Derivations of Operators on Hilbert Modules

A. Sahleh\(^1\) and L. Najarpisheh\(^2\)

\(^1\)\(^2\)Department of Mathematics, University of Guilan
P.O. Box 1914, Rasht, Iran
\(^1\)E-mail: sahlehj@guilan.ac.ir
\(^2\)E-mail: najarpisheh@phd.guilan.ac.ir

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Abstract

Let \(A\) be a \(C^*\)-algebra and \(X\) be a right Hilbert \(A\)-module. In this paper we study the relation between innerness of derivations on \(\mathcal{K}(X)\), compact operators on \(X\), and \(\mathcal{L}(X)\), adjointable operators on \(X\), also we show that with the certain conditions every derivation on \(\mathcal{K}(X)\) and \(\mathcal{L}(X)\) is zero.

Keywords: \(C^*\)-algebra, Hilbert \(C^*\)-module, Derivation.

1 Introduction

Hilbert \(C^*\)-modules were first introduced in the work of I. Kaplansky [5]. Hilbert \(C^*\)-modules are very useful in operator \(K\)-theory, operator algebra, Morita equivalence and others. Hilbert \(C^*\)-modules form a category in between Banach spaces and Hilbert spaces and obey the same axioms as a Hilbert space except that inner product takes values in a \(C^*\)-algebra rather than in the complex numbers. Let us recall some basic facts about the Hilbert \(C^*\)-modules.

Let \(A\) be a \(C^*\)-algebra. An right inner product \(A\)-module is a linear space \(X\) which is a right \(A\)-module (with compatible scalar multiplication: \(\lambda(x.a) = (\lambda x).a = x.(\lambda a)\) for \(x \in X, a \in A, \lambda \in \mathbb{C}\)), together with a map \((x, y) \mapsto \langle x, y \rangle_X : X \times X \rightarrow A\) such that for all \(x, y, z \in X, a \in A, \alpha, \beta \in \mathbb{C}\)

\begin{align*}
(i) \quad & \langle x, \alpha y + \beta z \rangle_X = \alpha \langle x, y \rangle_X + \beta \langle x, z \rangle_X; \\
(ii) \quad & \langle x, y.a \rangle_X = \langle x, y \rangle_X a; \\
(iii) \quad & \langle y, x \rangle_X = \langle x, y \rangle_X^*; \\
(iv) \quad & \langle x, x \rangle_X \geq 0; \quad \text{if} \quad \langle x, x \rangle_X = 0 \quad \text{then} \quad x = 0.
\end{align*}
A right pre-Hilbert $A$-module $X$ is called a right Hilbert $A$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle_X\|^{\frac{1}{2}}$. $X$ is said to be full if the linear span of the set $\{\langle x, y \rangle_X : x, y \in X\}$ is dense in $A$. One interesting example of full right Hilbert $C^*$-modules is any $C^*$-algebra $A$ as a right Hilbert $A$-module via $\langle a, b \rangle_A = a^*b$ ($a, b \in A$).

Likewise, a left Hilbert $A$-module with an $A$-valued inner product $\langle ., . \rangle$ can be defined.

Let $X$ be a right Hilbert $A$-module, we define $\mathcal{L}(X)$ to be the set of all maps $T : X \rightarrow X$ for which there is a map $T^* : X \rightarrow X$ such that $\langle Tx, y \rangle_X = \langle x, T^*y \rangle_X$ ($x, y \in X$). It is easy to see that $T$ must be bounded $A$-linear and $\mathcal{L}(X)$ is a $C^*$-algebra. For $x, y \in X$, define the operator $\theta_{x,y}$ on $X$ by $\theta_{x,y}(z) = x \cdot \langle y, z \rangle_X$ ($z \in X$). Denote by $\mathcal{K}(X)$ the closed linear span of $\{\theta_{x,y} : x, y \in X\}$, then $\mathcal{K}(X)$ is a closed two sided ideal in $\mathcal{L}(X)$. The reader is referred to [6] for more details on Hilbert $C^*$-modules.

