ and } c_0, \text{ where } m \in \mathbb{N}, \Delta^0 x = x_i, \Delta x = (x_i - x_{i-1}), \Delta^m x_i = (\Delta^m x_i) = (\Delta^{m-1} x_i - \Delta^{m-1} x_{i+1}) \text{ and so that }
\]

\[
\Delta^m x_i = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{i+v}.
\]

An Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \).

An Orlicz function \( M \) is said to be satisfy \( \Delta_2 \)-condition for all values of \( t \), if there exists a constant \( T > 0 \) such that \( M(2t) \leq TM(t) \) for all \( t \geq 0 \).

**Remark 1.** The \( \Delta_2 \)-condition is equivalent to the satisfaction of the inequality \( M(Lt) \leq TL^2 M(t) \) for all values of \( t \) and for \( L > 1 \). This inequality was used in some published articles [12], [13] and many others. But this is not true, which is shown by the simplest example such as \( M(t) = t^2 \). Then the Orlicz function \( M \) satisfies \( \Delta_2 \)-condition with \( T = 4 \), but for \( M(Lt) = L^2 t^2 > 4Lt^2 \) when \( L \geq 5 \).

**Remark 2.** An Orlicz function \( M \) satisfies the inequality \( M(\lambda t) \leq \lambda M(t) \) for all \( \lambda \) with \( 0 < \lambda < 1 \).

Lindenstrauss and Tzafriri [7] used the idea of Orlicz functions to construct Orlicz sequence spaces

\[
l_M = \left\{ x = (x_i) : \sum_i M\left( \frac{|x_i|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]

The sequence space \( l_M \) with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_i M\left( \frac{|x_i|}{\rho} \right) \leq 1 \right\}
\]
becomes a Banach space with is called an Orlicz Sequence Space. The space \( l_M \) is closely related to the space \( l_p \), which is an Orlicz Sequence Space with \( M(x) = x^p \) for \( 1 \leq p < \infty \).

Let \( M \) be an Orlicz function. Gönçür and Et [4] defined the following sequence spaces:

\[
\left[ \hat{c}, M \right] (\Delta^m) = \left\{ x = (x_i) : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M \left( \frac{\Delta^m x_{i+s} - L}{\rho} \right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \right\},
\]

\[
\left[ \hat{c}, M \right]_0 (\Delta^m) = \left\{ x = (x_i) : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M \left( \frac{\Delta^m x_{i+s}}{\rho} \right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\},
\]

\[
\left[ \hat{c}, M \right]_\infty (\Delta^m) = \left\{ x = (x_i) : \sup_{n,s} \frac{1}{n} \sum_{i=1}^{n} M \left( \frac{\Delta^m x_{i+s}}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]

Let \( M \) be an Orlicz function. We defined in [1] new generalized difference sequence spaces as follows:

\[
\left[ \hat{c}, M \right]^{\theta} (\Delta^m) = \left\{ x = (x_i) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s} - L}{\rho} \right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \right\},
\]

\[
\left[ \hat{c}, M \right]_0^{\theta} (\Delta^m) = \left\{ x = (x_i) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s}}{\rho} \right) = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\},
\]

\[
\left[ \hat{c}, M \right]_\infty^{\theta} (\Delta^m) = \left\{ x = (x_i) : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s}}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.
\]

If \( x = (x_i) \in \left[ \hat{c}, M \right]^{\theta} (\Delta^m) \), we say that \( x = (x_i) \) is lacunary strongly almost generalized \( \Delta^m \)-convergence to the number \( L \) with respect to Orlicz function \( M \). In this case we write \( \left[ \hat{c}, M \right]^{\theta} (\Delta^m) \rightleftharpoons \lim x = L \). When \( M(x) = x \), then we write \( \left[ \hat{c} \right]^{\theta} (\Delta^m) \), \( \left[ \hat{c} \right]_0^{\theta} (\Delta^m) \) and \( \left[ \hat{c} \right]_\infty^{\theta} (\Delta^m) \) for the spaces \( \left[ \hat{c}, M \right]^{\theta} (\Delta^m) \), \( \left[ \hat{c}, M \right]_0^{\theta} (\Delta^m) \) and \( \left[ \hat{c}, M \right]_\infty^{\theta} (\Delta^m) \).

