Some Theorems on Fixed Point

Sampada Navshinde\(^1\) and J. Achari\(^2\)

\(^1\)Asst. Prof. SGGSI&T, Nanded-431605, (Maharashtra) India
E-mail: snavshinde@gmail.com
\(^2\)Retired H.O.D.Mathematics, N.E.S.'Science College,
Nanded-431605,(Maharashtra) India

(Received: 24-12-10/ Accepted: 23-3-11)

Abstract

Fixed point theorems for a class of mappings using rational symmetric expression involving four points of the space under consideration have been studied.

Keywords: Clouser, Common fixed point, commuting mappings.

1 Introduction

The chief aim of this paper is to introduce a class of mappings by using rational symmetric expression and which involve four points of the space under consideration. A fixed point theorem with this mapping has been proved. Finally some related results with this type of mappings have been proved.

Let \((M,d)\) be a complete metric space. Let \(\psi_i : \bar{P} \rightarrow [0, \infty )\) (\(P\) is the range of \(d\) and \(\bar{P}\) is the closure of \(P\)) be an upper semicontinuous function from the right on \(P\) and satisfies the condition

\[
\psi_i(t) < \frac{t}{3} \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi_i(0) = 0 \quad i = 1, 2, 3.
\]
Also, let $f$ be a mapping of $M$ into itself such that
\[
d(fu_1, fu_2) \leq \frac{\psi_1(d(u_2, fu_4)[1 + \psi_1(d(u_1, fu_3)])}{1 + \psi_1(d(u_1, u_2))} + \frac{\psi_2(d(u_1, fu_4)[1 + \psi_2(d(u_2, fu_3)])}{1 + \psi_2(d(u_1, u_2))} + \frac{\psi_3(d(u_1, fu_3)[1 + \psi_3(d(u_2, fu_4)])}{1 + \psi_3(d(u_1, u_2))}
\]
for $u_1, u_2, u_3, u_4 \in M$.

2 The Main Results

**Theorem 2.1.** If $f$ be mapping of $M$ into itself satisfying (1.2), then $f$ has a unique fixed point.

**Proof.** Let $x, y \in M$ and we define $u_1 = fy$, $u_2 = fx$, $u_3 = x$, $u_4 = y$ Then (1.2) takes the form
\[
d(f(fy), f(fx)) \leq \frac{\psi_1(d(fy, f(x)))[1 + \psi_1(d(fy, fx)])}{1 + \psi_1(d(fy, fx))} + \frac{\psi_2(d(fy, f(x)))[1 + \psi_2(d(fx, fx)])}{1 + \psi_2(d(fy, fx))} + \frac{\psi_3(d(fy, f(x)))[1 + \psi_3(d(fx, f(y)])}{1 + \psi_3(d(fy, fx))}
\]
(2.1)

Let $x_0 \in M$ be arbitrary and construct a sequence $\{x_n\}$ defined by
\[
f_{x_{n-1}} = x_n, \quad f_{x_n} = x_{n+1}, \quad f_{x_{n+1}}, \quad n = 1, 2, \ldots
\]
Let us put $x = x_{n-1}$, $y = x_n$ in (2.1), then we have,
\[
d(f(f_{x_n}), f(f_{x_{n-1}})) \leq \psi_1(d(f_{x_{n-1}}, f_{x_n})) + \psi_3(d(f_{x_{n-1}}, f_{x_n}))
\]
\[i.e. \, d(x_{n+1}, x_{n+2}) \leq \psi_1(d(x_n, x_{n+1})) + \psi_3(d(x_n, x_{n+1})) \quad (2.2)
\]
Now set $C_n = d(x_{n-1}, x_n)$. Then
\[
C_{n+2} = d(x_{n+1}, x_{n+2}) \leq \psi_1(d(x_n, x_{n+1})) + \psi_3(d(x_n, x_{n+1})) \leq \psi_1(C_{n+1}) + \psi_3(C_{n+1})
\]
(2.3)