In this paper a derivation of an algebra $A$ is a linear mapping $D$ from $A$ into itself such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. For a fixed $b \in A$, the mapping $a \mapsto ab - ba$ is clearly a derivation, which is called an inner derivation implemented by $b$.

## 2 Derivations of $\mathcal{K}(X)$ and $\mathcal{L}(X)$

The aim of present section is to study the derivations of operators on a Hilbert $C^*$-module. Throughout this section $A$ is a $C^*$-algebra.

**Lemma 2.1** [7] Every derivation of a $C^*$-algebra is bounded.

**Theorem 2.2** Let $X$ be a full right Hilbert $A$-module. If every derivation of $\mathcal{K}(X)$ is inner, then any derivation of $\mathcal{L}(X)$ is also inner.

**Proof:** Let $D$ be a derivation of $\mathcal{L}(X)$ and let $x, y \in X$. By Cohen’s factorization Theorem [3] for $x$ there exists $a \in A$, $z \in X$ such that $x = za$. Moreover since $X$ is full, there exists $(a_n)$ in $\langle X, X \rangle_X$ such that $a = \lim_n a_n$. Each $a_n$ is of the form $a_n = \sum_{i=1}^{k_n} \langle x_{in}, y_{in} \rangle_X$ in which $x_{in}, y_{in} \in X$. Hence

$$D(\theta_{z,a_n,y}) = D(\theta_{z,\sum_{i=1}^{k_n} z \cdot \langle x_{in}, y_{in} \rangle_X, y})$$

$$= D(\sum_{i=1}^{k_n} \theta_{z, \langle x_{in}, y_{in} \rangle_X, y})$$

$$= \sum_{i=1}^{k_n} D(\theta_{z,x_{in}, \theta_{y_{in}, y}})$$

$$= \sum_{i=1}^{k_n} \theta_{z,x_{in}} D(\theta_{y_{in}, y}) + \sum_{i=1}^{k_n} D(\theta_{z,x_{in}}) \theta_{y_{in}, y} \in \mathcal{K}(X).$$
Since \( \| \sum_{i=1}^{k_n} z_i \langle x_{in}, y_{in} \rangle_X - x \| \leq \| \sum_{i=1}^{k_n} z_i \langle x_{in}, y_{in} \rangle_X \| y \| \), we get \( \sum_{i=1}^{k_n} z_i \langle x_{in}, y_{in} \rangle_X \) converges to \( \theta_{x,y} \) in norm topology, as \( n \) tends to \( \infty \). It follows from Lemma (2.1) that \( D \) maps \( \mathcal{K}(X) \) into itself. Now since every derivation of \( \mathcal{K}(X) \) is inner, there exists \( T \in \mathcal{K}(X) \) such that \( D(K) = KT - TK \) for all \( K \in \mathcal{K}(X) \). Now for \( S \in \mathcal{L}(X) \) and \( \theta_{x,y} \) we have \( D(S\theta_{x,y}) = S\theta_{x,y}T - TS\theta_{x,y} \).

On the other hand,

\[
D(S\theta_{x,y}) = SD(\theta_{x,y}) + D(S)\theta_{x,y} = S\theta_{x,y}T - ST\theta_{x,y} + D(S)\theta_{x,y}
\]

Consequently, we obtain \( D(S)\theta_{x,y} = (ST - TS)\theta_{x,y} \). So for all \( z \in X \), \( D(S)\theta_{x,y}(z) = (ST - TS)\theta_{x,y}(z) \). Now since \( X \) is full, for every \( u \in X \) there exist \( x \in X \) and \( (a_n) \subseteq A \) such that \( u = \lim_n x.a_n \) and every \( a_n \) is of the form \( a_n = \sum_{i=1}^{k_n} \langle x_{in}, y_{in} \rangle_X \) in which \( x_{in}, y_{in} \in X \). Now since \( D(S), (ST - TS) \in \mathcal{L}(X) \) we have

