Biharmonic Curves in $\mathbb{H}^2\times\mathbb{R}$

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Abstract

In this paper, we study biharmonic curves in the $\mathbb{H}^2\times\mathbb{R}$. We show that all of them are helices. By using the curvature and torsion of the curves, we give some characterizations biharmonic curves in the $\mathbb{H}^2\times\mathbb{R}$.

Keywords: Biharmonic curve, curvature, helices, torsion.

1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu. In the last decade there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.
Let \( f : (M, g) \to (N, h) \) be a smooth function between two Riemannian manifolds. The bienergy \( E_2(f) \) of \( f \) over compact domain \( \Omega \subset M \) is defined by

\[
E_2(f) = \int_{\Omega} h(\tau(f), \tau(f)) dv_g,
\]

where \( \tau(f) = \text{trace}_g \nabla df \) is the tension field of \( f \) and \( dv_g \) is the volume form of \( M \). Using the first variational formula one sees that \( f \) is a biharmonic function if and only if its bitension field vanishes identically, i.e.

\[
\tau_2(f) := -\Delta^f(\tau(f)) - \text{trace}_g R^N(df, \tau(f))df = 0,
\]

where

\[
\Delta^f = -\text{trace}_g (\nabla^f)^2 = -\text{trace}_g \left( \nabla^f \nabla^f - \nabla^f \nabla^f_M \right)
\]

is the Laplacian on sections of the pull-back bundle \( f^{-1}(TN) \) and \( R^N \) is the curvature operator of \( (N, h) \) defined by

\[
R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z.
\]

In this paper we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in the \( \mathbb{H}^2 \times \mathbb{R} \) must satisfy. Then we prove that the non-geodesic biharmonic curves in the \( \mathbb{H}^2 \times \mathbb{R} \) are helices. Finally, we deduce the explicit parametric equations of the non-geodesic biharmonic curves in the \( \mathbb{H}^2 \times \mathbb{R} \).

2 Left Invariant Metric in \( \mathbb{H}^2 \times \mathbb{R} \)

Let \( \mathbb{H}^2 \) be the upper half-plane model \( \{(x, y) \in \mathbb{R}^2 : y > 0\} \) of the hyperbolic plane endowed with the metric

\[
g_H = \frac{(dx^2 + dy^2)}{y^2}
\]

of constant Gauss curvature \(-1\). The space \( \mathbb{H}^2 \), with the group structure derived by the composition of proper affine maps, is a Lie group and the metric \( g_H \) is left invariant. Therefore the product \( \mathbb{H}^2 \times \mathbb{R} \) is a Lie group with the left invariant product metric

\[
g = \frac{(dx^2 + dy^2)}{y^2} + dz^2.
\]
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\[ e_1 = \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}. \]  

(2.2)

Now let \( C^k_{ij} \) be the structure's constants of the Lie algebra \( g \) of \( G \) that is

\[ [e_i, e_j] = C^k_{ij} e_k. \]

The corresponding Lie brackets are

\[ [e_1, e_2] = -e_1, \quad [e_1, e_3] = [e_2, e_3] = 0. \]

The Coshul formula for the Levi-Civita connection is:

\[ 2g(\nabla_{e_i} e_j, e_k) = C^l_{ij} - C^l_{jk} + C^l_{il} := L^l_{ij}, \]

where the non zero \( L^l_{ij} \) ’s are

\[ L^1_{12} = -2, \quad L^2_{12} = 2. \]  

(2.3)

We adopt the following notation and sign convention for Riemannian curvature operator:

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

The Riemannian curvature tensor is given by

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W). \]

Moreover we put

\[ R_{abc} = R(e_a, e_b) e_c, \quad R_{abcd} = R(e_a, e_b, e_c, e_d), \]

where the indices \( a, b, c, d \) take the values 1, 2, 3. Then,

\[ R_{121} = e_2, \quad R_{1212} = 1. \]  

(2.4)

### 3 Biharmonic Curves in \( H^2 \times \mathbb{R} \)

Let \( I \subset \mathbb{R} \) be an open interval and \( \gamma : I \to H^2 \times \mathbb{R} \) be a curve, parametrized by arc length, on a Riemannian manifold. Putting \( T = \gamma' \), we can write the tension field of \( \gamma \) as \( \tau(\gamma) = \nabla_\gamma \gamma' \) and the biharmonic map equation (1.2) reduces to

\[ \nabla_\gamma T + R(T, \nabla_\gamma T) T = 0. \]  