From (2.3) it follows that $C_n$ decreases with $n$ and hence $C_n \to C$ say as $n \to \infty$. Then since $\psi_i$ is upper semicontinous we obtain in the limit as $n \to \infty$
\[
C \leq \psi_1(C) + \psi_3(C) < \frac{2}{3}C
\]
Next, we shall show that the sequence \( \{x_n\} \) is Cauchy. Suppose that it is not so. Then there exist an \( \epsilon > 0 \) and sequence of integers \( \{m(k)\}, \{n(k)\} \) with \( m(k) > n(k) \geq k \) such that

\[
d_k = d(x_{m(k)}, x_{n(k)}) \geq \epsilon, k = 1, 2, 3, \ldots \tag{2.4}
\]

If \( m(k) \) is the smallest integer exceeding \( n(k) \) for which (2.4) holds, then from the well ordering principle we have,

\[
d(x_{m(k) - 1}, x_{n(k)}) \leq \epsilon \tag{2.5}
\]

Then \( d_k = d(x_{m(k)}, x_{m(k) - 1}) + d(x_{m(k) - 1}, x_{n(k)}) \leq C_m(k) + \epsilon < C_k + \epsilon \) which implies that \( d_k \rightarrow \epsilon \) as \( n \rightarrow \infty \).

Also we have,

\[
d_k = d(x_m, x_n) \\
\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\
\leq C_{m+1} + C_{n+1} + C d(f x_n, f x_m) \\
\leq C_{m+1} + C_{n+1} + \frac{\psi_1(d(x_m, f x_{m-1})[1 + \psi_1(d(x_n, f x_{n-1})])] + \psi_1(d(x_n, x_m))}{1 + \psi_1(d(x_n, x_m))} \\
+ \frac{\psi_2(d(x_n, f x_{m-1})[1 + \psi_2(d(x_m, f x_{n-1})])] + \psi_2(d(x_m, x_n))}{1 + \psi_2(d(x_m, x_n))} \\
+ \frac{\psi_3(d(x_n, f x_{n-1})[1 + \psi_3(d(x_m, f x_{m-1})])] + \psi_3(d(x_m, x_n))}{1 + \psi_3(d(x_n, x_m))}
\]

(By putting \( u_1 = x_n, u_2 = x_m, u_3 = x_{n-1}, u_4 = x_{m-1} \))

\[
d_k = d(x_m, x_n) \leq C_{m+1} + C_{n+1} + \psi_2(d(x_m x_m)) + \psi_3(d(x_n, x_m)) \\
\leq C_{m+1} + C_{n+1} + \psi_2(d_k) + \psi_3(d_k)
\]

letting \( k \rightarrow \infty \) we have

\[
\epsilon \leq \psi_2(\epsilon) + \psi_3(\epsilon) < \frac{2}{3} \epsilon
\]

which is a contradiction if \( \epsilon > 0 \).

This leads us to conclude that \( \{x_n\} \) is a Cauchy sequence and since \( M \) is complete, there exists a point \( z \in M \) such that \( x_n \rightarrow z \) as \( n \rightarrow \infty \). We shall show that \( z \) is a fixed point of \( f \).
Some Theorems on Fixed Point