The purpose of this paper is to examine some properties of these new sequence spaces as a concept of lacunary almost generalized \( \Delta^m \)-convergence using Orlicz function which also generalize the well known Orlicz sequence space \( l_M \), strongly summable sequence spaces \( [C, 1] \), \( [C, 1]_0 \) and \( [C, 1]_\infty \).
2 Main Results

In this section we prove some results involving the sequence spaces \([\hat{c}, M]^\theta (\Delta^m)\), \([\hat{c}, M]_0^\theta (\Delta^m)\) and \([\hat{c}, M]^\theta_\infty (\Delta^m)\).

**Theorem 2.1.** The spaces \([\hat{c}, M]^\theta (\Delta^m)\), \([\hat{c}, M]_0^\theta (\Delta^m)\) and \([\hat{c}, M]^\theta_\infty (\Delta^m)\) are linear spaces over the complex field \(C\).

**Proof.** Let \(x = (x_i), y = (y_i) \in [\hat{c}, M]^\theta_0 (\Delta^m)\) and \(\alpha, \beta \in C\). Then there exist positive numbers \(\rho_1\) and \(\rho_2\) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m x_{i+s}|}{\rho_1} \right) = 0, \text{ uniformly in } s
\]

and

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m y_{i+s}|}{\rho_2} \right) = 0, \text{ uniformly in } s.
\]

Let \(\rho_3 = \max (2|\alpha|\rho_1, 2|\beta|\rho_2)\). Since \(M\) is non-decreasing convex function, we have

\[
\frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m (\alpha x_{i+s} + \beta y_{i+s})|}{\rho_3} \right) \leq \frac{1}{h_r} \sum_{i \in I_r} M \left[ \frac{|\Delta^m (\alpha x_{i+s})|}{\rho_3} \right] + \frac{1}{h_r} \sum_{i \in I_r} M \left[ \frac{|\Delta^m (\beta y_{i+s})|}{\rho_2} \right] 
\]

\[
\leq \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m x_{i+s}|}{\rho_1} \right) + \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m y_{i+s}|}{\rho_2} \right).
\]

Therefore \(\alpha x + \beta y \in [\hat{c}, M]^\theta_0 (\Delta^m)\).

The proof for other two cases are routine work in view of above proof.

**Theorem 2.2.** For any Orlicz function \(M\), \([\hat{c}, M]^\theta_\infty (\Delta^m)\) is a semi-normed linear space, semi-normed by

\[
h_{\Delta^m}(x) = \sum_{i=1}^m |x_i| + \inf \left\{ \rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m x_{i+s}|}{\rho} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}.
\]

**Proof.** Clearly; \(h_{\Delta^m}(x) = h_{\Delta^m}(-x), x = 0\) implies \(\Delta^m x_{i+s} = 0\) for all \(i, s \in N\) and as such \(M(0) = 0\), where \(0 = (0, 0, \ldots)\). Therefore \(h_{\Delta^m}(0) = 0\).

