Complementary Nil Domination in Interval-valued Intuitionistic Fuzzy Graph

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Abstract

The aim of this paper is to introduce the concept of complementary nil domination in interval-valued intuitionistic fuzzy graph and to obtain some results related to this concept.

Keywords: Interval-valued Intuitionistic Fuzzy graph, degree, neighborhood of vertex, dominating set, Complementary nil domination set.

1 Introduction

Zadeh [8] introduced the notion of interval-valued fuzzy set as an extension of fuzzy set [7] which gives a more precise tool to modal uncertainty in real life situations. Some recent work of Zadeh in connection with the importance of fuzzy logic may be found in [9, 10]. Interval-valued fuzzy sets have widely used in many areas of science and engineering, e.g. in approximate reasoning, medical diagnosis, multi valued logic, intelligent control, topological spaces, etc.

In fact, interval-valued fuzzy graphs and interval-valued intuitionistic fuzzy graphs are two different models that extend the theory of fuzzy graph. Recently, complementary nil domination in fuzzy graph was introduced by Ismayil and Mohideen [18] in 2014. Further, In 2014, Hussain and Mohamed [15] introduced the complementary nil domination in intuitionistic fuzzy graph. Motivated by the concept of complementary nil domination and interval-valued intuitionistic fuzzy graph, we introduce complementary nil domination in interval-valued intuitionistic fuzzy graph and obtain some results related to this concept.

2 Preliminaries

**Definition 2.1** A fuzzy set $\sigma$ is a mapping $\sigma: V \rightarrow [0, 1]$. A fuzzy graph $G$ is a pair of functions $G = (\sigma, \mu)$ where $\sigma$ is a fuzzy subset of non-empty set $V$ and $\mu$ is a symmetric fuzzy relation on $\sigma$, i.e., $\mu(u, v) \leq \sigma(u) \land \sigma(v)$. The underlying crisp graph of $G = (\sigma, \mu)$ is denoted by $G^* = (V, E)$ where $E \subseteq V \times V$.

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$ and element of this set are denoted by uppercase letters. If $M \in D[0,1]$ then it can be represented as $M = [M_L, M_U]$, where $M_L$ and $M_U$ are the lower and upper limits of $M$.

**Definition 2.2** An interval valued intuitionistic fuzzy graph with underlying set $V$ is defined to be a pair $G = (A, B)$ where

i) The functions $M_A: V \rightarrow D[0,1]$ and $N_A: V \rightarrow D[0,1]$ denote the degree of membership and non membership of the element $x \in V$, respectively, such that

$$0 \leq M_A(x) + N_A(x) \leq 1 \text{ for all } x \in V.$$ 

ii) The functions $M_B: E \subseteq V \times V \rightarrow D[0,1]$ and $N_B: E \subseteq V \times V \rightarrow D[0,1]$ are defined by

$$M_{BL}((x, y)) \leq \min\{M_{AL}(x), M_{AL}(y)\} \text{ and }$$
$$N_{BL}((x, y)) \geq \max\{N_{AL}(x), N_{AL}(y)\}$$
$$M_{BU}((x, y)) \leq \min\{M_{AU}(x), M_{AU}(y)\} \text{ and }$$
$$N_{BU}((x, y)) \geq \max\{N_{AU}(x), N_{AU}(y)\}$$

such that $0 \leq M_{BU}(x, y) + N_{BU}(x, y) \leq 1 \text{ for all } (x, y) \in E.$
Example 2.3 Figure 2.1 is an example for IVIFG, $G = (A,B)$ defined on a graph $G^* = (V,E)$ such that $V = \{a, b, c, d\}$, $E = \{ab, bc, cd, da\}$, $A$ is an interval valued intuitionistic fuzzy set of $V$ and let $B$ is an interval-valued intuitionistic fuzzy set of $E \subseteq V \times V$. Here

$A = \{<a, [0.5,0.7],[0.1,0.2]>, <b, [0.3,0.5],[0.2,0.4]>, <c, [0.4,0.6],[0.2,0.4]>, <d, [0.2,0.5],[0.2,0.5]> \}$,

$B = \{<ab, [0.3,0.5],[0.2,0.4]>, <bc, [0.3,0.4],[0.2,0.4]>, <cd, [0.2,0.4],[0.3,0.5]>, <da, [0.2,0.5],[0.2,0.5]> \}$