(3.1)
A successful key to study the geometry of a curve is to use the Frenet frames along the curve which is recalled in the following. Let \( \gamma : I \to \mathbb{H}^2 \times \mathbb{R} \) be a curve on \( \mathbb{H}^2 \times \mathbb{R} \) parametrized by arc length. Let \( \{T, N, B\} \) be the Frenet frame fields tangent to \( \mathbb{H}^2 \times \mathbb{R} \) along \( \gamma \) defined as follows: \( T \) is the unit vector field tangent to \( \gamma \), \( N \) is the unit vector field in the direction of \( \nabla_\gamma T \) (normal to \( \gamma \)), and \( B \) is chosen so that \( \{T, N, B\} \) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas

\[
\begin{align*}
\nabla_\gamma T &= \kappa N \\
\nabla_\gamma N &= -\kappa T - \tau B \\
\nabla_\gamma B &= \tau N,
\end{align*}
\]

where \( \kappa \) is the curvature of \( \gamma \) and \( \tau \) its torsion. With respect to the orthonormal basis \( \{e_1, e_2, e_3\} \) we can write

\[
\begin{align*}
T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\
N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\
B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3.
\end{align*}
\]

\[\textbf{Theorem 3.1} \quad \gamma : I \to \mathbb{H}^2 \times \mathbb{R} \text{ is a biharmonic curve if and only if}
\]

\[
\kappa = \text{constant} \neq 0, \\
\kappa^2 + \tau^2 = B_3^2, \\
\tau = -N_3 B_3.
\]

\[\textbf{Proof.} \text{ From (3.1) we obtain}
\]

\[
\begin{align*}
\tau_3 (\gamma) &= \nabla_\gamma T + R(T, \nabla_\gamma T) T \\
&= (-3 \kappa \kappa') T + (\kappa^3 - \kappa^3 - \kappa \tau^2 + \kappa R(T, N, T, N)) N \\
&\quad + (-2 \kappa' \tau - \kappa \tau' + \kappa R(T, N, T, B)) B \\
&= 0.
\end{align*}
\]

We see that is a biharmonic curve if and only if

\[
\kappa \kappa' = 0, \\
\kappa' - \kappa^3 - \kappa \tau^2 + \kappa R(T, N, T, N) = 0, \\
-2 \kappa' \tau - \kappa \tau' + \kappa R(T, N, T, B) = 0,
\]

which is equivalent to
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\[ \kappa = \text{constant} \neq 0, \]
\[ \kappa^2 + \tau^2 = R(T, N, T, N), \tag{3.4} \]
\[ \tau = R(T, N, T, B). \]

A direct computation using (2.4) yields

\[ R(T, N, T, N) = B_3^2, \quad R(T, N, T, B) = -N_3B_3. \tag{3.5} \]

These, together with (3.4), complete the proof of the theorem.

**Corollary 3.2.** Let \( \gamma: I \rightarrow \mathbb{H}^2 \times \mathbb{R} \) be a curve with constant curvature and \( N_3B_3 \neq 0 \). Then, \( \gamma \) is not biharmonic.

**Proof.** We use the covariant derivatives of the vector fields \( T, N \) and \( B \), these, together with equations (3.2) we get

\[ T_3 = \kappa N_3, \]
\[ N_3 = -\kappa T_3 - \tau B_3, \]
\[ B_3 = \tau N_3. \tag{3.6} \]

Assume now that \( \gamma \) is biharmonic. Then \( \tau^\prime = -N_3B_3 \neq 0 \) and (3.3) we obtain

\[ \tau\tau^\prime = B_3B_3, \]

Since \( \tau^\prime = -N_3B_3 \), this is rewritten as

\[ \tau = -\frac{B_3}{N_3}. \tag{3.7} \]

From (3.6) we have

\[ B_3 = \tau N_3. \tag{3.8} \]

If we substitute \( B_3^\prime \) in equation (3.7)

\[ \tau = 0. \]

Therefore also \( \tau \) is constant and we have a contradiction. Complete the proof of the corollary.