Next, let \(\rho_1\) and \(\rho_2\) be such that

\[
\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{|\Delta^m x_{i+s}|}{\rho_1} \right) \leq 1
\]
Let $\rho = \rho_1 + \rho_2$. Then, we have
\[
\sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m (x_{i+s} + y_{i+s})}{\rho} \right) 
\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s}}{\rho_1} \right) + \frac{\rho_2}{\rho_1 + \rho_2} \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m y_{i+s}}{\rho_2} \right) \leq 1.
\]
Since the $\rho$'s non-negative, so we have
\[
h_{\Delta^m} (x+y) = \sum_{i=1}^m |x_i + y_i| + \inf \left\{ \rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m (x_{i+s} + y_{i+s})}{\rho} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}
\leq \sum_{i=1}^m |x_i| + \inf \left\{ \rho_1 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s}}{\rho_1} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}
\ + \sum_{i=1}^m |y_i| + \inf \left\{ \rho_2 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m y_{i+s}}{\rho_2} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}.
\]
So, $h_{\Delta^m} (x+y) \leq h_{\Delta^m} (x) + h_{\Delta^m} (y)$. Finally for $\lambda \in \mathbb{C}$, without loss of generality $\lambda \neq 0$, then
\[
h_{\Delta^m} (\lambda x) = \sum_{i=1}^m |\lambda x_i| + \inf \left\{ \rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m \lambda x_{i+s}}{\rho} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}
\]
\[
= |\lambda| \sum_{i=1}^m |x_i| + \inf \left\{ |\lambda| r > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} M \left( \frac{\Delta^m x_{i+s}}{r} \right) \leq 1, \ r = 1, 2, \ldots s = 1, 2, \ldots \right\}, \text{ where } r = \frac{\rho}{\lambda}
\]
\[
= |\lambda| h_{\Delta^m} (\lambda x).
\]
This completes the proof.

**Theorem 2.3.** If $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r > 1$, then
\[
[\bar{c}, M] (\Delta^m) \subset [\bar{c}, M]^\theta (\Delta^m)
\]
where
\[
[\hat{C},M] (\Delta^m) = \left\{ x = (x_i) : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M \left( \frac{|\Delta^m_{x_i+s} - L|}{\rho} \right) = 0, \right. \text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0 \right\}.
\]

**Proof.** Let \( \lim \inf q_r > 1 \). Then there exists \( \delta > 0 \) such that \( q_r > 1 + \delta \) and hence
\[
\frac{h_r}{k_r} = 1 - k_{r-1} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.
\]
Therefore,
\[
\frac{1}{k_r} \sum_{i=1}^{k_r} M \left( \frac{|\Delta^m_{x_i+s}|}{\rho} \right) \geq \frac{1}{k_r} \sum_{i \in I} M \left( \frac{|\Delta^m_{x_i+s}|}{\rho} \right) \geq \frac{\delta}{1+\delta} \sum_{i \in I} M \left( \frac{|\Delta^m_{x_i+s}|}{\rho} \right)
\]
and if \( x = (x_i) \in [\hat{C},M] (\Delta^m) \), then it follows that \( x = (x_i) \in [\hat{C},M]^{\theta} (\Delta^m) \).

**Theorem 2.4.** If \( \theta = (k_r) \) be a lacunary sequence with \( \lim \sup q_r < \infty \), then
\[
[\hat{C},M]^{\theta} (\Delta^m) \subset [\hat{C},M] (\Delta^m).
\]

**Proof.** Let \( x = (x_i) \in [\hat{C},M]^{\theta} (\Delta^m) \). Then for \( \varepsilon > 0 \), there exists \( j_0 \) such that for every \( j \geq j_0 \) and for all \( s \in N \)
\[
a_{js} = \frac{1}{h_j} \sum_{i \in I_j} M \left( \frac{|\Delta^m_{x_i+s}|}{\rho} \right) < \varepsilon
\]
that is, we can find some positive constant \( T \), such that
\[
a_{js} < T
\] (1)
for all \( j \) and \( s \). Given \( \lim \sup q_r < \infty \) implies that there exists some positive number \( K \) such that
\[
q_r < K
\] (2)
for all \( r \geq 1 \). Therefore, for \( k_{r-1} < n \leq k_r \), we have by (1) and (2)
\[
\frac{1}{k_{r-1}} \sum_{j=1}^{r} \sum_{i \in I_j} M \left( \frac{|\Delta^m_{x_i+s} - L|}{\rho} \right) \leq \frac{1}{k_{r-1}} \sum_{j=1}^{j_0} \sum_{i \in I_j} M \left( \frac{|\Delta^m_{x_i+s} - L|}{\rho} \right)
\]
\[
\leq \frac{1}{k_{r-1}} \left( \sup_{1 \leq \rho \leq j_0} a_{ps} \right) k_{j_0} + \varepsilon \frac{k_r - k_{j_0}}{k_{r-1}}
\]
\[ \leq T \frac{k_{j_0}}{k_{r-1}} + \varepsilon K. \]

Since \( k_{r-1} \to \infty \) as \( r \to \infty \), we get \( x = (x_i) \in \hat{c}, M (\Delta^m) \). This completes the proof.