Definition 2.4 The vertex cardinality of interval valued intuitionistic fuzzy graph $G = (A, B)$ of the graph $G^* = (V, E)$ is defined

$$|V| = \sum_{x \in V} \left( \frac{1}{2} \left( M_{AL}(x) - M_{LU}(x) \right) + \frac{1}{2} \left( N_{AU}(x) - N_{LU}(x) \right) \right) = p$$

And Edge cardinality of interval valued intuitionistic fuzzy graph is defined

$$|E| = \sum_{(x, y) \in E} \left( \frac{1}{2} \left( M_{LU}(x) - M_{BL}(x) \right) + \frac{1}{2} \left( N_{LU}(x) - N_{BL}(x) \right) \right) = q$$

The vertex cardinality of interval valued intuitionistic fuzzy graph (IVIFG) is called the order of $G$ and denoted by $O(G)$. The edge cardinality of IVIFG is called the size of $G$ and denoted by $S(G)$.

Definition 2.5 An edge $e = (x, y)$ of an interval valued intuitionistic fuzzy graph is called an effective edge if

$$M_{LU}(x, y) = \min \{ M_{AL}(x), M_{LU}(y) \} \quad \text{and} \quad N_{LU}(x, y) = \max \{ N_{AL}(x), N_{LU}(y) \}$$

$$M_{BL}(x, y) = \min \{ M_{LU}(x), M_{BL}(y) \} \quad \text{and} \quad N_{BL}(x, y) = \max \{ N_{LU}(x), N_{BL}(y) \}$$

In this case, the vertex $x$ is called a neighbor of $y$. 

Fig. 2.1
N(x) = \{y \in V: y \text{ is a neighbor of } x\}. N[x] = N(x) \cup N(y) \text{ is called the closed neighbourhood of } x.

**Definition 2.6** The complement of an IVIFG G = (A,B) of the graph G = (V,E) is the IVIFG \( \bar{G} = (\bar{A}, \bar{B}) \) where

i) \( \overline{M_A}(x) = M_A(x) \) and \( \overline{N_A}(x) = N_A(x) \)

ii) \( \overline{M}_{BL}(x,y) = \min\{M_A(x), M_A(y)\} - M_{BL}(x,y) \) and \( \overline{N}_{BL}(x,y) = \max\{N_A(x), N_A(y)\} - N_{BL}(x,y) \)

\( \overline{M}_{BU}(x,y) = \min\{M_A(x), M_A(y)\} - M_{BU}(x,y) \) and \( \overline{N}_{BU}(x,y) = \max\{N_A(x), N_A(y)\} - N_{BU}(x,y) \) for x, y in V.

**Definition 2.7** An interval valued intuitionistic fuzzy graph G = (A, B) of the graph G* = (V, E) is said to be complete if

\( M_{BL}((x,y)) = \min\{M_{AL}(x), M_{AL}(y)\} \) and \( N_{BL}((x,y)) = \max\{N_{AL}(x), N_{AL}(y)\} \)

\( M_{BU}((x,y)) = \min\{M_{AU}(x), M_{AU}(y)\} \) and \( N_{BU}((x,y)) = \max\{N_{AU}(x), N_{AU}(y)\} \)

for all (x,y) \in E.

**Example 2.8** Figure 2.2 is a complete IVIFG G = (A, B), where

\[ A = \{ <a,[0.3,0.5],[0.2,0.4]>, <b,[0.4,0.6],[0.1,0.3]>, <c,[0.2,0.4],[0.3,0.5]> \} \]

\[ B = \{ <ab,[0.3,0.5],[0.2,0.4]>, <bc,[0.2,0.4],[0.3,0.5]>, <ac,[0.2,0.4],[0.3,0.5]> \} \]

![Fig. 2.2](Image)

**Definition 2.9** An interval valued intuitionistic fuzzy graph G = (A, B) is said to be strong interval valued intuitionistic fuzzy graph if
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\[
M_{\text{bl}}((x, y)) = \min\{M_{\text{al}}(x), M_{\text{al}}(y)\} \quad \text{and} \quad N_{\text{bl}}((x, y)) = \max\{N_{\text{al}}(x), N_{\text{al}}(y)\}
\]

\[
M_{\text{bu}}((x, y)) = \min\{M_{\text{au}}(x), M_{\text{au}}(y)\} \quad \text{and} \quad N_{\text{bu}}((x, y)) = \max\{N_{\text{au}}(x), N_{\text{au}}(y)\}
\]

for all \((x, y) \in E\). i.e Every edge is effective edge.