Now putting \( u_1 = x_{n-1}, \ u_2 = z, \ u_3 = x_{n+1}, \ u_4 = x_n \) in (1.2) we have,
\[
d(fx_{n-1}, fz) \
\leq \frac{\psi_1(d(z, fx_n))[1 + \psi_1(d(x_{n-1}, fx_{n+1})]}{1 + \psi_1(d(x_{n-1}, z))} \]
\[+ \frac{\psi_2(d(x_{n-1}, fx_n))[1 + \psi_2(d(z, fx_{n+1})]}{1 + \psi_2(d(x_{n-1}, z))} \]
\[+ \frac{\psi_3(d(x_{n-1}, fx_n))[1 + \psi_3(d(z, fx_{n+1})]}{1 + \psi_3(d(x_{n-1}, z))} \]
\[\leq \frac{\psi_1(d(z, x_{n+1}))[1 + \psi_1(d(x_{n-1}, x_{n+2})]}{1 + \psi_1(d(x_{n-1}, z))} \]
\[+ \frac{\psi_2(d(x_{n-1}, x_{n+1}))[1 + \psi_2(d(z, x_{n+2})]}{1 + \psi_2(d(x_{n-1}, z))} \]
\[+ \frac{\psi_3(d(x_{n-1}, x_{n+2}))[1 + \psi_3(d(z, x_{n+1})]}{1 + \psi_3(d(x_{n-1}, z))} \]  
(2.6)

Letting \( n \to \infty \) we get \( d(z, fz) \leq 0 \) which implies \( z = fz \). Thus \( z \) is a fixed point of \( f \).

If possible, let there be another fixed point \( w(\neq z) \), then putting \( u_1 = u_4 = z, \ u_2 = u_3 = w \) in (1.2) we get,
\[
d(z, w) = d(fz, fw) \
\leq \frac{\psi_1(d(w, fz))[1 + \psi_1(d(z, fw))]}{1 + \psi_1(d(z, w))} \]
\[+ \frac{\psi_2(d(z, fz))[1 + \psi_2(d(w, fw))]}{1 + \psi_2(d(z, w))} \]
\[+ \frac{\psi_3(d(z, fw))[1 + \psi_3(d(w, fz))]}{1 + \psi_3(d(z, w))} \]
\[\leq \frac{\psi_1(d(z, w)) + \psi_2(d(z, w))}{1 + \psi_3(d(z, w))} < \frac{2}{3}d(z, w) \]

which is impossible. Hence \( z = w \).

**Theorem 2.2.** Let \((M, d)\) be a complete metric space and \( f_k \)
\((k = 1, 2, \ldots, n)\) be a family of mappings of \( M \) into itself. If \( f_k \) \((k = 1, 2, \ldots, n)\) satisfies

(i) \( f_k f_m = f_m f_k \) \((m, k = 1, 2, \ldots, n)\)

(ii) there is a system of positive integers \( m_1, m_2, \ldots, m_n \) such that
\[
d(f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_2) \
\leq \frac{\psi_1(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_1))[1 + \psi_1(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_1(d(u_1, u_2))} \]
\[+ \frac{\psi_2(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))[1 + \psi_2(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_2(d(u_1, u_2))} \]
\[+ \frac{\psi_3(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))[1 + \psi_3(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))]}{1 + \psi_3(d(u_1, u_2))} \]
for \( u_1, u_2, u_3, u_4 \in M \) and \( \psi_i(t) \) satisfies (1.1), then \( f_k (k = 1, 2, \cdots n) \) have a unique common fixed point.

**Proof.** Let \( f = f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} \). Then (ii) takes the form (iii)

\[
d(fu_1, fu_2) \leq \frac{\psi_1(d(u_2, fu_4)[1 + \psi_1(d(u_1, fu_3))]}{1 + \psi_1(d(u_1, u_2))} + \frac{\psi_2(d(u_1, fu_4)[1 + \psi_2(d(u_2, fu_3))]}{1 + \psi_2(d(u_1, u_2))} + \frac{\psi_3(d(u_1, fu_4)[1 + \psi_3(d(u_2, fu_4))]}{1 + \psi_3(d(u_1, u_2))}
\]

Then by Theorem [2.1], \( f \) has a unique fixed point \( z \) in \( M \). Therefore \( fz = z \), then we have,

\[
f_k(fz) = f_kz, \quad k = 1, 2, \cdots n
\]

By commutativity of \( f_k \) we have,

\[
f(f_kz) = f_kz, \quad k = 1, 2, \cdots n
\]