**Theorem 2.5.** Let \( \theta = (k_r) \) be a lacunary sequence with \( 1 \leq \lim \inf q_r \leq \lim \sup q_r < \infty \), then

\[ [\hat{c}, M]^\theta (\Delta^m) = [\hat{c}, M] (\Delta^m). \]

**Proof.** It follows from Theorem 2.3. and Theorem 2.4.

**Theorem 2.6.** Let \( x = (x_i) \in \hat{c}, M (\Delta^m) \cap [\hat{c}, M]^\theta (\Delta^m) \). Then

\[ [\hat{c}, M]^\theta (\Delta^m) - \lim x = [\hat{c}, M] (\Delta^m) - \lim x \]

and \([\hat{c}, M]^\theta (\Delta^m) - \lim x\) is unique for any lacunary sequence \( \theta = (k_r) \).

**Proof.** Let \( x = (x_i) \in [\hat{c}, M] (\Delta^m) \cap [\hat{c}, M]^\theta (\Delta^m) \) and \([\hat{c}, M]^\theta (\Delta^m) - \lim x = L_o \), \([\hat{c}, M] (\Delta^m) - \lim x = L\). Suppose that \( L \neq L_o \). We can see that

\[
M\left( \frac{|L - L_o|}{\rho} \right) \leq \frac{1}{h_r} \sum_{i \in I_r} M\left( \frac{|\Delta^m x_{i+s} - L|}{\rho} \right) + \frac{1}{h_r} \sum_{i \in I_r} M\left( \frac{|\Delta^m x_{i+s} - L_o|}{\rho} \right)
\]

\[
\leq \limsup_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M\left( \frac{|\Delta^m x_{i+s} - L|}{\rho} \right) + 0.
\]

Hence, there exists \( r_o \), such that for \( r > r_o \)

\[
\frac{1}{h_r} \sum_{i \in I_r} M\left( \frac{|\Delta^m x_{i+s} - L|}{\rho} \right) > \frac{1}{2} M\left( \frac{|L - L_o|}{\rho} \right).
\]

Since \([\hat{c}, M] (\Delta^m) - \lim x = L\), it follows that

\[
0 \geq \limsup_{r} \left( \frac{h_r}{k_r} \right) M\left( \frac{|L - L_o|}{\rho} \right) \geq \liminf_{r} M\left( \frac{|L - L_o|}{\rho} \right) \geq 0
\]

and so \( \lim_{r} q_r = 1 \). Hence by Theorem 2.2., \([\hat{c}, M]^\theta (\Delta^m) \subset [\hat{c}, M] (\Delta^m) \) and \([\hat{c}, M]^\theta (\Delta^m) - \lim x = L_o = [\hat{c}, M] (\Delta^m) - \lim x = L\). Further

\[
\frac{1}{n} \sum_{i=1}^{n} M\left( \frac{|\Delta^m x_{i+s} - L|}{\rho} \right) + \frac{1}{n} \sum_{i=1}^{n} M\left( \frac{|\Delta^m x_{i+s} - L_o|}{\rho} \right) \geq M\left( \frac{|L - L_o|}{\rho} \right) \geq 0
\]

and taking the limit on both sides as \( n \to \infty \), we have \( M\left( \frac{|L - L_o|}{\rho} \right) = 0 \), i.e., \( L = L_o \) for any Orlicz function \( M \). This completes the proof.


References