**Example 2.10** Figure 2.3 is an SIVIFG \(G = (A,B)\), where

\[
A = \{ < a, [0.4,0.7],[0.1,0.2] >, < b, [0.3,0.6],[0.1,0.4] >, < c, [0.1,0.3],[0.4,0.6] >, < d, [0.2,0.5],[0.3,0.4] > \}
\]

\[
B = \{ < ab, [0.3,0.6],[0.1,0.4] >, < bc, [0.1,0.3],[0.4,0.6] >, < cd, [0.1,0.3],[0.4,0.6] >, < da, [0.2,0.5],[0.3,0.4] > \}
\]

![Figure 2.3](image)

**Definition 2.11** Let \(G = (A,B)\) be an IVIFG on \(V\) and let \(u,v \in V\), we say that \(v\) in \(G\) if there exist a strong arc between them. A subset \(D \subseteq V\) is said to be dominating set in \(G\) if for every \(v \notin D\) there exist \(u \in D\) such that \(u\) dominates \(v\). The minimum cardinality of a dominating set in \(G\) is called domination number of \(G\) and denoted by \(\gamma(G)\). The maximum cardinality of a minimal domination set is called upper domination number and is denoted by \(\Gamma(G)\).

### 3 Complementary Nil Domination Set in IVIFG

**Definition 3.1** Let \(G = (A,B)\) be an IVIFG of graph \(G = (V,E)\). A set \(X \subset V\) is said to be a complementary nil domination set (or simply cnd-set) of an IVIFG \(G\), if \(X\) is a dominating set and its complement \(V-X\) is not a dominating set. The minimum scalar cardinality over all cnd-set is called a complementary nil domination number and is denoted by \(\gamma_{\text{cnd}}\), the corresponding minimum cnd-set is denote by \(\gamma_{\text{cnd}}\)-set.

**Example 3.2** Let \(G = (A,B)\) be an IVIFG of graph \(G^* = (V,E)\) be defined as follows:
Here $X_1 = \{a, b, c\}$ and $X_2 = \{a, c, d\}$ are minimal cnd-sets.

**Definition 3.3** Let $X \subseteq V$ in connected IVIFG $G = (A, B)$ be an IVIFG of graph $G = (V, E)$. A vertex $x \in X$ is said to be an enclave of $X$ if

$$M_{bl}((x, y)) < \min\{M_{al}(x), M_{al}(y)\} \text{ and } N_{bl}((x, y)) < \max\{N_{al}(x), N_{al}(y)\}$$

$$M_{bu}((x, y)) < \min\{M_{au}(x), M_{au}(y)\} \text{ and } N_{bu}((x, y)) < \max\{N_{au}(x), N_{au}(y)\}$$

for all $y \in V - X$. That is $N[x] \subseteq X$

In above figure, $b$ is enclave of the set $X_1$ and $d$ is enclave of the set $X_2$.

**Theorem 3.4** A dominating set $X$ of interval-valued IFG is a cnd-set if and only if it contains at least one enclave.

**Proof:** Let $X$ be a cnd-set of interval valued intuitionistic fuzzy graph $G = (A, B)$ of the graph $G^* = (V, E)$. The $V-X$ is not a dominating set which implies that there exist a vertex $x \in X$ such that

$$M_{bl}((x, y)) < \min\{M_{al}(x), M_{al}(y)\} \text{ and } N_{bl}((x, y)) < \max\{N_{al}(x), N_{al}(y)\}$$

$$M_{bu}((x, y)) < \min\{M_{au}(x), M_{au}(y)\} \text{ and } N_{bu}((x, y)) < \max\{N_{au}(x), N_{au}(y)\}$$

for all $y \in V - X$. Therefore $x$ is enclave of $X$.

Hence $X$ contains at least one enclave.

**Conversely:** Suppose the dominating set $X$ contains enclaves. Without loss of generality let us take $x$ be the enclave of $X$.

$$M_{bl}((x, y)) < \min\{M_{al}(x), M_{al}(y)\} \text{ and } N_{bl}((x, y)) < \max\{N_{al}(x), N_{al}(y)\}$$

$$M_{bu}((x, y)) < \min\{M_{au}(x), M_{au}(y)\} \text{ and } N_{bu}((x, y)) < \max\{N_{au}(x), N_{au}(y)\}$$
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for all \( y \in V - X \).

Hence \( V - X \) is not a dominating set.

Hence dominating set \( X \) is cnd-set.