\[
D(S)(u) = D(S)(\lim_n \sum_{i=1}^{k_n} x_{in} \langle x_{in}, y_{in} \rangle_X) = \lim_n D(S)(\sum_{i=1}^{k_n} x_{in} \langle x_{in}, y_{in} \rangle_X) = \lim_n \sum_{i=1}^{k_n} D(S)(x_{in} \langle x_{in}, y_{in} \rangle_X) = \lim_n \sum_{i=1}^{k_n} (ST - TS)(x_{in} \langle x_{in}, y_{in} \rangle_X) = (ST - TS)(\lim_n \sum_{i=1}^{k_n} x_{in} \langle x_{in}, y_{in} \rangle_X) = (ST - TS)(u).
\]

Hence \( D(S) = ST - TS \) and this completes the proof.

The following definition of a Hilbert bimodule is orginally due to Brown, Mingo and Shen [2].

**Definition 2.3** Let \( X \) be an \( A \)-bimodule. \( X \) is said to be a Hilbert \( A \)-bimodule, when \( X \) is a left and right Hilbert \( A \)-module and satisfies the relation \( X \langle x, y \rangle.z = x \langle y, z \rangle_X \).

**Example 2.4** Let \( A \) be a \( C^* \)-algebra. Then \( A \) is a Hilbert \( A \)-bimodule with left and right inner products given by \( \langle a, b \rangle = ab^* \) and \( \langle a, b \rangle_A = a^*b \) \( (a, b \in A) \).

**Proposition 2.5** Let \( X \) be a Hilbert \( A \)-bimodule. If \( A \) is commutative then \( \mathcal{K}(X) \) is commutative.
Proof: Since $K(E)$ is the closed linear span of $\{\theta_{x,y} : x, y \in X\}$, we show that $\theta_{x,y}\theta_{u,v}(z) = \theta_{u,v}\theta_{x,y}(z)$ for every $x, y, u, v, z \in X$.

$$
\theta_{x,y}\theta_{u,v}(z) = \theta_{x,y}\langle y, u \rangle_X \langle z, v \rangle_X = x\langle y, u \rangle_X \langle v, z \rangle_X = x\langle y, u \rangle_X \langle v, z \rangle.
$$

$$
\theta_{u,v}\theta_{x,y}(z) = \theta_{u,v}\langle x, y \rangle_X \langle z, v \rangle_X = u\langle x, y \rangle_X \langle y, z \rangle_X = u\langle x, y \rangle_X \langle y, z \rangle.
$$

Therefore $\theta_{x,y}\theta_{u,v} = \theta_{u,v}\theta_{x,y}$, as claimed.

Remark 2.6 Let $A$ be a $C^*$-algebra. In [4, theorem 2] R. V. Kadison showed that each derivation of $A$ annihilates its center.

Corollary 2.7 Let $X$ be a Hilbert $A$-bimodule. If $A$ is commutative then every derivation on $K(X)$ is zero.

Proof: Since $A$ is commutative, the $C^*$-algebra $K(X)$ is commutative. So by remark (2.6) every derivation on $K(X)$ is zero.

Let $X$ be a Hilbert $A$-bimodule and $T \in L(X)$. Then there exists an operator $T^* \in L(X)$ such that $\langle Tx, y \rangle_X = \langle x, T^*y \rangle_X (x, y \in X)$. Here there exist one interesting point about $T$ and $T^*$, in fact we can't conclude that $\langle Tx, y \rangle = \langle x, T^*y \rangle$. For example let $A$ be a noncommutative $C^*$-algebra and $Z(A)$ be the center of $A$. Then for Hilbert $A$-bimodule $A$, the operator $T_c$ (c$\notin Z(A)$) defined by $T_c(a) = ca$ on $A$ is a adjointable operator and $T_c^* = T_c^*$, because

$$
\langle T_c(a), b \rangle_A = \langle ca, b \rangle_A = \langle ca \rangle^*b = a^*c^*b = \langle a, c^*b \rangle_A = \langle a, T_c^*b \rangle_A.
$$

But since $\langle T_c(a), b \rangle = \langle ca, b \rangle = cab^*$ and $\langle a, T_c^*b \rangle = \langle a, c^*b \rangle = a(c^*b)^* = ab^*c$, we have $\langle T_c(a), b \rangle \neq \langle a, T_c^*b \rangle$.