By using Theorem 3.1 and Corollary 3.2, we have the following corollary:

**Corollary 3.3.** \( \gamma: I \rightarrow \mathbb{H}^2 \times \mathbb{R} \) is biharmonic if and only if

\[ \kappa = \text{constant} \neq 0, \]
Corollary 3.4.

(i) If $N_3 \neq 0$ then $\gamma$ is not biharmonic.

(ii) If $N_3 = 0$, then

$$T(s) = \sin \alpha_0 \cos \beta(s)e_1 + \sin \alpha_0 \sin \beta(s)e_2 + \cos \alpha_0 e_3, \quad (3.9)$$

where $\alpha_0 \in \mathbb{R}$.

Proof.

(i) Using the above Corollary 3.3 it is easy to see that $\gamma$ is not biharmonic.

(ii) Since $\gamma$ is parametrized by arc length, we can write

$$T(s) = \sin \alpha \cos \beta(s)e_1 + \sin \alpha \sin \beta(s)e_2 + \cos \alpha e_3. \quad (3.10)$$

From (3.6) we obtain

$$T_3' = \kappa N_3.$$  

Since $N_3 = 0$

$$T_3' = 0.$$  

Then $T_3$ is constant. We use (3.10), we have

$$T_3 = \cos \alpha_0 = \text{constant}.$$  

Theorem 3.5. The parametric equations of all biharmonic curves of $H^2 \times \mathbb{R}$,

$$x(s) = c_2 \sin \alpha_0 \int \sin \beta(s)e^{e^{0}} ds + c_1,$$

$$y(s) = c_2 \int \cos \beta(s) ds,$$

$$z(s) = \sin \alpha_0 e^{c_3}, \quad (3.11)$$

where $c_1, c_2, c_3$ is arbitrary constants.

Proof. We note that

$$\frac{dy}{ds} = T(s) = T_1 e_1 + T_2 e_2 + T_3 e_3,$$

and our left-invariant vector fields are

$$T_1 = \sin \alpha_0 \cos \beta(s)e_1 + \sin \alpha_0 \sin \beta(s)e_2 + \cos \alpha_0 e_3,$$

$$T_2 = \sin \alpha \cos \beta(s)e_1 + \sin \alpha \sin \beta(s)e_2 + \cos \alpha e_3,$$

$$T_3 = \cos \alpha_0 = \text{constant}.$$

Thus, $\gamma$ is not biharmonic.
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\[ e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \]

then

\[ \frac{dy}{ds} = (y(s)T_1, y(s)T_2, T_3). \]

By using equation (3.9), we have

\[ \frac{dy}{ds} = T(s) = (y(s) \sin \alpha_0 \cos \beta(s), y(s) \sin \alpha_0 \sin \beta(s), \cos \alpha_0). \]

In order to find the explicit equations for \( \gamma(s) = (x(s), y(s), z(s)) \), we must integrate the system \( \frac{dy}{ds} = T(s) \), that in our case is

\[
\begin{align*}
\frac{dx}{ds} &= y(s) \sin \alpha_0 \cos \beta(s), \\
\frac{dy}{ds} &= y(s) \sin \alpha_0 \sin \beta(s), \\
\frac{dz}{ds} &= \cos \alpha_0.
\end{align*}
\]

The integration is immediate and yields (3.11).

**Corollary 3.7.** If \( \beta(s) = s \), then the parametric equations of all biharmonic curves are

\[
\begin{align*}
x(s) &= e^{-s \sin \alpha_0 \cos s}(\sin \alpha_0 \cos s + 1) + c_1, \\
y(s) &= c_2 e^{-s \sin \alpha_0 \cos s}, \\
z(s) &= s \cos \alpha_0 + c_3,
\end{align*}
\]

where \( c_1, c_2, c_3 \) is arbitrary constants.

The picture of biharmonic curve \( \gamma(s) \) at \( c_1 = c_2 = c_3 = 1 \):

**4 Conclusion**

In this work, we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in the \( \mathbb{H}^2 \times \mathbb{R} \) must satisfy. Then we prove that the non-geodesic biharmonic curves in the \( \mathbb{H}^2 \times \mathbb{R} \) are helices. Finally, we deduce the explicit parametric equations of the non-geodesic biharmonic curves in the \( \mathbb{H}^2 \times \mathbb{R} \).

**References**