**Remark 3.5** For any IVIFG \( G = (A,B) \) of graph \( G = (V,E) \).

1. Every super set of a cnd-set is also a cnd-set.
2. Complement of a cnd-set is not a cnd-set.
3. Complement of a domination set is not a cnd-set.
4. A cnd-set need not be unique.

**Theorem 3.6** In any interval valued intuitionistic fuzzy graph \( G = (A, B) \) of the graph \( G^* = (V, E) \), every complementary nil dominating set of \( G \) intersects with every dominating set of \( G \).

**Proof:** Let \( X \) be complementary nil dominating set and \( D \) be a dominate set of interval valued intuitionistic fuzzy graph \( G = (A, B) \) of the graph \( G^* = (V, E) \).

Suppose \( D \cap X = \phi \), then \( D \subseteq V - X \) and \( V - X \) contains a dominating set \( D \).

Therefore \( V - X \), a super of \( D \), is dominating set.

Which is contradiction.

Hence \( X \cap D \neq \phi \).

**Theorem 3.7** If \( X \) is a complementary nil dominating set of an interval valued intuitionistic fuzzy graph \( G = (A, B) \) of the graph \( G^* = (V, E) \), then there is a vertex \( x \in X \) such that \( X - \{x\} \) is dominating set.

**Proof:** Let \( X \) be a cnd-set. By theorem 3.4, every cnd-set contains at least one enclave of \( X \). Then

\[
M_{BL}((x, y)) < \min\{M_{AL}(x), M_{AL}(y)\} \quad \text{and} \quad N_{BL}((x, y)) < \max\{N_{AL}(x), N_{AL}(y)\}
\]

\[
M_{BU}((x, y)) < \min\{M_{AU}(x), M_{AU}(y)\} \quad \text{and} \quad N_{BU}((x, y)) < \max\{N_{AU}(x), N_{AU}(y)\}
\]

for all \( y \in V - X \).

Since \( G \) is connected interval valued intuitionistic fuzzy graph, there exist at least a vertex \( z \in X \) such that

\[
M_{BL}((x, z)) = \min\{M_{AL}(x), M_{AL}(z)\} \quad \text{and} \quad N_{BL}((x, z)) = \max\{N_{AL}(x), N_{AL}(z)\}
\]

\[
M_{BU}((x, z)) = \min\{M_{AU}(x), M_{AU}(z)\} \quad \text{and} \quad N_{BU}((x, z)) = \max\{N_{AU}(x), N_{AU}(z)\}
\] .

Hence \( X - \{x\} \) is a dominating set.
Theorem 3.8 A complementary nil dominating set of an interval valued intuitionistic fuzzy graph $G = (A, B)$ of the graph $G^* = (V, E)$ is not singleton.

Proof: If $X$ is a complementary nil dominating set. By theorem 3.4, every cnd-set contains at least one enclave of $X$.

Let $x \in X$ be an enclave of $X$. Then

$$M_{bl}((x, y)) < \min\{M_{AL}(x), M_{AL}(y)\} \text{ and } N_{bl}((x, y)) < \max\{N_{AL}(x), N_{AL}(y)\}$$

$$M_{bu}((x, y)) < \min\{M_{AU}(x), M_{AU}(y)\} \text{ and } N_{bu}((x, y)) < \max\{N_{AU}(x), N_{AU}(y)\}$$

for all $y \in V-X$.

Suppose $X$ contains only one vertex $x$, then it must be isolated in $G$.

This is contradiction to connectedness.

Hence complementary nil dominating set contains more than one vertex.

Theorem 3.9 An interval valued intuitionistic fuzzy graph $G = (A, B)$ of the graph $G^* = (V, E)$ and $X$ be a cnd-set of $G$. If $x$ and $y$ are two enclaves in $X$ then

(i) $N[x] \cap N[y] \neq \emptyset$ and

(ii) $x$ and $y$ are adjacent.

Proof: Let $X$ be a minimum cnd-set of IVIFG, $G = (A,B)$ of graph $G = (V,E)$. Let $x$ and $y$ are two enclaves of $X$.

Suppose $N[x] \cap N[y] = \emptyset$, then $x$ is an enclave of $X-N[y]$ which implies that $V-(X-N[y])$ is not a dominating set.

Therefore $X-N[y]$ is not a cnd-set of $G$ and $|X-N[y]| < |x| = \gamma_{cnd}(G)$ which is contradiction to the minimality of $X$.