Remark 2.8 Let $A$ be a commutative $C^*$-algebra and $X$ a Hilbert $C^*$-bimodule over $A$. In [1, Proposition 1.4] B. Abadie and R. Exel proved that $\langle x, y \rangle_z = \langle x, y \rangle_z \cdot x$ for all $x, y, z \in X$. By this Proposition, for all $x, y, z, t \in X$ we have:

$$
\langle x, \langle z, y \rangle, z, t \rangle = \langle x, \langle z, y \rangle, x, t \rangle = \langle z, y \rangle \langle x, t \rangle = \langle x, t \rangle \langle z, y \rangle = \langle x, t, z \rangle \langle x, z, y \rangle = \langle x, t, z \rangle, y \rangle.
$$
Proof: Suppose that $u = \langle x, T \rangle y - \langle x, T^* y \rangle$, we prove that $uu^* = 0$.

\[
\begin{align*}
uu^* & = \langle x, T \rangle x \langle T \rangle y - \langle x, T^* y \rangle \langle T \rangle x \langle T^* y, x \rangle \\
& = \langle x, T \rangle x \langle y, T x \rangle - \langle x, T^* y, x \rangle \\
& - \langle x, T^* y \rangle x \langle y, T x \rangle + \langle x, T^* y \rangle \langle y, T x \rangle \\
& = \langle x, T \rangle x \langle y, T x \rangle - \langle x, T^* y, x \rangle \\
& - \langle x, T^* y \rangle x \langle y, T x \rangle + \langle x, T^* y \rangle \langle y, T x \rangle \\
& = 0.
\end{align*}
\]

Thus we conclude that $\langle x, T \rangle y - \langle x, T^* y \rangle = 0$ as claimed.

Theorem 2.10 Let $X$ be a Hilbert $A$-bimodule. If $A$ is commutative then every derivation on $\mathcal{L}(X)$ is zero.

Proof: Let $D$ be a derivation of $\mathcal{L}(X)$. First notice that for every $x, y$ in $X$, the operator $\theta_{x,y}$ belongs to the center of $\mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$, then $\theta_{x,y}T(z) = \theta_{x,T^* y}(z) = x, T^* y, x \rangle = x, T^* y, x \rangle$. Now by Proposition (2.9) we have

\[
\begin{align*}
\theta_{x,y}T(z) & = \langle x, T \rangle y z = \langle x, T \rangle \langle y, z \rangle x = \theta_{T,\theta}(z) = T \theta_{x,y}(z).
\end{align*}
\]

So Remark (2.6) implies that for every $x, y$ in $X$, $D(\theta_{x,y}) = 0$. Now we prove that for every operator $T \in \mathcal{L}(X)$, $D(T) = 0$. For this goal, let $x \in X$. Thus $D(T)\theta_{x,D(T)(x)} = D(T)\theta_{x,D(T)(x)} - TD(\theta_{x,D(T)(x)}) = 0$.

Hence for every $z \in X$ we conclude that

\[
D(T)\theta_{x,D(T)(x)}(z) = D(T)(x, D(T)(x), z) = D(T)(x, D(T)(x), z) = 0
\]

Now by setting $z = D(T)(x)$ we have $D(T)(x, D(T)(x), D(T)(x)) = 0$ and so $D(T)(x, D(T)(x), D(T)(x)) = 0$. This implies that $D(T)(x, D(T)(x)) = 0$, consequently we obtain $D(T)(x) = 0$ and the proof is complete.
References