Since \( f \) has a unique common fixed point \( z \), we obtain \( f_kz, \quad k = 1, 2, \cdots n \). Hence \( z \) is a common fixed point of the family \( f_k \). Let \( z, w \) be common fixed point of \( f_k \), then by (ii) we have by putting \( u_1 = u_4 = z, \ u_2 = u_3 = w \)

\[
d(z, w) = d(fz, fw)
\]

\[
\leq \frac{\psi_1(d(w, fz)[1 + \psi_1(d(z, fw))]}{1 + \psi_1(d(z, w))} + \frac{\psi_2(d(z, fz)[1 + \psi_2(d(w, fw))]}{1 + \psi_2(d(z, w))} + \frac{\psi_3(d(z, fz)[1 + \psi_3(d(w, fz))]}{1 + \psi_3(d(z, w))}
\]

\[
\leq \psi_1(d(z, w)) + \psi_2(d(z, w)) < \frac{2}{3}d(z, w)
\]

which implies \( z = w \). Hence the proof.

**Theorem 2.3.** Let \( M \) be a metric space with \( d \) and \( p \) and \( f_k (k = 1, 2, \cdots, n) \) be a family of mappings of \( M \) into itself.

Suppose that

(i) \( d(x,y) \leq p(x,y) \) to all \( x, y \in M \)

(ii) \( M \) is \( f \)-orbitally complete w.r.t. \( d \).

(iii) \( f_k f_m = f_m f_k \) \( (m, k = 1, 2, \cdots n) \)

(iv) there is a system of positive integers \( m_1, m_2, \cdots, m_n \) such that
Some Theorems on Fixed Point

\[ p(f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_1, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_2) \]
\[ \leq \frac{\psi_1(p(u_2, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_4)[1 + \psi_1(p(u_1, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_3))]}{1 + \psi_1(p(u_1, u_2))} + \frac{\psi_2(p(u_1, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_4)[1 + \psi_2(p(u_2, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_3))]}{1 + \psi_2(p(u_1, u_2))} + \frac{\psi_3(p(u_1, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_4)[1 + \psi_3(p(u_2, f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} u_3))]}{1 + \psi_3(p(u_1, u_2))} \]

for \( u_1, u_2, u_3, u_4 \in M \) and \( \psi_i(t) < \frac{t}{3} \) for \( t > 0 \) and \( \psi_i(0) = 0 \), \( i = 1, 2, 3 \) Then \( f_k \) \((k = 1, 2, \cdots, n)\) have a unique common fixed point.

**Proof.** As in Theorem [2.1] put \( f = f_{m_1}^{f_2} \cdots f_{m_n}^{f_4} \) then (iv) takes the form

\[ p(f u_1, f u_2) \leq \frac{\psi_1(p(u_2, f u_4)[1 + \psi_1(p(u_1, f u_3))]}{1 + \psi_1(p(u_1, u_2))} + \frac{\psi_2(p(u_1, f u_4)[1 + \psi_2(p(u_2, f u_3))]}{1 + \psi_2(p(u_1, u_2))} + \frac{\psi_3(p(u_1, f u_4)[1 + \psi_3(p(u_2, f u_3))]}{1 + \psi_3(p(u_1, u_2))} \]

for \( u_1, u_2, u_3, u_4 \in M \)

Following the lines of arguments of the proof of Theorem [2.1], it can be shown that the sequence of iterates \( \{x_n\} \) is Cauchy with respect to \( p \). Since \( d(x, y) \leq p(x, y) \) for all \( x, y \in M \), so \( \{x_n\} \) is Cauchy with respect to \( d \) also. Again \( M \) being \( f \)-orbitally complete with respect to \( d \), so we have \( \{x_n\} \) has a limit \( u \) in \( M \). From the proof of Theorem [2.1] it can be easily shown that \( u \) is the unique common fixed point of the family \( f_k \).

**Acknowledgements**

The authors thank the referee for his/her suggestions and comments.

**References**