Then $N[x] \cap N[y] \neq \emptyset$

Suppose

$$M_{bl}((x, y)) < \min\{M_{AL}(x), M_{AL}(y)\} \text{ and } N_{bl}((x, y)) < \max\{N_{AL}(x), N_{AL}(y)\}$$

$$M_{bu}((x, y)) < \min\{M_{AU}(x), M_{AU}(y)\} \text{ and } N_{bu}((x, y)) < \max\{N_{AU}(x), N_{AU}(y)\}$$

for all $y \in V-X$. 
that is x and y are non-adjacent. Then \( x \notin N[y] \) and x is an enclave of \( X-\{y\} \).

\( V-(X-\{y\}) \) is not a dominating set.

Hence \( X-\{y\} \) is a cnd-set, which is contradiction to minimality of \( X \).

Hence x and y are adjacent.

**Theorem 3.10** For any Interval-valued fuzzy graph \( G = (A, B) \) of graph \( G = (V,E) \), \( \Gamma(G) + \gamma_{\text{cnd}}(G) \leq O(G) + \max\{|v_i|\} \) for every \( v_i \in V \).

**Proof:** If \( G \) is a IVIFG then cnd-set \( \subseteq V \) and super set of \( \gamma \)-set

i.e. \( \gamma_{\text{cnd}}(G) \leq O(G) \)

and \( \Gamma(G) = \max \{ \) minimal dominating set \( \}

i.e. \( \Gamma(G) \leq \max\{|v_i|\} \)

Therefore, we have

\[ \Gamma(G) + \gamma_{\text{cnd}}(G) \leq O(G) + \max\{|v_i|\}. \]

**Example 3.11** Let \( G = (A, B) \) of graph \( G^* = (V, E) \), where \( V = \{a, b, c, d, e, f\} \) and \( E = \{ab, ad, af, be, cd, cf, ef\} \). Let \( A \) is an interval valued intuitionistic fuzzy set of \( V \) and let \( B \) is an interval-valued intuitionistic fuzzy set of \( E \subseteq V \times V \) defined by

\[
A = \{(a, \{0.2,0.4\},\{0.1,0.5\}), (b, \{0.1,0.5\},\{0.2,0.3\}), (c, \{0.4,0.5\},\{0.1,0.3\}), (d, \{0.3,0.6\},\{0.2,0.4\}), (e, \{0.4,0.5,0.1,0.1,0.3\}), (f, \{0.2,0.4\},\{0.1,0.5\})\}
\]

\[
B = \{(ab, \{0.1,0.4\},\{0.2,0.5\}), (ad, \{0.2,0.4\},\{0.2,0.5\}), (af, \{0.2,0.4\},\{0.1,0.5\}), (be, \{0.1,0.5\},\{0.2,0.4\}), (cd, \{0.3,0.4\},\{0.2,0.5\}), (cf, \{0.3,0.5\},\{0.1,0.4\}), (ef, \{0.3,0.5\},\{0.2,0.4\})\}.
\]

![Fig. 3.2](image-url)
Here $X_1 = \{a, b, c, d\}$ and $X_2 = \{b, c, e, f\}$ are minimal complementary nil dominating set. Also minimal dominating set ($\gamma$-set) = \{b, c\}.

(i) $|a| = 1.3, |b| = 1.25, |c| = 1.15, |d| = 1.25, |e| = 1.2, |f| = 1.25, \ O(G) = p = 7.4$

(ii) $\gamma_{cnd}(G) = 4.85$ and $\Gamma_{cnd}(G) = 4.95$

(iii) $\Gamma(G) = \Gamma(G) = 2.4$

(iv) The vertices a and d are two enclaves with respect to $X_1$. The vertices e and f are two enclaves with respect to $X_2$.

(v) $N[a] = \{a, b, d\}, N[d] = \{a, c, d\}$

i.e. $N[a] \cap N[d] \neq \phi$

Also a and d are adjacent.

(vi) The vertex $d \in X_1$ and $X_1\{d\}$ is a dominating set.

(vii) $Min[|v_i|] = 1.15, Max[|v_i|] = 1.3$ for every $v_i \in V$

Also $\Gamma(G) + \gamma_{cnd}(G) = 4.85 + 2.4 = 7.25$

$O(G) + Max[|v_i|] = 7.4 + 1.3 = 8.7$

i.e $\Gamma(G) + \gamma_{cnd}(G) \leq O(G) + Max[|v_i|]$

References


